GOSFORD HIGH SCHOOL
MATHEMATICS
Extension 2
Assessment task: 30/05/06

Time: 60 minutes.

Question 1.

For the parabola $x^2+9y^2=9$ find:

a) The eccentricity. (2).

b) The length of the major and minor axes. (2)

c) The co-ordinates of the foci. (1)

d) The equation of the directrices. (1)

e) Sketch the ellipse. (2)

Question 2.

a) By using cylindrical shells find the volume of the solid of revolution when the area enclosed by the graph of $y=e^{2x}$, the 'y' axis and the horizontal line $y=e^2$ is rotated about the 'y' axis. (3)

b) Calculate the volume when the region between the curve $y=2x-x^2$, the 'x' axis from $x=0$, to $x=2$ is rotated about 'y' axis. (4)

Question 3.

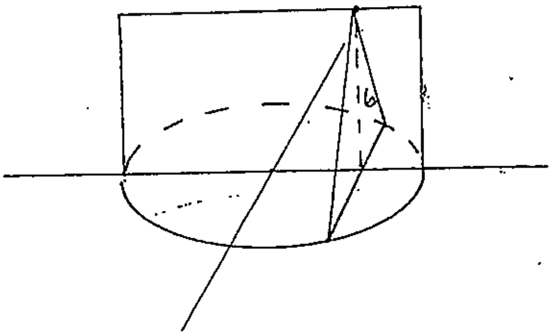
a) i) Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is given by $\frac{\cos\theta x}{a}+\frac{\sin\theta y}{b}=1$ (3)

ii) the tangent at a point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts the x-axis at M, while the normal at P cuts the x-axis at N.
Prove that $OM.ON= a^2e^2$,where O is the centre of the ellipse.
(you may assume the equation of the normal at P is given by $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$). (3)

b) Prove that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that the chord of contact from a point on the directrix is a focal chord. (3)

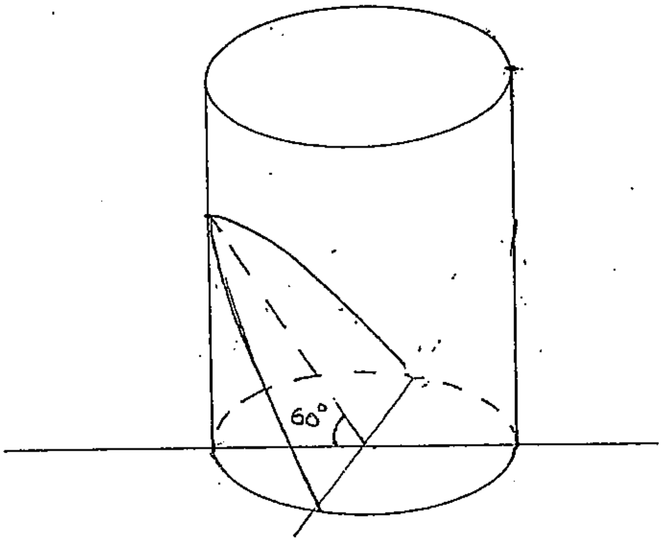
Question 4.

a)



A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. Find the volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6 units. (3)

b)



Two cuts are made on a circular log of radius 8 centimetres, the first perpendicular to the axis of the log and the second inclined at an angle of 60° with the first. If the two cuts meet on a line through the centre, find the volume of the wood cut out. (4)

Ext 2. Assessment Task Solutions. 30/5/06

Q1) $x^2 + 9y^2 = 9$

$$\frac{x^2}{9} + y^2 = 1$$

$a = 3, \ b = 1$

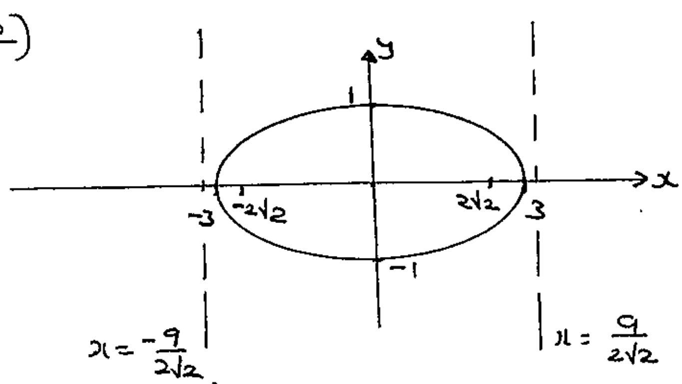
a)
$$b^2 = a^2(1-e^2)$$
$$1 = 9(1-e^2)$$
$$\frac{1}{9} = 1 - e^2$$
$$e^2 = \frac{8}{9}$$
$$e = \frac{2\sqrt{2}}{3}$$

b) major axis $= 6$
minor axis $= 2$.

c) foci $(\pm ae, 0)$
$$= \left(\pm 3 \times \frac{2\sqrt{2}}{3}, 0\right)$$
$$= (\pm 2\sqrt{2}, 0)$$

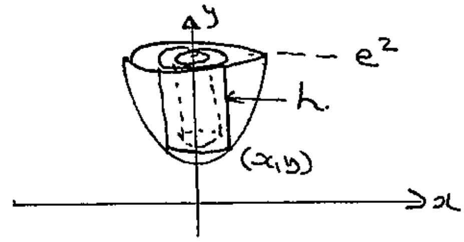
d) directrices.
$$x = \pm \frac{a}{e}$$
$$= \pm \frac{3}{\frac{2\sqrt{2}}{3}}$$
$$x = \pm \frac{9}{2\sqrt{2}}$$

e)



Q2.)

a)



Volume of shell $= 2\pi r h \cdot \Delta x$
$$= 2\pi x(e^2 - y)\Delta x.$$

$$\therefore V \doteqdot \sum_{x=0}^{1} 2\pi x(e^2 - y)\Delta x$$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x(e^2 - y)\Delta x$$

$$= 2\pi \int_0^1 x(e^2 - y)\, dx$$

$$= 2\pi \int_0^1 x(e^2 - e^{2x})\, dx$$

$$= 2\pi \int_0^1 xe^2 - xe^{2x}\, dx.$$

$$= 2\pi \left[\left[\frac{e^2}{2}x^2\right]_0^1 - \int_0^1 x \cdot \frac{d}{dx}\left(\tfrac{1}{2}e^{2x}\right) dx \right]$$

$$= 2\pi \left[\frac{e^2}{2} - \left(\frac{x}{2}e^{2x} - \int_0^1 \tfrac{1}{2}e^{2x}\right) \right]$$

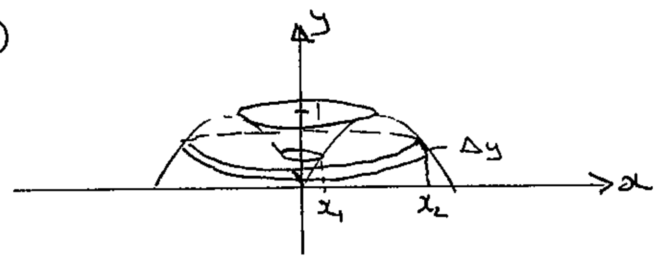
$$= 2\pi \left[\frac{e^2}{2} - \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^1 \right]$$

$$= 2\pi \left[\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{e^2}{4} - \left(0 - \frac{1}{4}\right)\right) \right]$$

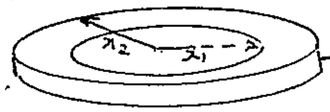
$$= 2\pi \left[\frac{e^2}{4} - \frac{1}{4} \right]$$

$$= \frac{\pi}{2}\left[e^2 - 1\right] \text{ cubic units}$$

b)



taking a slice perpendicular to the 'y' axis gives an annulus.



Volume of a Slice $= \pi\left(x_2^2 - x_1^2\right)\Delta y$

$$V \doteqdot \sum_{y=0}^{1} \pi\left(x_2^2 - x_1^2\right)\Delta y$$

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi\left(x_2^2 - x_1^2\right)\Delta y$$

$$= \pi \int_0^1 x_2^2 - x_1^2 \, dy.$$

Now $y = 2x - x^2$

$$x^2 - 2x + y = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4y}}{2}$$

$$= \frac{2 \pm 2\sqrt{1-y}}{2}$$

$$= 1 \pm \sqrt{1-y}$$

$\therefore x_2 = 1 + \sqrt{1-y}, \quad x_1 = 1 - \sqrt{1-y}$

$$V = \pi \int_0^1 \left(1 + \sqrt{1-y}\right)^2 - \left(1 - \sqrt{1-y}\right)^2 dy$$

$$= \pi \int_0^1 1 + 2\sqrt{1-y} + 1 - y - \left(1 - 2\sqrt{1-y} + 1 - y\right) dy$$

$$= \pi \int_0^1 4\sqrt{1-y}\, dy$$

$$= 4\pi \int_0^1 \sqrt{1-y}\, dy$$

$$= 4\pi \left[\frac{-2(1-y)^{3/2}}{3} \right]_0^1$$

$$= 4\pi \left[0 - \left(-2/3\right) \right]$$

$$= \frac{8\pi}{3} \text{ cubic units}$$

Q3) a) i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$$

at $(a\cos\theta, b\sin\theta)$

$$\frac{dy}{dx} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$\therefore$ equation of the tangent

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$$

$$b\cos\theta x + a\sin\theta y = ab\left(\sin^2\theta + \cos^2\theta\right)$$

$$b\cos\theta x + a\sin\theta y = ab$$

$$\frac{\cos\theta x}{a} + \frac{\sin\theta y}{b} = 1.$$

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ii) For M put $y = 0$

$$\frac{\cos\theta x}{a} + 0 = 1$$

$$\frac{\cos\theta x}{a} = 1$$

$$x = \frac{a}{\cos\theta}$$

$\therefore M\left(\dfrac{a}{\cos\theta}, 0\right)$

eqn of normal at P

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

for N, $y = 0$

$$\frac{ax}{\cos\theta} = a^2 - b^2$$

$$x = \frac{(a^2 - b^2)\cos\theta}{a}$$

$\therefore\ N\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$

Now $OM.ON$

$$= \frac{a}{\cos\theta} \times \frac{(a^2-b^2)\cos\theta}{a}$$

$$= a^2-b^2$$

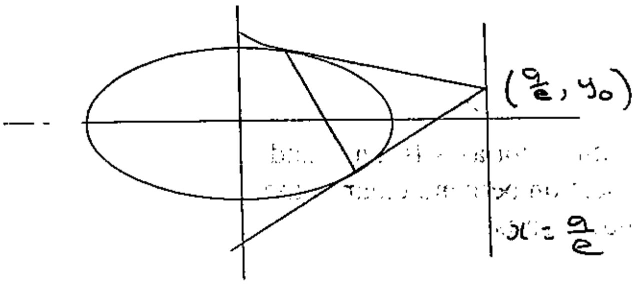
but $b^2 = a^2(1-e^2)$

$$= a^2 - a^2(1-e^2)$$

$$= a^2 - a^2 + a^2e^2$$

$$= a^2e^2$$

b)



Let the point on the directrix be $\left(\frac{a}{e}, y_0\right)$

∴ equation of the chord of contact.

$$\frac{x.\frac{a}{e}}{a^2} + \frac{y y_0}{b^2} = 1$$

$$\frac{x}{ae} + \frac{y y_0}{b^2} = 1$$

If passes through the focus then $(ae, 0)$ must satisfy the equation of the chord.

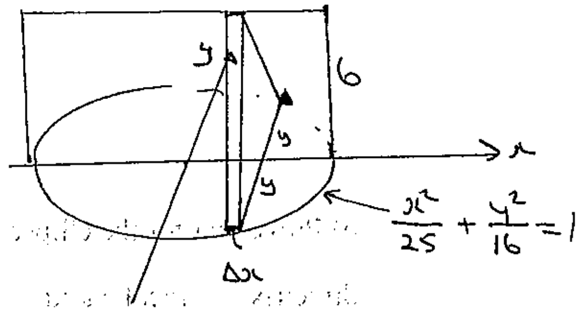
$$\frac{ae}{ae} + 0 = 1$$

$$1 = 1 \quad \text{- True}$$

∴ the chord passes through the focus.

Q4)

a)



Volume of a slice $= \frac{1}{2}(2y)6 \,.\, \Delta x$

$$= 6y\,\Delta x.$$

$$V \doteqdot \sum_{x=-5}^{5} 6y\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=-5}^{5} 6y\Delta x$$

$$= \int_{-5}^{5} 6y\,dx.$$

$$= \int_{-5}^{5} 6\left(\frac{4}{5}\sqrt{25-x^2}\right) dx$$

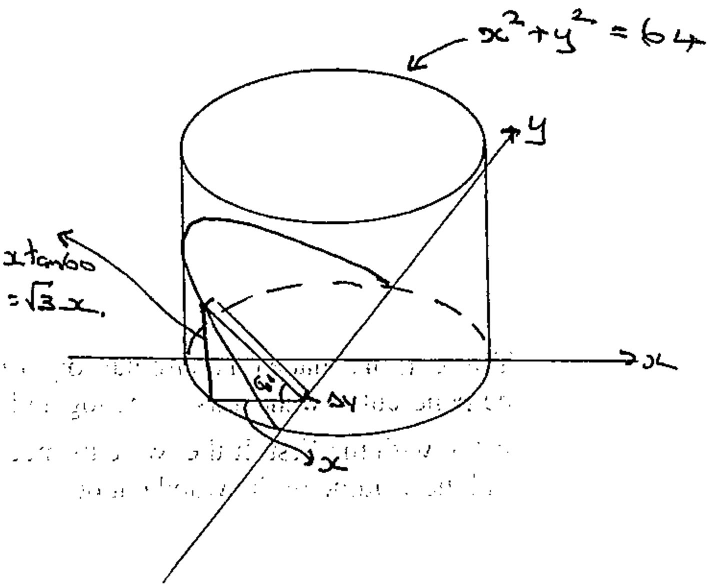
$$y^2 = 16\left(1-\frac{x^2}{25}\right) = 16\left(\frac{25-x^2}{25}\right) = \frac{16}{25}(25-x^2)$$

$$y = \frac{4}{5}\sqrt{25-x^2}$$

$$= \frac{24}{5}\int_{-5}^{5} \sqrt{25-x^2}\,dx$$

$$= \frac{24}{5}\left(\frac{1}{2}\pi 5^2\right) \leftarrow \text{(semi-circle)}$$

$$= 60\pi \text{ cubic units.}$$



Volume of a Slice $= \frac{1}{2}x \times \sqrt{3}x \,.\, \Delta y.$

$$= \frac{\sqrt{3}}{2}x^2\,\Delta y$$

(4)

$$V \doteq \sum_{y=-8}^{8} \frac{\sqrt{3}}{2} x^2 \Delta y$$

$$V \doteq \lim_{\Delta y \to 0} \sum_{y=-8}^{8} \frac{\sqrt{3}}{2} x^2 \Delta y$$

$$= \int_{-8}^{8} \frac{\sqrt{3}}{2} x^2 \, dy.$$

$$= \sqrt{3} \int_0^8 x^2 \, dy$$

$$= \sqrt{3} \int_0^8 64 - x^2 \, dy$$

$$= \sqrt{3} \left[64y - \frac{y^3}{3} \right]_0^8$$

$$= \sqrt{3} \left[\left(512 - \frac{512}{3}\right) - 0 \right)$$

$$= 1024\sqrt{3} \text{ cubic units}$$