GIRRAWEEN HIGH SCHOOL
MATHEMATICS

Year 12 Extension 2 Task 3 — Friday 15th June 2007

Instructions:
a) Write all your answers on your own paper.
b) Show all necessary working.
c) Marks may be deducted for careless or badly arranged work.

Time Allowed: 90 minutes

Question 1 *(27 marks)* — *Marks*

Find the following integrals:

(i) $\int \cos^3 dx$ — *4*

(ii) $\int \frac{dx}{4x^2 - 9}$ — *3*

(iii) $\int \frac{x^2}{\sqrt{9 - x^2}} dx$ — *4*

(iv) $\int \sin^{-1} x dx$ — *4*

(v) $\int \frac{dx}{1 + \cos^2 x}$ — *4*

(vi) $\int \frac{x+1}{\sqrt{x^2 + x + 1}} dx$ — *4*

(vii) $\int \frac{x^2 dx}{(x-1)^2(x^2+1)}$ — *4*

Question 2 *(13 marks)*

a) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are NOT required to evaluate the integrals)

(i) $\int_0^1 e^{-\frac{1}{2}x^2} dx = 0$ — *1*

(ii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7 x dx = 0$ — *1*

(iii) $\int_0^{\pi} \cos^9 x dx > 0$ — *1*

(iv) $\int_0^1 \frac{dx}{\sqrt{1 + x^7}} > \int_0^1 \frac{dx}{\sqrt{1 + x^8}}$ — *1*

b) By using the fact that $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x} = \log 2$, evaluate $\int_0^{\frac{\pi}{2}} \frac{x dx}{1 + \cos x + \sin x}$ — *3*

Question 2...*continued* *Marks*

c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

(i) Show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$, for $n \geq 2$ *3*

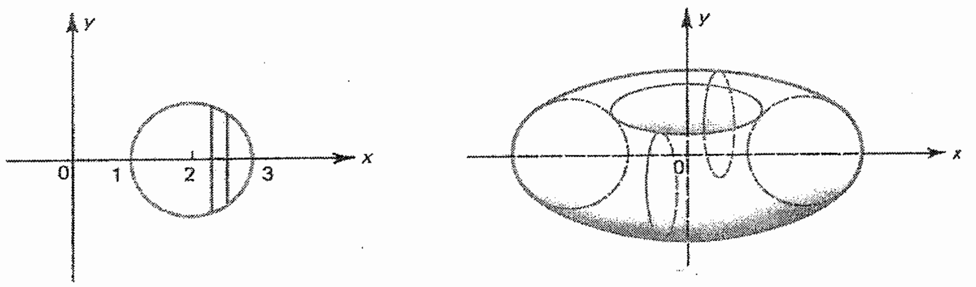
(ii) Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ *3*

Question 3 *(21 marks)*

a) The region between the curve $y = x^2 + 2$, the line $y = x + 8$ is rotated around the x-axis. Find the volume formed by taking slices perpendicular to the x-axis. *5*

b) The base of a certain solid is the region between the line $y = x$ and the curve $y = x^2$. Find the volume of the solid if cross sections perpendicular to the x-axis are squares. *5*

c) The circular disk $(x-2)^2 + y^2 \leq 1$ is rotated about the y-axis to form a torus.

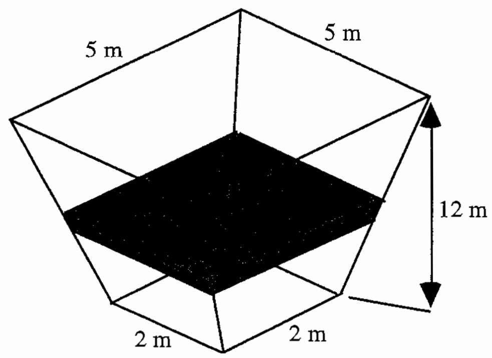


(i) Using the method of cylindrical shells, show that the volume ΔV of a typical shell distant x units from the origin and with thickness Δx is given by *3*

$$\Delta V = 4\pi x\sqrt{1-(x-2)^2}\,\Delta x$$

(ii) Hence, find the total volume of the solid formed. *3*

d) The tank illustrated below has plane sides with square horizontal cross-sections whose sides vary in length from 2 m at the base to 5 m at the top; the height is 12 m.



By taking slices parallel to the base of the tank find the volume of the tank. *5*

Year 12 Extension 2 Task 3 2007

Question 1 (27)

(i) $\int \cos^3 x\,dx$ $\qquad u = \sin x$, $du = \cos x\,dx$

$= \int \cos x(1-\sin^2 x)\,dx$

$= \int (1-u^2)\,du$

$= u - \frac{1}{3}u^3 + c$

$= \sin x - \frac{1}{3}\sin^3 x + c$ (4)

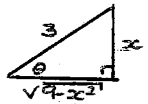
(ii) $\int \frac{dx}{4x^2-9}$

$= \frac{1}{4}\int \frac{dx}{x^2-\frac{9}{4}}$

$= \frac{1}{4}\cdot\frac{1}{3}\log\left(\frac{x-\frac{3}{2}}{x+\frac{3}{2}}\right) + c$

$= \frac{1}{12}\log\left(\frac{2x-3}{2x+3}\right) + c$ (3)

(iii) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ $\qquad x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$



$= \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{3\cos\theta}$

$= 9\int \sin^2\theta\,d\theta$

$= \frac{9}{2}\int (1-\cos 2\theta)\,d\theta$

$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c$

$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + c$

$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c$

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + c$ (4)

(iv) $\int \sin^{-1}x\,dx$ $\qquad u = \sin^{-1}x$, $v = x$; $du = \frac{dx}{\sqrt{1-x^2}}$, $dv = dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$ (4)

(v) $\int \frac{dx}{1+\cos^2 x}$

$= \int \frac{dx}{1+\frac{1}{2}(1+\cos 2x)}$

$= 2\int \frac{dx}{3+\cos 2x}$ $\qquad t = \tan x$, $dx = \frac{dt}{1+t^2}$

$= 2\int \frac{\frac{dt}{1+t^2}}{3+\left(\frac{1-t^2}{1+t^2}\right)}$

$= 2\int \frac{dt}{3+3t^2+1-t^2}$

$= 2\int \frac{dt}{4+2t^2}$

$= \int \frac{dt}{2+t^2}$

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c$ (4)

(vi) $\int \frac{x+1}{\sqrt{x^2+x+1}}\,dx$

$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$ (4)

$= \sqrt{x^2+x+1} + \frac{1}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + c$

(vii) $\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{x^2}{(x-1)^2(x^2+1)}$

$A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 = x^2$

$x=1$: $2B = 1$, $B = \frac{1}{2}$

$x=i$: $2C - 2iD = -1$, $C = -\frac{1}{2}$, $D = 0$

$x=0$: $-A+B+D = 0$, $-A+\frac{1}{2} = 0$, $A = \frac{1}{2}$

$\int \frac{x^2}{(x-1)^2(x^2+1)}\,dx$

$= \frac{1}{2}\int\left(\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{x}{x^2+1}\right)dx$ (4)

$= \frac{1}{2}\log(x-1) - \frac{1}{2(x-1)} - \frac{1}{4}\log(x^2+1) + c$

Question 2 (13)

a)(i) $\int_0^1 e^{-\frac{1}{2}x^2}\,dx = 0$

FALSE (1)

The exponential curve is completely above the x axis $\therefore \int > 0$

(ii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7 x\,dx = 0$

TRUE (1)

odd function $\therefore$ curve has rotational symmetry.

(iii) $\int_0^{\pi} \cos^9 x\,dx > 0$

FALSE (1)

function has rotational symmetry about $x = \frac{\pi}{2}$, $\therefore \int = 0$

(iv) $\int_0^1 \frac{dx}{\sqrt{1+x^7}} > \int_0^1 \frac{dx}{\sqrt{1+x^8}}$

FALSE (1)

for the domain $0 < x < 1$

$\frac{1}{\sqrt{1+x^7}} < \frac{1}{\sqrt{1+x^8}}$

$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^7}} < \int_0^1 \frac{dx}{\sqrt{1+x^8}}$

b) $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$

$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2}-x)\,dx}{1+\cos(\frac{\pi}{2}-x)+\sin(\frac{\pi}{2}-x)}$

$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2}-x)\,dx}{1+\sin x+\cos x}$

$\therefore 2\int_0^1 \frac{x\,dx}{1+\sin x+\cos x} = \frac{\pi}{2}\int_0^1 \frac{dx}{1+\sin x+\cos x}$

$= \frac{\pi}{2}(\log 2)$

$\int_0^1 \frac{x\,dx}{1+\sin x+\cos x} = \frac{\pi}{4}\log 2$ (3)

c) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ $\qquad u = \sin^{n-1}x$, $v = -\cos x$; $du = (n-1)\sin^{n-2}x\cos x\,dx$, $dv = \sin x\,dx$

$= \left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x(1-\sin^2 x)\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2}x - \sin^n x)\,dx$

$= (n-1)I_{n-2} - (n-1)I_n$

$\therefore nI_n = (n-1)I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$ (3)

(ii) $\int_0^{\frac{\pi}{2}} \sin^{2n}dx = I_{2n}$

$= \frac{2n-1}{2n} I_{2n-2}$

$= \frac{(2n-1)}{2n} \times \frac{(2n-3)}{(2n-2)} I_{2n-4}$

$= \frac{(2n-1)(2n-3)(2n-5)}{2n(2n-2)(2n-4)} I_{2n-6}$

$= \frac{(2n-1)(2n-3)(2n-5)\ldots(7)(5)(3)}{2n(2n-2)(2n-4)\ldots(8)(6)(4)} I_0$

$= \frac{(2n-1)(2n-3)(2n-5)\ldots(7)(5)(3)}{2(n)2(n-1)2(n-2)\ldots 2(4)2(3)2(2)} I_0$

$= \frac{(2n-1)(2n-3)(2n-5)\ldots(7)(5)(3)}{2^n(n!)} I_0$

$= \frac{(2n-1)(2n-3)(2n-5)\ldots(7)(5)(3)}{2^n(n!)} I_0 \times \frac{2n(2n-2)(2n-4)\ldots(6)(4)(2)}{2n(2n-2)(2n-4)\ldots(6)(4)(2)}$

$= \frac{2n(2n-1)(2n-2)\times\ldots(3)(2)}{2^{2n}(n!)^2} I_0$

$= \frac{(2n)!}{2^{2n}(n!)^2}\int_0^{\frac{\pi}{2}} dx$

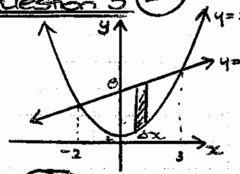
$= \frac{(2n)!}{2^{2n}(n!)^2}\left[x\right]_0^{\frac{\pi}{2}}$

$= \frac{(2n)!}{2^{2n}(n!)^2} \times \frac{\pi}{2}$

$= \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ (3)

Question 3 (21)

a)

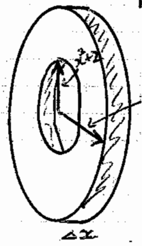


$x^2+2 = x+8$

$x^2-x-6=0$

$(x-3)(x+2)=0$

$x=3$ or $x=-2$



$A(x)=\pi\left[(x+8)^2-(x^2+2)^2\right]$

$\Delta V=\pi\left(x^2+16x+64-x^4-4x^2-4\right)\Delta x$

$V=\lim_{\Delta x\to 0}\sum_{x=-2}^{3}\pi\left(-x^4-3x^2+16x+60\right)\Delta x$

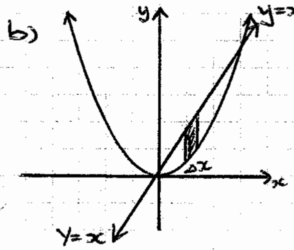
$=\pi\int_{-2}^{3}\left(-x^4-3x^2+16x+60\right)dx$

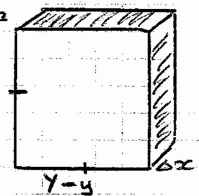
$=\pi\left[-\frac{1}{5}x^5-x^3+8x^2+60x\right]_{-2}^{3}$

$=\pi\left(-\frac{243}{5}-27+72+180-\frac{32}{5}-8-32+120\right)$

$=\underline{\underline{250\pi \text{ units}^3}}$ (5)

b)





$Y-y$

$= x-x^2$

$A(x) = (x-x^2)^2$

$= x^2-2x^3+x^4$

$\Delta V = (x^2-2x^3+x^4)\,\Delta x$

$V=\lim_{\Delta x\to 0}\sum_{x=0}^{1}(x^2-2x^3+x^4)\,\Delta x$

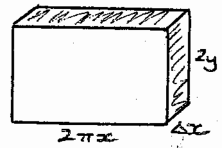
$=\int_0^1 (x^2-2x^3+x^4)\,dx$

$=\left[\frac{1}{3}x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\right]_0^1$

$=\frac{1}{3}-\frac{1}{2}+\frac{1}{5}$

$=\underline{\underline{\frac{1}{30}\text{ units}^3}}$ (5)

c)



$(x-2)^2+y^2=1$

$y^2 = 1-(x-2)^2$

$y=\pm\sqrt{1-(x-2)^2}$

$A(x) = 2\pi x \cdot 2\sqrt{1-(x-2)^2}$

$= 4\pi x\sqrt{1-(x-2)^2}$

$\underline{\underline{\Delta V = 4\pi x\sqrt{1-(x-2)^2}\,\Delta x}}$ (3)

(ii) $V=\lim_{\Delta x\to 0}\sum_{x=1}^{3}4\pi x\sqrt{1-(x-2)^2}\,\Delta x$

$=4\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$

$x-2=\sin\theta$

$dx=\cos\theta\, d\theta$

$V=4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2+\sin\theta)\cos^2\theta\, d\theta$

$=16\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$ (NOTE: $\sin\theta\cos^2\theta$ is odd function)

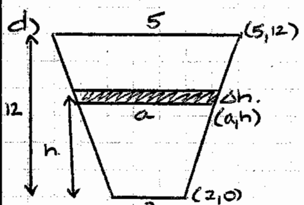
$=8\pi\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$

$=8\pi\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$=8\pi\left(\frac{\pi}{2}+0-0\right)$

$=\underline{\underline{4\pi^2\text{ units}^3}}$ (3)

d)



$m=\frac{12-0}{5-2}$

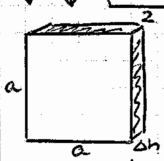
$=4$

$h-0=4(a-2)$

$h=4a-8$

$4a=h+8$

$a=\frac{1}{4}h+2$



$A(h)=a^2$

$=\left(\frac{1}{4}h+2\right)^2$

$\Delta V=\left(\frac{1}{4}h+2\right)^2\Delta h$

$V=\lim_{\Delta h\to 0}\sum_{h=0}^{12}\left(\frac{1}{4}h+2\right)^2\Delta h$

$=\frac{1}{16}\int_0^{12}(h+8)^2\,dh$

$=\frac{1}{48}\left[(h+8)^3\right]_0^{12}$

$=\underline{\underline{156\text{ m}^3}}$ (5)