NAME: ______________________________

TEACHER: ____________________________

# GOSFORD HIGH SCHOOL

## 2012
## EXTENSION 2 MATHEMATICS
## HSC ASSESSMENT TASK 3.

**Time Allowed:** 60 minutes (plus 5 min. reading time)

- Write using black or blue pen.
- Board-approved calculators may be used.
- Each Section should be started on a new page.
- All necessary working should be shown in Section II and Section III.

**SPECIAL INSTRUCTIONS:** Tear off back page (**multiple choice/standard integrals**)

| SECTION | QUESTION TYPE | MARKS | RESULT |
|---|---|---|---|
| I | MULTIPLE CHOICE | 4 | |
| II | EXTENDED RESPONSE | 18 | |
| III | EXTENDED RESPONSE | 18 | |
| | TOTAL | 40 | |

## SECTION I : MULTIPLE CHOICE. (4 MARKS)

1. The equation $x^2 + (2 - r^2)y^2 = 1$ represents a conic section. For what values of $r$ is the curve a hyperbola

   A. $r = \pm 1$.   B. $r = \pm\sqrt{2}$.   C. $-\sqrt{2} < r < \sqrt{2}$.   D. $r < -\sqrt{2}$ or $r > \sqrt{2}$.

2. The eccentricity of $\frac{x^2}{4} - \frac{y^2}{3} = 1$ is

   A. $\frac{1}{2}$.   B. $\frac{1}{4}$   C. $\frac{\sqrt{7}}{2}$   D. $\frac{7}{4}$

3. If $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$ then

   A. $A = 1, B = 1$   B. $A = \frac{1}{2}, B = \frac{1}{2}$   C. $A = 2, B = 2$   D. $A = -1, B = 2$

4. $\int sin^{-1}x \ dx =$

   A. $x \, sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} \, dx$   C. $\frac{1}{\sqrt{1-x^2}} - \int x sin^{-1}x \, dx$

   B. $\frac{sin^{-1}x}{\sqrt{1-x^2}} - \int x \, dx$   D. $\frac{x}{\sqrt{1-x^2}} - \int sin^{-1}x \ dx$

**SECTION II :** (18 MARKS) Show all necessary working.

1. Consider the ellipse $\mathcal{E}$ with equation $3x^2 + 4y^2 = 12$.

   (i) Calculate the eccentricity of $\mathcal{E}$. (1)

   (ii) Find the coordinates of the foci $S$ and $S'$ of $\mathcal{E}$. (1)

   (iii) Find the equations of the directrices of $\mathcal{E}$. (1)

   (iv) Show that the equation of the tangent at the point $P_0(x_0, y_0)$ on $\mathcal{E}$ is $\frac{xx_0}{4} + \frac{yy_0}{3} = 1$ (3)

   (v) If $P$ is an arbitrary point on $\mathcal{E}$ prove that the sum of the distances $SP$ and $S'P$ is a constant. (2)

2. The point $P(asec\theta, btan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

   (i) Show that the equation of the normal at $P$ is given by $atan\theta x + bsec\theta y = (a^2 + b^2)sec\theta tan\theta.$ (3)

   (ii) If the normal at $P$ meets the x-axis at $G$ and $N$ is the foot of the perpendicular from $P$ to the x-axis show that $NG : ON = b^2 : a^2$ where $O$ is the origin. (3)

3. Let $P\ (ct, \frac{c}{t})$, where $t > 0$, be any point on the rectangular hyperbola $xy = c^2$. The tangent at $P$ meets the x-axis at $Q$. Show that the locus of the midpoint $M$ of $PQ$ is another rectangular hyperbola. (4)

**SECTION III:** (18 MARKS) **START A NEW PAGE**. Show all necessary working.

1. (a) Evaluate $\int_0^2 \frac{2}{\sqrt{x^2+16}}\,dx$ . (1)

(b) Find $\int \frac{4}{x^2-4x+3}\,dx$ . (4)

(c) Show that $\int_4^{12} \frac{dx}{\sqrt{x}\,(4+x)} = \frac{\pi}{12}$ . (4)

(d) Use the substitution $t = tan\frac{x}{2}$ to show that $\int \frac{dx}{1+sinx} = 2\int \frac{dt}{(1+t)^2}$ . (3)

(e) Use integration by parts to find $\int xe^{3x}\,dx$ . (3)

2. The equation of a curve is $x^2 + xy + 4y^2 = 4$ . Use implicit differentiation to find the equation of the tangent to the curve at the point $(1, -1)$ on it. (3)

Name:_______________________ Teacher:___________________

# Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct ↓

A ⊗ B ⊗ C ○ D ○

**Start here** →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

# STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

# SOLUTIONS

## SECTION I

1. D
2. C
3. B
4. A

## SECTION II

1. (i) If $3x^2+4y^2=12$

$\frac{3x^2}{12}+\frac{4y^2}{12}=\frac{12}{12}$

ie $\frac{x^2}{4}+\frac{y^2}{3}=1$

Now $b^2=a^2(1-e^2)$

$3=4(1-e^2)$

$3=4-4e^2$

$4e^2=1$ ①

$e^2=\frac{1}{4}$

$e=\frac{1}{2}$

(ii) Foci are $(\pm ae, 0)$

$ae = 2\times\frac{1}{2}$

$=1$ ①

∴ S is $(1,0)$ , S' is $(-1,0)$

(iii) Directrices are $x=\pm\frac{a}{e}$

$\frac{a}{e}=\frac{2}{\frac{1}{2}}$ ①

$=4$

∴ Directrices are $x=\pm 4$.

(iv) Given $\frac{x^2}{4}+\frac{y^2}{3}=1$

Diff. w.r.t $x$

$\frac{2x}{4}+\frac{2y}{3}\cdot y'=0$

$\frac{2y}{3}\cdot y' = -\frac{x}{2}$

$y' = \frac{x}{2}\Big/\frac{2y}{3}$

$=\frac{-3x}{4y}$

At $P(x_0,y_0)$ $y'=\frac{-3x_0}{4y_0}$

∴ Eqn of the tangent is

$y-y_0=\frac{-3x_0}{4y_0}(x-x_0)$

$4yy_0-4y_0^2=-3xx_0+3x_0^2$

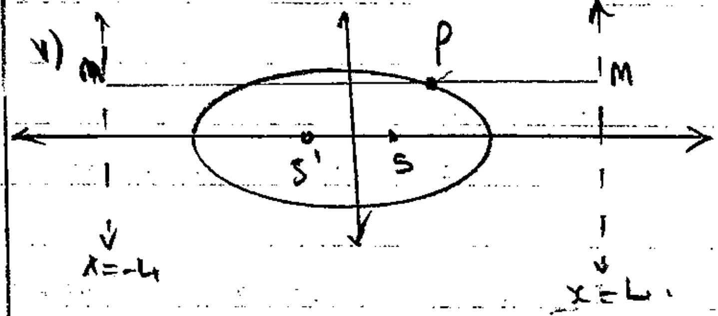
$3xx_0+4yy_0=3x_0^2+4y_0^2$

Now $3x_0^2+4y_0^2=12$ ③

∴ $3xx_0+4yy_0=12$

ie $\frac{xx_0}{4}+\frac{yy_0}{3}=1$.

v)



$\frac{PS}{PM}=e$ & $\frac{PS'}{PM'}=e$

∴ $PS=ePM$ & $PS'=ePM'$

$PS+PS'=e(PM+PM')$

$=\frac{1}{2}\times 8$ ②

$=4$

which is a constant.

2(i) If $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

If $y = b\tan\theta$

$\frac{dy}{d\theta} = b\sec^2\theta$

Now $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

∴ the gradient of the normal is $\frac{-a\tan\theta}{b\sec\theta}$

③

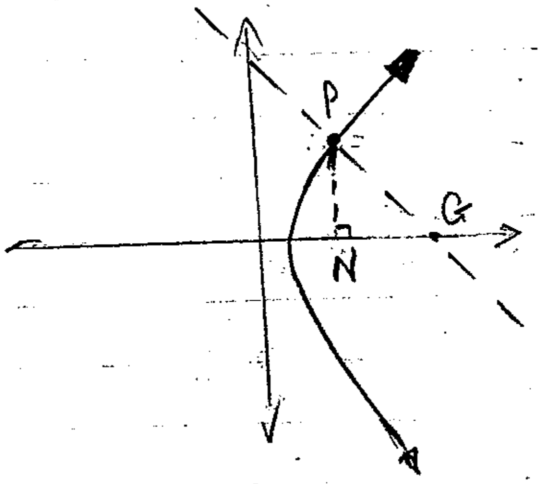
Hence the eqn is

$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

ie $by\sec\theta - b^2\sec\theta\tan\theta$
$= -ax\tan\theta + a^2\sec\theta\tan\theta$

ie $a\tan\theta x + b\sec\theta y = (a^2+b^2)\sec\theta\tan\theta$

(ii)



When $y = 0$,

$a\tan\theta x = (a^2+b^2)\sec\theta\tan\theta$

∴ $x = \frac{(a^2+b^2)}{a}\sec\theta$

∴ G is $\left(\frac{(a^2+b^2)}{a}\sec\theta, 0\right)$

N is $(a\sec\theta, 0)$

∴ GN = $\left[\frac{(a^2+b^2)}{a} - a\right]\sec\theta$

$= \left[\frac{a^2+b^2-a^2}{a}\right]\sec\theta$

$= \frac{b^2}{a}\sec\theta$

③

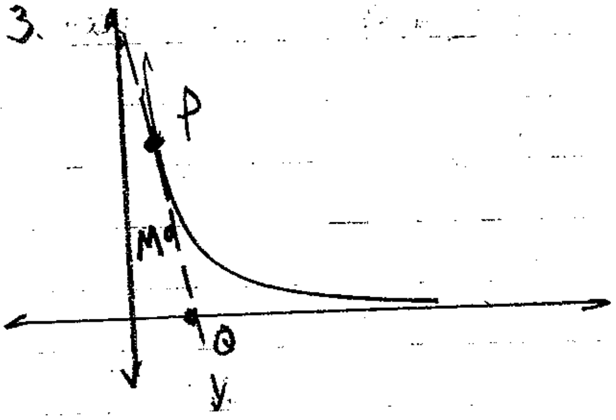
ON = $a\sec\theta$

$\frac{NG}{ON} = \frac{\frac{b^2}{a}\cancel{\sec\theta}}{a\cancel{\sec\theta}}$

$= \frac{b^2}{a^2}$

ie NG : ON = $b^2 : a^2$

3.



If $xy = c^2$

$y = c^2x^{-1}$

$y' = -c^2x^{-2}$

When $x = ct$

$$y' = \frac{-c^2}{c^2t^2}$$

$$= -\frac{1}{t^2}$$

∴ Eq$^n$ of tangent is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = -x + ct$$

$$x + t^2y = 2ct$$

When $y = 0$ $x = 2ct$

∴ M is $\left(\frac{3ct}{2}, \frac{c}{2t}\right)$

If $x = \frac{3ct}{2}$, $y = \frac{c}{2t}$

$$\frac{2x}{3c} = t$$

$$\therefore y = \frac{c}{\frac{4x}{3c}}$$

$$y = \frac{3c^2}{4x}$$

or $xy = \frac{3c^2}{4}$

which is another hyperbola

(4)

SECTION III

1(a) $\int_0^2 \frac{2}{\sqrt{x^2+16}} dx$

$$= 2\left[\ln\left(x + \sqrt{x^2+16}\right)\right]_0^2$$

$$= 2\left\{\ln(2+\sqrt{20}) - \ln 4\right\}$$ ①

$$= 2\ln\left(\frac{2+2\sqrt{5}}{4}\right)$$

$$= 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

b) $\int \frac{4\ dx}{x^2-4x+3} = \int \frac{4\ dx}{(x-1)(x-3)}$

Let $\frac{4}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$

$$\therefore 4 = A(x-3) + B(x-1)$$

If $x = 3$, $4 = 2B$

$B = 2$

If $x = 1$ $4 = -2A$

$A = -2$

$$\therefore I = \int \frac{-2}{x-1} + \frac{2}{x-3}\ dx$$

(4)

$$= -2\ln(x-1) + 2\ln(x-3) + C$$

$$= 2\ln\left|\frac{x-3}{x-1}\right| + c$$

c) $\int_4^{12} \frac{dx}{\sqrt{x}(4+x)}$

Let $u = \sqrt{x}$ or $x^{\frac{1}{2}}$

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx$$

$$= \frac{1}{2\sqrt{x}}dx$$

If $x = 12$, $u = 2\sqrt{3}$

$x = 4$, $u = 2$

$$I = 2\int_4^{12} \frac{dx}{2\sqrt{x}(4+x)}$$

$$= 2\int_2^{2\sqrt{3}} \frac{du}{u^2+4u}$$

$= \cancel{2}\left[\frac{1}{\cancel{2}}\tan^{-1}\frac{u}{2}\right]_2^{2\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1}1$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ ④

d) $\int \frac{dx}{1+\sin x}$ Let $t = \tan\frac{x}{2}$

$dx = \frac{2}{1+t^2}\,dt$

& $\sin x = \frac{2t}{1+t^2}$

$\therefore I = \int \frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$= \int \frac{1}{\frac{1+t^2+2t}{\cancel{1+t^2}}}\cdot\frac{2}{\cancel{1+t^2}}\,dt$

$= 2\int \frac{1}{t^2+2t+1}\,dt$ ③

$= 2\int \frac{1}{(1+t)^2}\,dt$

e) $\int x e^{3x}\,dx$ $u = x$, $v' = e^{3x}$

$u' = 1$, $v = \frac{1}{3}e^{3x}$

$I = \frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx$

$= \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + c$

$= \frac{3xe^{3x} - e^{3x}}{9} + c$

$= \frac{e^{3x}(3x-1)}{9} + c$ ③

2. $x^2 + xy + 4y^2 = 4$

Diff w.r.t $x$

$2x + y\cdot 1 + x\cdot y' + 8y\cdot y' = 0$

$y'(x+8y) = -2x-y$

$y' = \frac{-2x-y}{x+8y}$

If $x = 1$, & $y = -1$, $y' = \frac{-2+1}{1-8}$

$= \frac{-1}{-7}$

$= \frac{1}{7}$ ③

∴ the eqn is

$y+1 = \frac{1}{7}(x-1)$

$7y+7 = x-1$

$0 = x-7y-8$

<u>or</u> $x-7y-8=0$