STUDENT NAME ....

YEAR 12
Mathematics Extension 2
HSC Course
Assessment Task 3
June 2010

1. There are 2 questions.
2. Marks allocated to each question are indicated in brackets
3. Answer each question on your own paper showing all necessary working
4. Start each question on a new page
5. Calculators may be used
6. Time allowed - **70 minutes**

Topic	Mark
Question 1 (Integration)	/22
Question 2 (Conics)	/22

TOTAL **/44**

Question 1 Integration (marks are shown in brackets)

a) Find $\int \frac{x}{\sqrt{x^2+4}}\, dx$ (2)

b) Find $\int \frac{\cos^{-1}x}{\sqrt{1-x^2}}\, dx$ (2)

c) Find $\int \frac{dx}{e^x+e^{-x}}$ (2)

d) Find $\int \frac{1}{\sqrt{x^2+4x-12}}\, dx$ (3)

e) Find $\int \frac{1}{5+3\cos x}\, dx$ (3)

f) Let $I_n = \int_0^2 x^n e^{-x}\, dx$, where n is a **non-negative** integer.

i. Show that $I_n = -2^n e^{-2} + nI_{n-1}$ (3)

ii. Evaluate I_2 (2)

g) Sketch the graph of the curve $y = \frac{1}{x(3-x)}$ *for* $0 < x < 3$.

Find the area bounded by the curve, the X axis and the lines $x = 1$ and $x = 2$. (5)

Question 2 Conics

a) The equation of a conic is $\frac{x^2}{4}+\frac{y^2}{3}=1$ Find

i. The eccentricity

ii. The coordinates of the foci (3)

iii. The equations of the directrices.

b) Find the equation of the chord of contact of the tangents to the hyperbola $x^2 - 16y^2 = 16$ from the point with coordinates $(2, -4)$. (2)

c) Show the area of the ellipse $x = a\cos\theta, y = b\sin\theta$ is πab (4)

d) The points $P\ (cp, \frac{c}{p})$ and $Q(-cp, \frac{-c}{p})$ lie on the hyperbola with equation $xy = c^2$. The normal at P meets the hyperbola again at R. Show that $\angle PQR$ is a right angle. (4)

e) The point $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P intersects the X axis at A and Y axis at B.

i. Find the coordinates of A and B (2)

ii. Show that $(AB)^2 = \frac{a^4}{(x_1)^2}+\frac{a^2b^2}{a^2-(x_1)^2}$ (2)

iii. Given that $\frac{d^2}{d(x_1)^2}(AB)^2 > 0\ for\ 0 < |x| < a,$

show that $(AB)^2$ is a minimum when $(x_1)^2 = \frac{a^3}{a+b}$ (3)

iv. Hence, find the minimum length of AB (2)

(end of examination)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

ANSWERS Assess task 3 ext 2 2010

Q1

(a) $\int x(x^2+4)^{-\frac{1}{2}}\,dx$

$= +\frac{1}{2}\int 2x(x^2+4)^{-\frac{1}{2}}\,dx$ (2)

$= \frac{1}{2} \times (x^2+4)^{\frac{1}{2}} \times 2 + C$

$= \sqrt{x^2+4} + c$

(b) $\int \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$

$= -\int \frac{-1}{\sqrt{1-x^2}}\cos^{-1}x\,dx$ (2)

$= -\frac{1}{2}(\cos^{-1}x)^2 + C$

(c) $\int \frac{1}{e^x + \frac{1}{e^x}}\,dx = \int \frac{1}{\frac{e^{2x}+1}{e^x}}\,dx$

$= \int \frac{e^x}{1+(e^x)^2}\,dx$ (2)

$= \tan^{-1}(e^x) + c$

(d) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

$= \int \frac{1}{\sqrt{x^2+4x+4-4-12}}\,dx$

$= \int \frac{1}{\sqrt{(x+2)^2-16}}\,dx$

Let $u = x+2$

$\frac{du}{dx} = 1$ (3)

$du = dx$

$= \int \frac{1}{\sqrt{u^2-4^2}}\,du$

$= \ln\left|u + \sqrt{u^2-4^2}\right| + C$

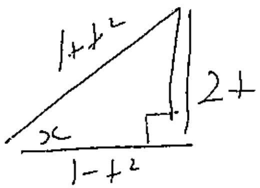
$= \ln\left|x+2+\sqrt{(x+2)^2-4^2}\right| + C$

(e) $\displaystyle\int \frac{1}{5+3\cos x}\,dx$ Let $t = \tan\frac{x}{2}$

$t = \frac{1}{2}.2\tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$



$\frac{dt}{dx} = \frac{1+t^2}{2}$

$dx = \frac{2\,dt}{1+t^2}$

$$\therefore = \int \frac{1}{5+\frac{3(1-t^2)}{(1+t^2)}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{5+5t^2+3-3t^2}{(1+t^2)}} \times \frac{2dt}{(1+t^2)}$$

$$= \int \frac{\cancel{(1+t^2)}}{\cancel{2}(4+t^2)} \times \frac{\cancel{2}dt}{\cancel{(1+t^2)}}$$

$$= \int \frac{1}{t^2+4}\,dt$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{\tan\frac{x}{2}}{2}\right] + C$$

(3)

(f) Let $u = x^n$

$\frac{du}{dx} = n x^{n-1}$ $\frac{dV}{dx} = e^{-x}$

$V = -e^{-x}$

$$= \left[x^n . -e^{-x}\right]_0^2 + \int_0^2 e^{-x} . n x^{n-1}\,dx$$

$$= -2^n e^{-2} + n\int_0^2 x^{n-1}e^{-x}\,dx$$

$$= -2^n e^{-2} + n\,I_{n-1}$$

(3)

$$I_2 = -2^2 e^{-2} + 2I_1$$

$$= -4e^{-2} + 2\left[-2e^{-2} + I_0\right]$$

$$= -4e^{-2} - 4e^{-2} + 2I_0$$

$$= -8e^{-2} + 2\times\int_0^2 e^{-x}\,dx$$

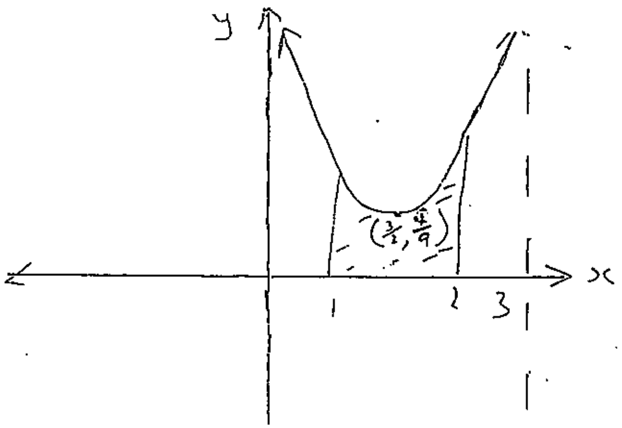
$$= -8e^{-2} + 2\left[-e^{-x}\right]_0^2$$

$$= -8e^{-2} + 2\left(-e^{-2} - -1\right)$$

$$= -10e^{-2} + 2$$

(2)

(g)



(2)

It would be of some help to find any stat. pts.

Stat pts occur when $y' = 0$

$y = (3x - x^2)^{-1}$

$y' = -(3x - x^2)^{-2} \cdot (3 - 2x)$

$= \dfrac{2x - 3}{(3x - x^2)^2}$

$y' = 0$ when $x = 3/2$ $\therefore$ $(3/2, 4/9)$ is a stat pt.

Test for nature:

x	1	$\frac{3}{2}$	2
y'	-ve	0	+ve

$\therefore$ Min T.P.

$\therefore$ Area $= \displaystyle\int_1^2 \frac{1}{x(3-x)}\,dx$

$\dfrac{1}{x(3-x)} = \dfrac{a}{x} + \dfrac{b}{3-x}$

$1 = a(3-x) + bx$

$(x=3)$ $b = \frac{1}{3}$

$(x=0)$ $a = \frac{1}{3}$

$\therefore = \dfrac{1}{3x} + \dfrac{1}{3(3-x)}$

$= \frac{1}{3}\displaystyle\int_1^2 \frac{1}{x} + \frac{1}{3-x}\,dx$

$= \frac{1}{3}\Big[\ln x - \ln|3-x|\Big]_1^2$

$= \frac{1}{3}\Big[(\ln 2 - 1) - (\ln 1 - \ln 2)\Big]$

$= \frac{1}{3}(2\ln 2) = \frac{2}{3}\ln 2$ units^2

Conics

Question 2

(a) $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ $\quad a = 2 \quad b = \sqrt{3}$

(i) $b^2 = a^2(1 - e^2)$

$3 = 4(1 - e^2)$

$3 = 4 - 4e^2$

$4e^2 = 1$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$

(3)

(ii) Foci: $(\pm ae, 0)$

$= (\pm 1, 0)$

(iii) $x = \pm \frac{a}{e}$

$= \pm 4$

(b) $x^2 - 16y^2 = 16$

$\dfrac{x^2}{16} - y^2 = 1$ $\quad a^2 = 16 \quad b^2 = 1$

$(2, -4)$

$\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$

$\dfrac{2x}{16} + \dfrac{4y}{1} = 1$

$2x + 64y = 16$

$x + 32y = 8$

(2)

(c) $x = a\cos\theta \qquad y = b\sin\theta$

$\frac{x}{a} = \cos\theta \qquad \frac{y}{b} = \sin\theta$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$

$y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$

$\therefore A = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}$

$= \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2} \longrightarrow$ Area of semi circle

so $\frac{2b}{a} \times \frac{\pi \times a^2}{2}$

$= \frac{2\pi a^2 b}{2a}$

$= \pi ab$

(4)

(d) Equation of normal is

$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right).$

As $y = \frac{c^2}{x}$

$px - \frac{c^2}{px} = c\left(p^2 - \frac{1}{p^2}\right)$

$p^2x^2 - c^2 = cpx\left(p^2 - \frac{1}{p^2}\right)$

$p^2x^2 - c^2 = cp^3x - \frac{cx}{p}$

$p^2x^2 - cp^3x + \frac{cx}{p} + c^2 = 0$

$p^2x(x-cp) + \frac{c}{p}(x-cp) = 0$

$$(x-cp)\left(p^2x+\frac{c}{p}\right)=0 \qquad x=cp \text{ OR } -\frac{c}{p^3}$$

$$\therefore R\left(-\frac{c}{p^3}, -cp^3\right)$$

$$m_{PQ} = \frac{\frac{c}{p}+\frac{c}{p}}{cp+cp} = \frac{\frac{2c}{p}}{2cp} = \frac{2c}{p}\times\frac{1}{2cp} = \frac{1}{p^2}$$

$$m_{QR} = \frac{-\frac{c}{p}+cp^3}{-cp+\frac{c}{p^3}} = \frac{\frac{-c+cp^4}{p}}{\frac{-cp^4+c}{p^3}} = \frac{c(p^4-1)}{\frac{-c(p^4-1)}{p^2}} = -p^2$$

$$\therefore m_{PQ}\times m_{QR} = -1$$

$\therefore \angle PQR$ is a right angle

(4)

(a) Equation of tangent is given by

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

Cuts x axis when $y=0$

$$\frac{x_1x}{a^2}=1$$

$$x_1x=a^2$$

$$x=\frac{a^2}{x_1} \qquad \therefore A\left(\frac{a^2}{x_1},0\right)$$

Cuts y axis when $x=0$ $\qquad \therefore B\left(0,\frac{b^2}{y_1}\right)$

(2)

(ii) $$(AB)^2=\frac{a^4}{(x_1)^2}+\frac{b^4}{(y_1)^2}$$

but $$\frac{y_1^2}{b^2}=1-\frac{x_1^2}{a^2} = \frac{a^2-x^2}{a^2}$$

So $$\frac{b^2}{y^2}=\frac{a^2}{a^2-x_1^2}$$

$$\therefore (AB)^2=\frac{a^4}{(x_1)^2}+b^2\left(\frac{a^2}{a^2-(x_1)^2}\right)$$

(2)

(iii) $$\frac{d}{dx}(AB)^2=\frac{-2a^4}{(x_1)^3}+\frac{a^2b^2(-1)(-2x_1)}{(a^2-x_1^2)^2} = \frac{-2a^4}{(x_1)^3}+\frac{2a^2b^2x_1}{(a^2-x_1^2)^2}$$

$$= \frac{2a^2\left[b^2x_1^4 - a^2(a^2-x_1^2)^2\right]}{x_1^3(a^2-x_1^2)^2}$$

For st pts $\frac{d}{d(x_1)}(AB)^2 = 0$

$\therefore \; b^2x_1^4 - a^2(a^2-x_1^2)^2 = 0$

$\left[bx_1^2 - a(a^2-x_1^2)\right]\left[bx_1^2 + a(a^2-x^2)\right] = 0$

$\left[(b+a)x_1^2 - a^3\right]\left[(b-a)x_1^2 + a^3\right] = 0$

$\therefore \; x_1^2 = \frac{a^3}{(a+b)}$ or $x_1^2 = \frac{a^3}{(a-b)}$

$x_1^2 = a^2 \cdot \frac{a}{a-b}$ (3)

which is greater than a^2

$\therefore$ outside domain

$\therefore \; x_1^2 = \frac{a^3}{(a+b)} \Rightarrow a^2 \cdot \frac{a}{a+b} < a^2$

$\therefore$ since $\frac{d^2}{d(x_1)^2}(AB)^2 > 0$ for $0 < |x| < a$

then $x_1^2 = \frac{a^3}{a+b}$ gives a minimum

(iv) $$(AB)^2 = \frac{a^4(a+b)}{a^3} + \frac{a^2b^2}{a^2 - \frac{a^3}{a+b}}$$

$$= a(a+b) + \frac{a^2b^2(a+b)}{(a+b)a^2 - a^3}$$

$$= a(a+b) + \frac{a^2b^2(a+b)}{a^3 + a^2b - a^3}$$

$$= a^2 + ab + b(a+b)$$ (2)

$$= a^2 + 2ab + b^2$$

$$= (a+b)^2$$

$\therefore$ Min length is $a+b$