GOSFORD HIGH SCHOOL

Year 12

MATHEMATICS EXTENSION 2

HSC Course

Assessment Task #3

June 2011

Time Allowed: 60 minutes + 5 minutes reading time

Instructions:

- There are 3 questions.
- Marks allocated to each question are indicated.
- Answer each question on your own paper showing all necessary working.
- Start each question on a new sheet of paper.
- Calculators may be used.

Topic		Mark
QUESTION 1:	Integration	/18
QUESTION 2:	Volumes	/12
QUESTION 3:	a, b, d (Conics)	/8
	c (Integration)	/4
TOTAL		/42

QUESTION 1: (18 Marks) Use a separate sheet of paper

Marks

a. Evaluate 2

$$\int_{-1}^{1} x^3 + x^2 \, sinx \, dx$$

(Giving a brief reason for your answer)

b. Using the table of standard integrals, show that 2

$$\int_{\frac{1}{4}}^{1} \frac{dx}{\sqrt{16x^2 - 1}} = \frac{1}{4}\ln\left(4 + \sqrt{15}\right)$$

c. Use integration by parts to evaluate 3

$$\int_{0}^{\pi} e^x \, sinx \, dx$$

d. It can be shown that: 2

$$\frac{9-x}{1+x+x^2+x^3} = \frac{4-5x}{1+x^2} + \frac{5}{1+x}$$

(Do not prove this!)

Use this result to find

$$\int \frac{9-x}{1+x+x^2+x^3} \, dx$$

Question 1 continues

Question 1 continued **Marks**

e. By using an appropriate substitution prove that 3

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

f. Use the substitution $t = \tan\frac{\theta}{2}$ 3

to find

$$\int \frac{1}{1+sin\theta}\,d\theta$$

g. Find 3

$$\int \frac{3x+4}{\sqrt{8-6x-9x^2}}\,dx$$

End of Question 1

QUESTION 2: (12 Marks) Use a separate sheet of paper

Marks

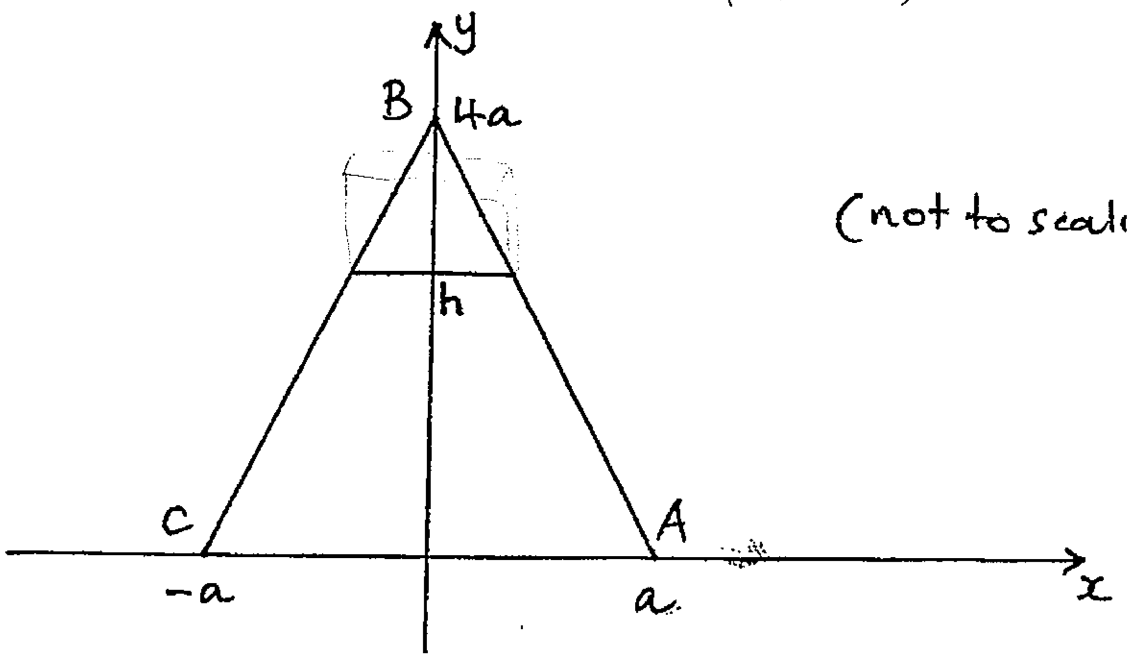
a. The region, in the first quadrant, bounded by $y = x(2-x)^2$, the x axis and x = 2 is rotated about the y axis to form a solid. This volume is to be found by the method of cylindrical shells.

i. Draw a neat diagram to represent the above information. (Indicating a typical cylindrical shell.) 1

ii. Use this method to find the volume of the solid. 3

b. The base of a solid is an isosceles triangle ABC as shown:

(Not to scale)



Vertical cross-sections taken at right angles to the y axis are squares. (The line interval indicated at h represents one side of a typical square.)

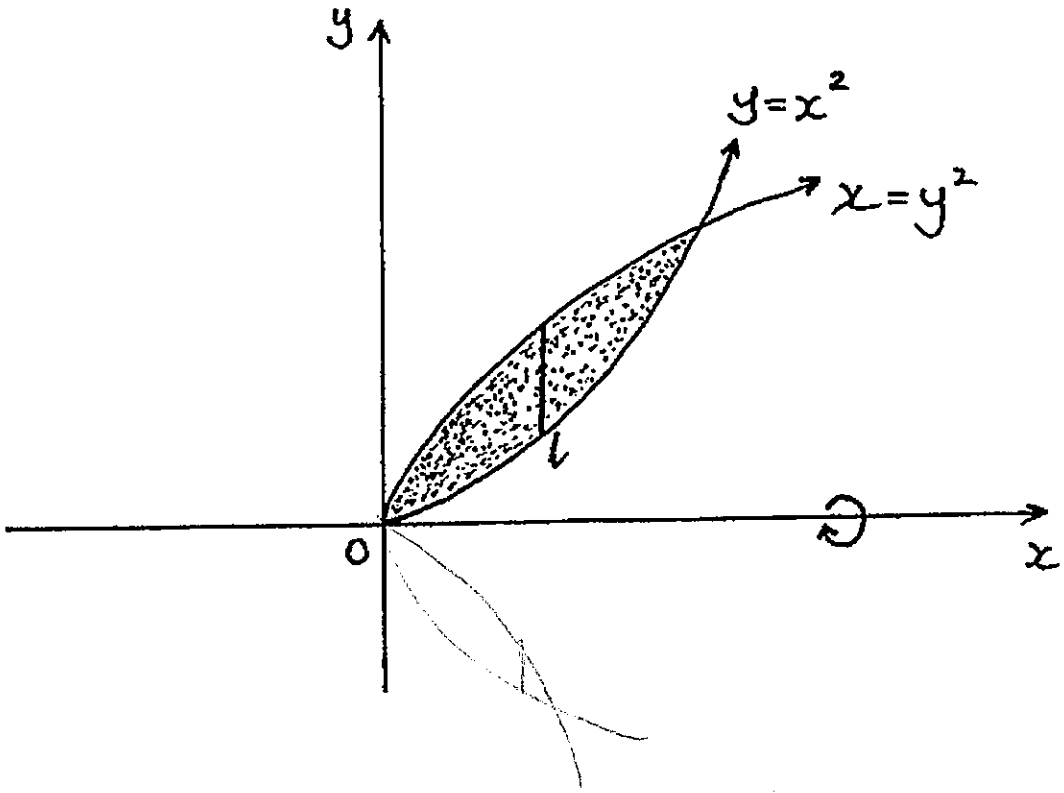
i. Show that the area of the square cross-section at $y = h$ is given by $4\left(a - \frac{h}{4}\right)^2$ 2

ii. Hence, find the volume of the solid. 2

Question 2 continues

Question 2 continued **Marks**

c. The shaded region indicated is bounded by the curves $y = x^2$ and $x = y^2$:



This region is rotated about the x axis to form a solid.

When the region is rotated the vertical line segment l sweeps out an annulus.

i. By taking annular slices perpendicular to the x axis, express the volume of the resulting solid as an integral. 2

ii. Evaluate the integral in Part (i) 2

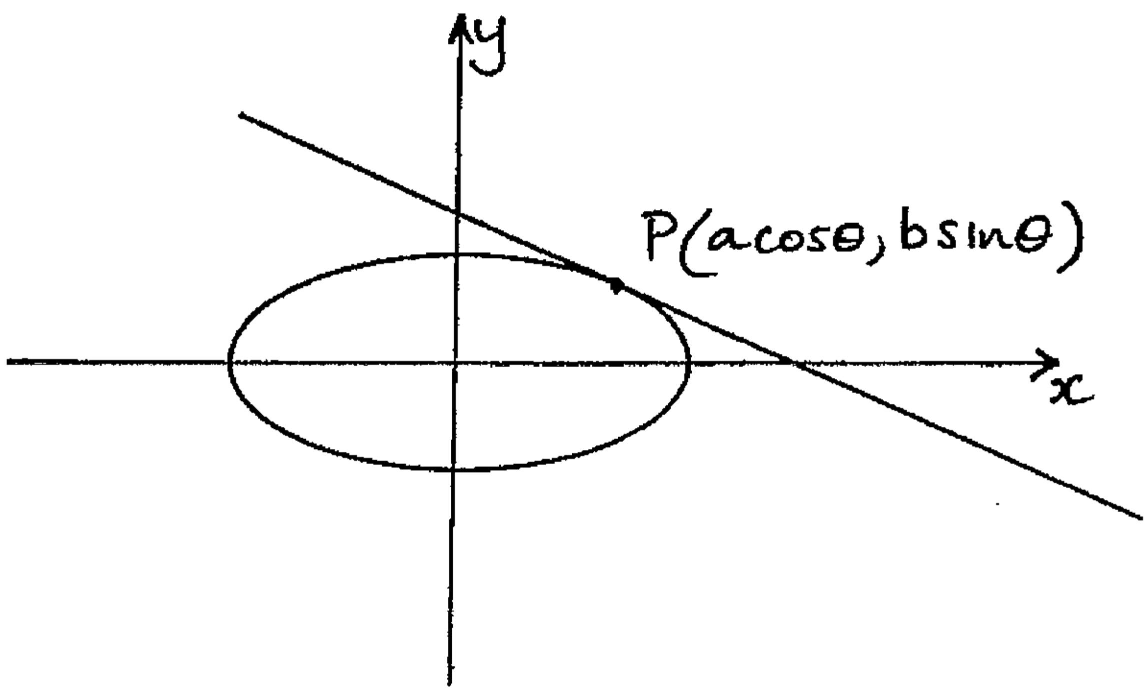
End of Question 2

QUESTION 3: (12 Marks) Use a separate sheet of paper

Marks

a Sketch the ellipse $9x^2 + 4y^2 = 36$, indicating the foci and directrices 2

b Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



If P$(acos\theta), bsin\theta$ is a variable point on the ellipse show that the tangent through P has equation: 2

$$bx\,cos\theta + ay\,sin\theta - ab = 0$$

Question 3 continues

Question 3 continued **Marks**

c Given that:

$$I_n = \int_1^e (1nx)^n \, dx$$

where n is a non-negative integer

i. Prove that $I_n = e - nI_{n-1}$ $(n \neq 0)$ 2

ii. Hence, evaluate I_4 2

d i. For $y = mx + c$ to be a tangent to the ellipse 2

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that

$c^2 = a^2m^2 + b^2$

ii. Hence, show that the pair of tangents drawn from the point (3,4) to the ellipse 2

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other

END OF ASSESSMENT TASK

Q1

a. $\int_{-1}^{1} x^3 + x^2 \sin x \, dx = 0$

since $x^3 + x^2 \sin x$ is an odd function.

b. $\int_{\frac{1}{4}}^{1} \frac{dx}{\sqrt{16x^2-1}} = \int_{1/4}^{1} \frac{dx}{\sqrt{16(x^2-\frac{1}{16})}}$

$= \frac{1}{4}\int_{1/4}^{1} \frac{dx}{\sqrt{x^2-\frac{1}{16}}}$

Using table of s.I :

$= \frac{1}{4}\left[\ln\left(x+\sqrt{x^2-\frac{1}{16}}\right)\right]_{1/4}^{1}$

$= \frac{1}{4}\left[\ln\left(1+\frac{\sqrt{15}}{4}\right) - \ln\left(\frac{1}{4}\right)\right]$

$= \frac{1}{4}\left[\ln\left(\frac{4+\sqrt{15}}{4}\right) + \ln 4\right]$

$= \frac{1}{4}\ln\left(4+\sqrt{15}\right)$

c. $I = \int_0^{\pi} e^x \sin x \, dx = \int_0^{\pi} \sin x . e^x dx$

Using integration by parts

$= \left[\sin x . e^x\right]_0^{\pi} - \int_0^{\pi} \cos x . e^x dx$

$= 0 - \int_0^{\pi} \cos x . e^x dx$

$= -\left[\left[\cos x . e^x\right]_0^{\pi} + \int_0^{\pi} \sin x . e^x dx\right]$

$\therefore I = e^{\pi} + 1 - I$

$2I = e^{\pi} + 1$

$I = \frac{e^{\pi}+1}{2}$

d. $\int \frac{9-x}{1+x+x^2+x^3} dx$

$= \int \frac{4-5x}{1+x^2}dx + \int \frac{5}{1+x} dx$

$= \int \frac{4}{1+x^2} dx - \frac{5}{2}\int \frac{2x}{1+x^2} dx + \int \frac{5}{1+x} d$

$= 4\tan^{-1} x - \frac{5}{2}\ln(1+x^2) + 5\ln|1+x| +$

e. $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} dx$

let $x = \sin\theta$, $\frac{dx}{d\theta} = \cos\theta$

$x = \frac{1}{2}, \theta = \frac{\pi}{6}$

$x = 0, \theta = 0$

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} . \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} . \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 - \cos 2\theta \, d\theta$

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - 0\right] = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

f. (i) $t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$

$= \frac{1}{2}\left(1+t^2\right)$

(ii) $\sin\theta = 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}$

$= \frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}.\cos^2\frac{\theta}{2}$

$= 2\tan\frac{\theta}{2}.\frac{1}{\sec^2\frac{\theta}{2}}$

$= 2t.\frac{1}{1+\tan^2\frac{\theta}{2}}$

$= \frac{2t}{1+t^2}$ as req.

i) $\int \frac{1}{1+\sin\theta}.d\theta$ $\quad t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1+t^2}{2}$

$\frac{2dt}{1+t^2} = d\theta$

$= \int \frac{1}{1+\frac{2t}{1+t^2}}.\frac{2dt}{1+t^2}$

$= \int \frac{\frac{2dt}{1+t^2}}{\frac{1+t^2+2t}{1+t^2}}$

$= \int \frac{2dt}{(1+t)^2} = 2\int (1+t)^{-2}dt$

$= 2\left[\frac{(1+t)^{-1}}{-1}\right] + c$

$= \frac{-2}{1+t} + c$

$= \frac{-2}{1+\tan\frac{\theta}{2}} + c$

g. $\int \frac{3x+4}{\sqrt{8-6x-9x^2}}\,dx$

$= -\frac{1}{6}\int \frac{-18x-6}{\sqrt{8-6x-9x^2}}dx + \int \frac{3}{\sqrt{8-6x-9x^2}}dx$

$= I + J$

$\therefore I = -\frac{1}{6}\int \frac{-18x-6}{\sqrt{8-6x-9x^2}}dx$ $\quad$ let $u = 8-6x-9x^2$, $\frac{du}{dx} = -6-18x$

$= -\frac{1}{6}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{6}\int u^{-1/2}du$

$= -\frac{1}{6}\left[\frac{u^{1/2}}{1/2}\right] + C_1$

$= -\frac{1}{3}\sqrt{u} + C_1$

$= -\frac{1}{3}\sqrt{8-6x-9x^2} + C_1$

$J = \int \frac{3}{\sqrt{9\left(\frac{8}{9}-\frac{2}{3}x-x^2\right)}}dx$

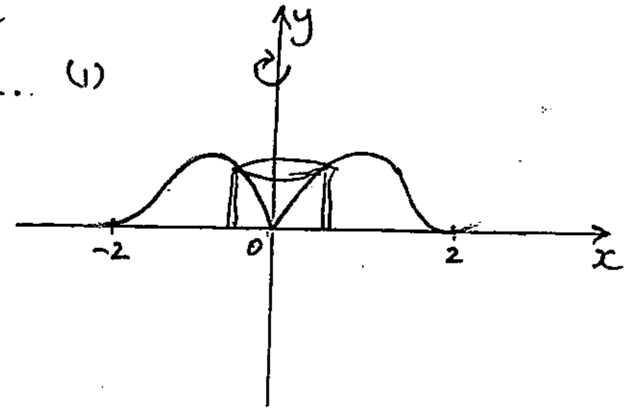
$= \int \frac{dx}{\sqrt{-\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+1}}$

$$= \int \frac{dx}{\sqrt{1-\left(x+\frac{1}{3}\right)^2}}$$

$$= \sin^{-1}\left(x+\frac{1}{3}\right) + c_2$$

$$I + J = \sin^{-1}\left(x+\frac{1}{3}\right) - \frac{1}{3}\sqrt{8-6x-9x^2} + C$$

2/ a. (i)



(ii) $V = 2\pi \int_0^2 x \cdot y \cdot dx$

where $y = x(2-x)^2$

$$V = 2\pi \int_0^2 x^2(2-x)^2\, dx$$

$$= 2\pi \int_0^2 x^2(4-4x+x^2)\, dx$$

$$= 2\pi \int_0^2 4x^2 - 4x^3 + x^4\, dx$$

$$= 2\pi \left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$

$$= 2\pi\left[\left(\frac{32}{3} - 16 + \frac{32}{5}\right) - 0\right]$$

$$= 2\pi\left(\frac{16}{15}\right) = \frac{32\pi}{15} \text{ units}^3$$

b. (i) Eqn AB $\Rightarrow \frac{x}{a} + \frac{y}{4a} = 1$

$$4x + y = 4a$$

$$y = -4x + 4a$$

$\therefore$ when $y = h$,

$$h = -4x + 4a$$

$$4x = 4a - h$$

$$x = a - \frac{h}{4}$$

$\therefore$ Area of square $= \left[2\left(a-\frac{h}{4}\right)\right]^2$

$$= 4\left(a-\frac{h}{4}\right)^2$$

(ii) $V = \int_0^{4a} 4\left(a-\frac{h}{4}\right)^2 dh$

$$= 4\int_0^{4a} a^2 - \frac{ah}{2} + \frac{h^2}{16}\, dh$$

$$= 4\left[a^2h - \frac{ah^2}{4} + \frac{h^3}{48}\right]_0^{4a}$$

$$= 4\left[\left(4a^3 - 4a^3 + \frac{4a^3}{3}\right) - 0\right]$$

$$= \frac{16a^3}{3} \text{ units}^3$$

c. (i) $A(x) = \pi(R^2 - r^2)$

$$= \pi\left((\sqrt{x})^2 - (x^2)^2\right)$$

$$= \pi(x - x^4)$$

$$\therefore \Delta V = \pi(x - x^4)\Delta x$$

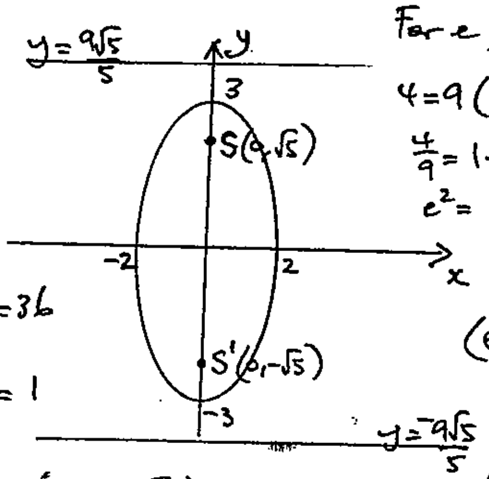
$$\therefore V = \pi\int^1 x - x^4\, dx$$

(ii) $V = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= \pi\left[\left(\frac{1}{2}-\frac{1}{5}\right) - 0\right]$

$= \frac{3\pi}{10}$ units^3

a.



For e,

$4 = 9(1-e^2)$

$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9}$

$e = \frac{\sqrt{5}}{3}$

$(e > 0)$

$x^2 + 4y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

ci ⇒ $S = (0, \pm\sqrt{5})$

irectrices ⇒ $y = \pm\frac{9}{\sqrt{5}} \Rightarrow \pm\frac{9\sqrt{5}}{5}$ (2)

).

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\frac{2y}{b^2}\cdot\frac{dy}{dx} = -\frac{2x}{a^2}$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

(at P) $\frac{dy}{dx} = \frac{-b^2 \cdot a\cos\theta}{a^2 b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

∴ eqn of tangent at P:

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta - ab(\sin^2\theta + \cos^2\theta) = 0$

$\therefore bx\cos\theta + ay\sin\theta - ab = 0$ (2)

as req.

c. (i) $I_n = \int_1^e (\ln x)^n \, dx$

Using integration by parts:

$I_n = \int_1^e (\ln x)^n \cdot 1 \, dx$

$= \left[(\ln x)^n \cdot x\right]_1^e - \int_1^e n \cdot \frac{1}{x}(\ln x)^{n-1} \cdot x \, dx$

$= (e - 0) - n\int_1^e (\ln x)^{n-1} dx$

$\therefore I_n = e - nI_{n-1}$ (2)

(ii) $I_4 = e - 4I_3$

$I_3 = e - 3I_2$

$I_2 = e - 2I_1$

Now $I_1 = \int_1^e \ln x \, dx$

$u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $\frac{dv}{dx} = 1$, $v = x$

$I_1 = \left[x\ln x\right]_1^e - \int_1^e 1 \, dx$

$= e - (e - 1)$

$= 1$

$$I_4 = e - 4I_3$$
$$= e - 4(e - 3I_2)$$
$$= e - 4e + 12I_2$$
$$= -3e + 12I_2$$
$$= -3e + 12(e - 2I_1)$$
$$= -3e + 12e - 24I_1$$
$$= 9e - 24I_1$$
$$= 9e - 24(1)$$
$$= 9e - 24$$

2\. (I) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ — ①

$y = mx + c$ — ②

Sub ② in ①

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\frac{x^2}{a^2} + \frac{m^2x^2 + 2mcx + c^2}{b^2} = 1$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2$$

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

to be a tangent this equation has one real root i.e $\Delta = 0$

$$4a^4m^2c^2 - 4a^2(b^2 + a^2m^2)(c^2 - b^2) = 0$$

$$4a^4m^2c^2 - (4a^2b^2 + 4a^4m^2)(c^2 - b^2) = 0$$

$$4a^4m^2c^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4m^2b^2 = 0$$

$$4a^2b^4 + 4a^4m^2b^2 = 4a^2b^2c^2$$

$(\div 4a^2b^2)$

$\therefore \quad b^2 + a^2m^2 = c^2$ as req

(2)

(II) Since $(3,4)$ lies on

$y = mx + c$

$\therefore 4 = 3m + c \Rightarrow 4 - 3m = c$

and since $c^2 = b^2 + a^2m^2$

$c^2 = 9 + 16m^2$

$$\therefore (4 - 3m)^2 = 9 + 16m^2$$

$$16 - 24m + 9m^2 = 9 + 16m^2$$

$$\therefore 7m^2 + 24m - 7 = 0$$

If m_1 and m_2 are the roots of this equation then $m_1m_2 = -\frac{7}{7}$

$= -1$

Hence the two tangents are at right angles since $m_1m_2 = -1$.

2/