# MATHEMATICS
# Extension 2
# Assessment task 3: 25/06/09

Time: 65 minutes.

## SECTION 1.

### Question 1.

For the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ find:

**a )** The eccentricity. **2**

**b)** The co-ordinates of the vertices. **1**

**c)** The co-ordinates of the foci. **1**

**d)** The equation of the directrices. **1**

**e)** The equation of the asymptotes. **1**

**f)** Sketch the hyperbola. **2**

### Question 2.

Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

at the point $P(a\cos\theta, b\sin\theta)$ is given by $\frac{\cos\theta x}{a}+\frac{\sin\theta y}{b}=1$ **3**

### Question 3.

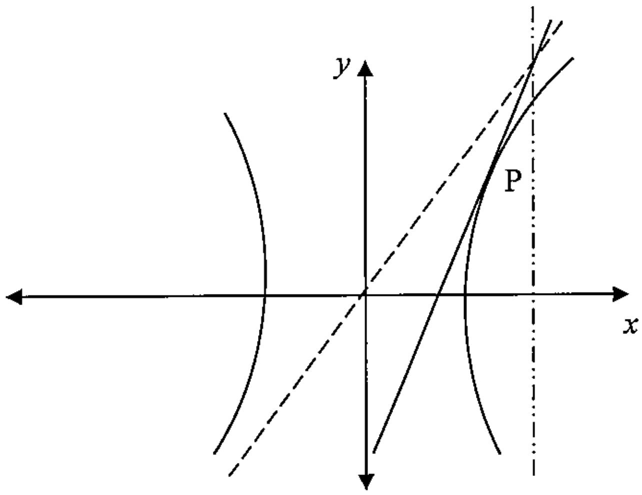
**a)** Show that the equation of the tangent to the hyperbola $xy=c^2$

at the point $(cp,\frac{c}{p})$ is $x+p^2y=2cp$. **3**

**b)** If the tangents at the points P and Q meet at the point R $(x_0, y_0)$

prove that $pq=\frac{x_0}{y_0}$ and $p+q=\frac{2c}{y_0}$ **4**

**Question 4.**




The point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ with focus, S, is such that the tangent at P, the latus rectum through S, and one asymptote are concurrent. Prove that SP is parallel to the other asymptote. **4**

( you may assume the equation of the tangent at $P$ is given by $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$).

**SECTION 2.**

**Question 1**

**a)** Sketch the curve $y=\sin^{-1}x$, for $-1\le x\le 1$ **1**

**d)** By taking slices perpendicular to the axis of rotation use the method of slicing to find the volume of the solid generated by rotating the region bounded by the curve $y=\sin^{-1}x$, the '$x$' axis and the ordinate $x=1$ about the '$y$' axis. **3**

**Question 2**

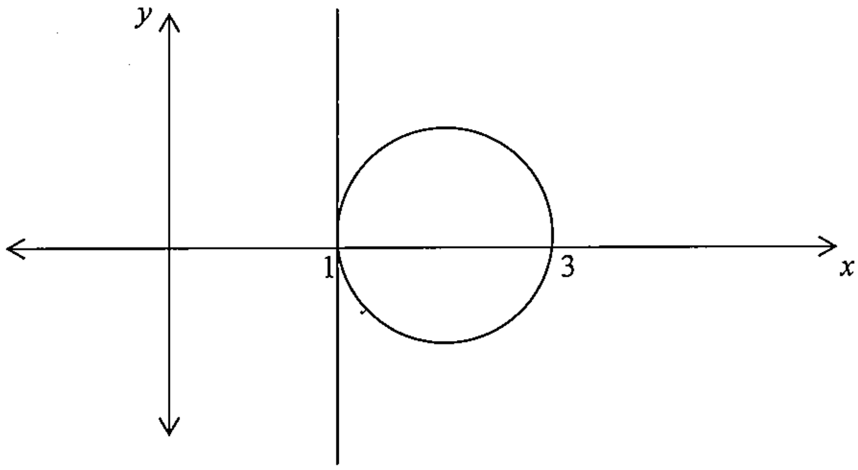
The area between the curve $y=8x-x^2$, the $x$ axis and the line $x=4$ is rotated about the line $x=4$. Find the volume generated by using:

**a)** slicing **3**

**b)** cylindrical shells. **3**

**Question 3.**

**a)**

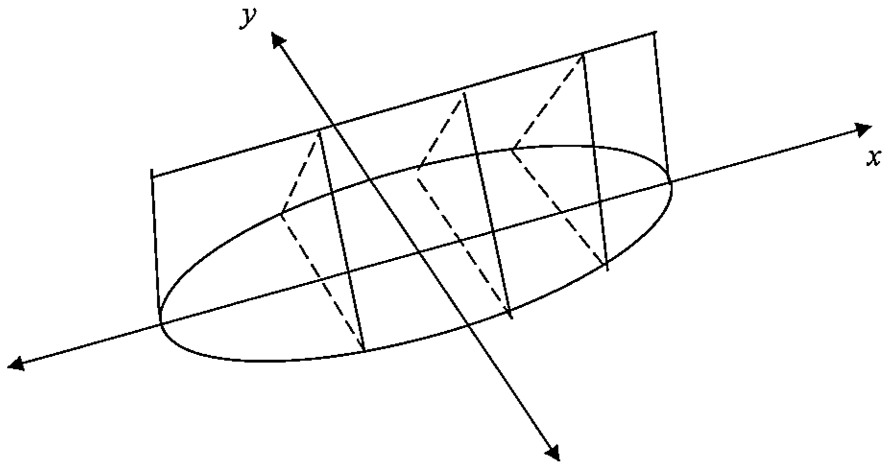



In the diagram above the circle $(x-2)^2+y^2=1$ is drawn. The region bounded by the circle is rotated about the line $x=1$. Use the method of cylindrical shells to show that the volume of the solid of revolution so formed is given by .

$$V=4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$ **3**

**ii)** By using the substitution $x-2=\sin u$, or otherwise, calculate the volume of the solid of revolution. **3**

**Question 4**




A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. Find the volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6 units. **3**

EXTENSION 2. ASSESSMENT 3 SOLUTIONS (2009)

QUES 1) $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

a) $b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

b) $(4,0)$ , $(-4,0)$

c) foci $(ae, 0)$ , $(-ae, 0)$

$(5,0)$ , $(-5,0)$

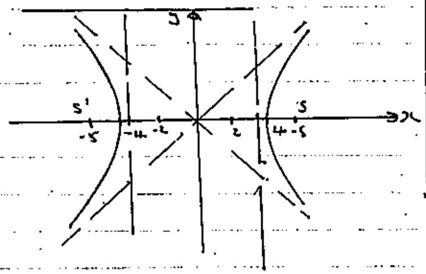
d) $x = \frac{a}{e}$ , $x = -\frac{a}{e}$

$x = \frac{16}{5}$ , $x = -\frac{16}{5}$

e) $y = \frac{bx}{a}$ , $y = -\frac{bx}{a}$

$y = \frac{3}{4}x$ , $y = -\frac{3}{4}x$

f)




Q2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{2y}{b^2} \cdot \frac{dy}{dx} = \frac{-2x}{a^2}$

$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

at $(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx} = \frac{-b^2 . a\cos\theta}{a^2 b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

$\therefore$ eqn of the tangent

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$

$b\cos\theta x + a\sin\theta y = ab(\sin^2\theta + \cos^2\theta)$

$b\cos\theta x + a\sin\theta y = ab$

$\frac{\cos\theta x}{a} + \frac{\sin\theta y}{b} = 1$.

Q3) a) $xy = c^2$

$x . \frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

at $(cp, \frac{c}{p})$, $\frac{dy}{dx} = \frac{-\frac{c}{p}}{cp}$

$= -\frac{1}{p^2}$

$\therefore$ equation of the tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$

$x + p^2 y = 2cp$.

b) tangent at P: $x + p^2 y = 2cp$

tangent at Q: $x + q^2 y = 2cq$

Now $(x_0, y_0)$ satisfies both equations

$\therefore$ $x_0 + p^2 y_0 = 2cp$ --- (1)

$x_0 + q^2 y_0 = 2cq$ --- (2)

(1) − (2) $(p^2 - q^2) y_0 = 2cp - 2cq$

$(p+q)(p-q) y_0 = 2c(p-q)$

$y_0 = \frac{2c}{p+q}$

$p + q = \frac{2c}{y_0}$

(1) $\times q^2$ : $q^2 x_0 + p^2 q^2 y_0 = 2cpq^2$ --- (3)

(2) $\times p^2$ : $p^2 x_0 + p^2 q^2 y_0 = 2cqp^2$ --- (4)

(4) − (3) $(p^2 - q^2) x_0 = 2cpq(p-q)$

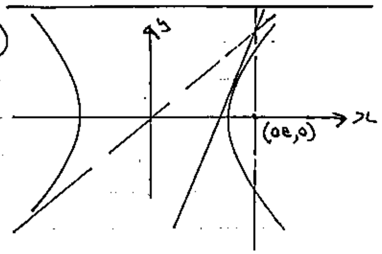
$x_0 = \frac{2cpq(p-q)}{(p+q)(p-q)}$

$x_0 = \frac{2cpq}{p+q}$

$\therefore pq = \frac{x_0(p+q)}{2c}$

$= \frac{x_0}{y_0}$ $\quad \left(y_0 = \frac{2c}{p+q}\right)$

Q4)




eqn tangent at P: $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

eqn of asymptote: $y = \frac{bx}{a}$ ---- (1)

equation Latus rectum: $x = ae$ --- (2)

Solving (1) and (2)

gives point of intersection

$(ae, be)$

Sub into the equation of the tangent

$\frac{ae\sec\theta}{a} - \frac{be\tan\theta}{b} = 1$

$e\sec\theta - e\tan\theta = 1$

$e(\sec\theta - \tan\theta) = 1$

$e = \frac{1}{\sec\theta - \tan\theta}$ ---- (A)

Now gradient SP.

$= \frac{b\tan\theta}{a\sec\theta - ae}$

$= \frac{b\tan\theta}{a\sec\theta - a\left(\frac{1}{\sec\theta - \tan\theta}\right)}$ from (A)

$= \frac{b\tan\theta}{a\sec\theta - \frac{a}{\sec\theta - \tan\theta}}$

$= \frac{b\tan\theta(\sec\theta - \tan\theta)}{a\sec\theta(\sec\theta - \tan\theta) - a}$

$= \frac{b\tan\theta\sec\theta - b\tan^2\theta}{a\sec^2\theta - a\sec\theta\tan\theta - a}$

$= \frac{b\tan\theta\sec\theta - b(\sec^2\theta - 1)}{a(\sec^2\theta - \sec\theta\tan\theta - 1)}$

$= \frac{b(\tan\theta\sec\theta - \sec^2\theta + 1)}{a(\sec^2\theta - \sec\theta\tan\theta - 1)}$

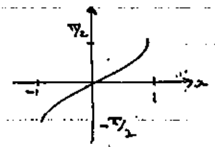
$= \frac{-b}{a}$

which is the gradient of the other asymptote.

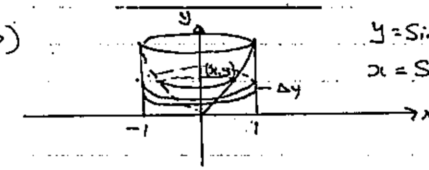
$\therefore$ SP $\parallel$ to the other asymptote.

SECTION 2.

Q1) a)




b)




$y = \sin^{-1}x$

$x = \sin y$

Volume of Slice $= \pi(1^2 - x^2)\Delta y$

$V \doteq \sum_{y=0}^{\pi/2} \pi(1 - x^2)\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{\pi/2} \pi(1 - x^2)\Delta y$

$= \pi\int_0^{\pi/2} 1 - x^2 \, dy$

$$\pi\int_0^{\pi/2}(1-\sin^2 y)\,dy.$$

$$\pi\int_0^{\pi/2}\cos^2 y\,dy.$$

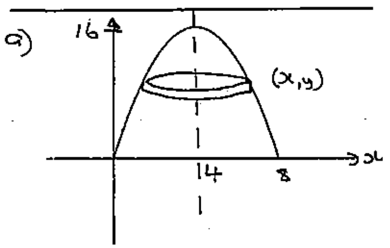
$$=\frac{\pi}{2}\int_0^{\pi/2}\cos 2y+1\,dy.$$

$$=\frac{\pi}{2}\left[\tfrac{1}{2}\sin 2y+y\right]_0^{\pi/2}$$

$$=\frac{\pi}{2}\left[(0+\tfrac{\pi}{2})-0\right)$$

$$=\frac{\pi^2}{4}\text{ cubic Units.}$$

Q2. a)




$$\text{Volume of a Slice} = \pi(x-4)^2\Delta y$$

$$V \doteq \sum_{y=0}^{16}\pi(x-4)^2\Delta y$$

$$V=\lim_{\Delta y\to 0}\sum_{y=0}^{16}\pi(x-4)^2\Delta y$$

$$V=\pi\int_0^{16}(x-4)^2\,dy.$$

$$y=8x-x^2$$

$$x^2-8x+y=0$$

$$x=\frac{8\pm\sqrt{64-4y}}{2}$$

$$=4\pm\sqrt{16-y}$$

$$V=\pi\int_0^{16}\left(4+\sqrt{16-y}-4\right)^2dy.$$

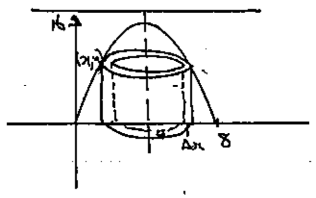
$$=\pi\int_0^{16}16-y\,dy$$

$$=\pi\left[16y-\frac{y^2}{2}\right]_0^{16}$$

$$\pi\left[(256-128)-0\right]$$

$$=128\pi\text{ cubic units}$$

b)




$$\text{Volume of a shell} = 2\pi rh.\Delta x$$

$$=2\pi.(4-x)y$$

$$V\doteq\sum_{x=0}^{4}2\pi(4-x)y\,\Delta x$$

$$V=\lim_{\Delta x\to 0}\sum_{x=0}^{4}2\pi(4-x)y\,\Delta x.$$

$$=2\pi\int_0^4(4-x)y\,dx$$

$$=2\pi\int_0^4(4-x)(8x-x^2)\,dx$$

$$=2\pi\int_0^4 x^3-12x^2+32x\,dx$$

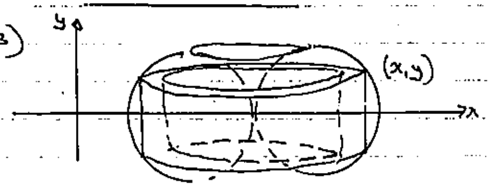
$$=2\pi\left[\frac{x^4}{4}-4x^3+16x^2\right]_0^4$$

$$=2\pi\left[(64-256+256)-0\right]$$

$$=2\pi\times 64$$

$$=128\pi\text{ cubic units.}$$

Q3) a)




$$\text{Volume of shell} = 2\pi rh.\Delta x$$

$$=2\pi(x-1)2y\,\Delta x$$

$$V\doteq\sum_{x=1}^{3}2\pi(x-1)2y\,\Delta x$$



$$V=\lim_{\Delta x\to 0}\sum_{x=1}^{3}4\pi(x-1)y\,\Delta x$$

$$=4\pi\int_1^3(x-1)y\,dx.$$

$$=4\pi\int_1^3(x-1)\sqrt{1-(x-2)^2}\,dx$$

$$\left(\begin{aligned}(x-2)^2+y^2&=1\\ y^2&=1-(x-2)^2\\ y&=\sqrt{1-(x-2)^2}\end{aligned}\right).$$

b) $V=4\pi\int_1^3(x-1)\sqrt{1-(x-2)^2}\,dx.$

$x-2=\sin u$ : $x=1$ $u=-\frac{\pi}{2}$

$x=\sin u+2$   $x=3$ $u=\frac{\pi}{2}$

$$\frac{dx}{du}=\cos u$$

$$dx=\cos u\,du$$

$$V=4\pi\int_{-\pi/2}^{\pi/2}(\sin u+1)\sqrt{1-\sin^2 u}.\cos u\,du$$

$$=4\pi\int_{-\pi/2}^{\pi/2}(\sin u+1)\sqrt{\cos^2 u}.\cos u\,du$$

$$=4\pi\int_{-\pi/2}^{\pi/2}(\sin u+1)\cos^2 u\,du$$

$$=4\pi\int_{-\pi/2}^{\pi/2}\sin u\cos^2 u+\cos^2 u\,du$$

$$=4\pi\int_{-\pi/2}^{\pi/2}\sin u\cos^2 u+\tfrac{1}{2}(\cos 2u+1)\,du$$

$$=4\pi\int_{-\pi/2}^{\pi/2}\sin u\cos^2 u+\tfrac{1}{2}\cos 2u+\tfrac{1}{2}\,du$$

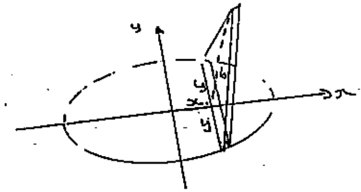
$$=4\pi\left[-\frac{\cos^3 u}{3}+\frac{\sin 2u}{4}+\frac{u}{2}\right]_{-\pi/2}^{\pi/2}$$

$$=4\pi\left[\left(0+0+\tfrac{\pi}{4}\right)-\left(0+0-\tfrac{\pi}{4}\right)\right]$$

$$=4\pi\times\frac{\pi}{2}$$

$$=2\pi^2\text{ cubic units.}$$

Q4)




equation of the ellipse : $\frac{x^2}{25}+\frac{y^2}{16}=1.$

$$\text{Volume of a Slice} = \tfrac{1}{2}\times 2y\times 6.\Delta x$$

$$V\doteq\sum_{x=-5}^{5}6y\,\Delta x$$

$$V=\lim_{\Delta x\to 0}\sum_{x=-5}^{5}6y\,\Delta x.$$

$$V=6\int_{-5}^{5}y\,dx.$$

$$\left\{\begin{aligned}\frac{x^2}{25}+\frac{y^2}{16}&=1\\ y^2&=16\left(1-\frac{x^2}{25}\right)\\ &=16\left(\frac{25-x^2}{25}\right)\\ y&=\frac{4}{5}\sqrt{25-x^2}\end{aligned}\right\}$$

$$V=\frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx.$$

$$=\frac{24}{5}\times\left(\tfrac{1}{2}\pi r^2\right)$$

$$=\frac{12}{5}\pi\times 5^2$$

$$=60\pi\text{ cubic Units.}$$