GOSFORD HIGH SCHOOL

2013

HSC ASSESSMENT TASK 3

EXTENSION 2 MATHEMATICS

General Instructions:

Total marks: - 40

Attempt all Questions

- Working time: 60 minutes
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question in Section 2 should be Started on a separate page.
- All necessary working should be shown in every question.

SECTION 1	**/ 4**
SECTION 2 Q1	**/ 12**
SECTION 2 Q2	**/ 12**
SECTION 2 Q3	**/ 12**
TOTAL	**/ 40**

SECTION 1. (4 MARKS) Answer on the answer sheet provided.

Question 1.

If $z = 1 + i$ is a root of $z^2 - aiz + b = 0$, where a and b are real numbers then

A. $a = 2, b = -2$

B. $a = -2, b = -2$

C. $a = 2, b = 2$

D. $a = -2, b = 2$

Question 2.

If $= \sqrt{2} - i\sqrt{2}$, then z^5 written in mod-arg form is

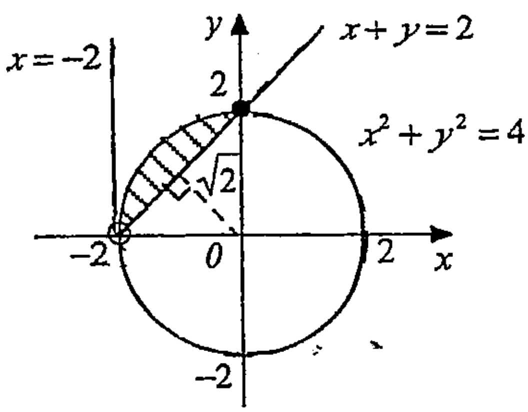
A. $2\sqrt{2}\ \text{cis}\ \frac{-\pi}{4}$

B. $2\sqrt{2}\ cis\ \frac{3\pi}{4}$

C. $32\ cis\ \frac{-\pi}{4}$

D. $32\ cis\ \frac{3\pi}{4}$

Question 3.



The shaded region in the Argand diagram above shows all the points representing complex numbers z such that

A. $|z| \leq 2$ and $\frac{\pi}{4} \leq \arg(z-2) \leq \frac{\pi}{2}$.

B. $|z| \leq 2$ and $\frac{\pi}{4} \leq \arg(z+2) \leq \frac{\pi}{2}$.

C. $|z| \geq 2$ and $\frac{\pi}{4} \leq \arg(z-2) \leq \frac{\pi}{2}$.

D. $|z| \geq 2$ and $\frac{\pi}{4} \leq \arg(z+2) \leq \frac{\pi}{2}$.

Question 4.

If $\left|\frac{z-1}{z}\right| = 2$ then the locus of z is given by

A. $3x^2 + 3y^2 + 2x - 1 = 0$

B. $3x^2 + 3y^2 - 2x - 1 = 0$

C. $x^2 + y^2 + 2x - 1 = 0$

D. $x^2 + y^2 - 2x + 1 = 0$

Name:____________________ Teacher:____________________

Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

A ☒ B ☒ (correct) C ○ D ○

Start here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

HSC ASSESSMENT TASK 3

EXTENSION 2 MATHEMATICS

SECTION 2. (36 marks) Answer on your own paper. Show all necessary working

Question 1. (12 marks) **START A NEW PAGE.**

(a) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where m and n are real. It is known that $1 - i$ is a root of this equation.

(i) Find the other roots of the equation. (2)

(ii) Find the values of m and n. (2)

(b) α, β, and γ are roots of the equation $x^3 - 6x^2 + 12x - 35 = 0$.

(i) Form a cubic equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2$. (2)

(ii) Hence, or otherwise solve the equation $x^3 - 6x^2 + 12x - 35 = 0$ over the complex field. (2)

(c) Solve the equation $x^4 - 6x^2 + 8x - 3 = 0$, given that it has a triple root (4)

Question 2. (12 marks) **START A NEW PAGE.**

(a) Find (i) $\int \frac{\ln x}{x}\, dx$ (1)

(ii) $\int \frac{\ln x}{x^2}\, dx$ (3)

(b) (i) If $\frac{x-11}{x^2+3x-4} = \frac{A}{x+4} + \frac{B}{x-1}$, find A and B. (2)

(ii) Hence, evaluate $\int_2^3 \frac{x-11}{x^2+3x-4}\, dx$. (2)

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$. (4)

Question 3. (12 marks) **START A NEW PAGE.**

(a) Find $\int \frac{x^2}{1+9x^2}\, dx$. (3)

(b) Use the substitution $x = \tan\theta$ to evaluate $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$ (4)

(c) (i) If $I_n = \int_{-1}^{0} x^n \sqrt{1+x}\, dx$, for $n = 0,1,2,\ldots$ show that

$$I_n = \frac{-2n}{2n+3} I_{n-1}, \quad \text{for } n = 1,2,3,\ldots$$ (3)

(ii) Hence evaluate $\int_{-1}^{0} x^3 \sqrt{1+x}\, dx$. (2)

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SOLUTIONS

SECTION 1

1/ If $z = 1+i$ is a root

$(1+i)^2 - ai(1+i) + b = 0 + 0i$

$1+2i+i^2 - ai - ai^2 + b = 0 + 0i$

$1+2i - 1 - ai + a + b = 0 + 0i$

$(a+b) + i(2-a) = 0 + 0i$

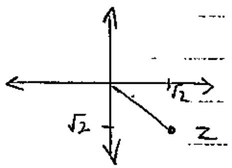
$\therefore 2-a = 0$

$a = 2$

$b = -2$

HENCE: (A)

2. $z = \sqrt{2} - i\sqrt{2}$



$\therefore z = 2\,\text{cis}\left(-\frac{\pi}{4}\right)$

$z^5 = 2^5\,\text{cis}\left(-\frac{5\pi}{4}\right)$

$= 32\,\text{cis}\,\frac{3\pi}{4}$

HENCE: (D)

3. Shaded region is inside $x^2+y^2=4$

$\therefore |z| \leq 2$

Shaded region is above and to the left of $x+y=2$

$\therefore \frac{\pi}{4} \leq \arg(z+2) \leq \frac{\pi}{2}$

HENCE: (B)

4. $\left|\frac{z-1}{z}\right| = 2.$

Square both sides

$|z-1|^2 = 4|z|^2$

Let $z = x+iy$

$|x+iy-1|^2 = 4|x+iy|$

$(x-1)^2 + y^2 = 4(x^2+y^2)$

$x^2 - 2x + 1 + y^2 = 4x^2 + 4y^2$

$0 = 3x^2 + 3y^2 + 2x - 1$

HENCE (A).

SECTION 2.

1/ (i) Since the coefficients are real, if $1-i$ is a root then so is $1+i$

Let the third root be α

Now $\sum \alpha\beta\gamma = \alpha(1-i)(1+i)$

$= \alpha(1-i^2)$

$= 2\alpha$

$\therefore 2\alpha = -\frac{d}{a}$

$2\alpha = -6$

$\alpha = -3$

$\therefore$ the roots are $-3,\ 1+i,\ 1-i$ (2)

(ii) $\sum \alpha = -3 + 1 + i + 1 - i$

$= -1$

$\therefore -1 = -\frac{b}{a}$

$-1 = -m$

$m = 1$ (1)

+ $\sum \alpha\beta = (1+i)(1-i) + -3(1-i) + -3(1+i)$

$= 2 - 3 + 3i - 3 - 3i$

$= -4$

$\therefore -4 = \frac{c}{a}$

$-4 = n$

$n = -4$ (1)

b)

(i) Let $y = x-2$ ie $x = y+2$

So $(y+2)^3 - 6(y+2)^2 + 12(y+2) - 35 = 0$

$y^3 + 3(y^2)(2) + 3(y)(2)^2 + (2)^3 - 6[y^2+4y+4] + 12y + 24 - 35 = 0$

ie $y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 12y + 24 - 35 = 0$

$\therefore \quad y^3 - 27 = 0.$ (2)

(ii) If $y^3 - 27 = 0$

$(y-3)(y^2+3y+9) = 0$

$y = 3$ or $y = \dfrac{-3 \pm \sqrt{9-36}}{2}$

$y = 3$ or $\dfrac{-3 \pm 3\sqrt{3}\,i}{2}$

Now the roots of $x^3 - 6x^2 + 12x - 35 = 0$ are two more than the roots of $y^3 - 27 = 0$

$\therefore$ the required roots are $5,\ \frac{1}{2} + \frac{3\sqrt{3}i}{2},\ \frac{1}{2} - \frac{3\sqrt{3}i}{2}$ (2)

c) Let $P(x) = x^4 - 6x^2 + 8x - 3$

$P'(x) = 4x^3 - 12x + 8$

$P''(x) = 12x^2 - 12$

If $P''(x) = 0$

$12x^2 - 12 = 0$

$x^2 - 1 = 0$

$x = \pm 1$

Now $P'(1) = 4 - 12 + 8$
$= 0$

& $P(1) = 1 - 6 + 8 - 3$
$= 0$

$\therefore$ $x = 1$ is a triple root

Let the other root be α

So $P(x) = (x-1)^3(x-\alpha)$

Now $\Sigma\alpha\beta\gamma\delta = \dfrac{e}{a}$ (4)

$\therefore (1)^3\alpha = -3$

$\alpha = -3$

$\therefore$ $x = 1$ is a triple root & -3 is a single root

2/ a) i) $\displaystyle\int \frac{\ln x}{x}\,dx = \int \frac{1}{x} \times \ln x \, dx$

$= \dfrac{(\ln x)^2}{2} + c$ (1)

ii) $\displaystyle\int \frac{\ln x}{x^2}\,dx = I$

Using integration by parts $u = \ln x$, $v' = x^{-2}$

$u' = \frac{1}{x}$, $\therefore v = \frac{-1}{x}$

$I = -\dfrac{\ln x}{x} - \displaystyle\int -\frac{1}{x^2}\,dx$

$= -\dfrac{\ln x}{x} + \displaystyle\int x^{-2}\,dx$

$= -\dfrac{\ln x}{x} - x^{-1} + c$

$= \dfrac{-\ln x - 1}{x} + c$ (3)

b) (i) $\frac{x-11}{x^2+3x-4} = \frac{x-11}{(x+4)(x-1)}$

$\therefore \frac{x-11}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$

Hence $x-11 = A(x-1) + B(x+4)$

If $x=1$, $-10 = 5B$
$B = -2$

If $x=-4$, $-15 = -5A$
$A = 3$

$\frac{x-11}{(x+4)(x-1)} = \frac{3}{x+4} - \frac{2}{x-1}$ (2)

(ii) $\int_2^3 \frac{x-11}{x^2+3x-4}dx = \int_2^3 \frac{3}{x+4} - \frac{2}{x-1}\,dx$

$= \left[3\ln(x+4) - 2\ln(x-1)\right]_2^3$

$= (3\ln 7 - 2\ln 2) - (3\ln 6 - 2\ln 1)$

$= 3\ln 7 - 2\ln 2 - 3\ln 6.$

$= \ln\left(\frac{343}{864}\right).$ (2)

c) $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$

Let $t = \tan\frac{\theta}{2}$

$\therefore d\theta = \frac{2}{1+t^2}dt$ (working not needed)

If $\theta = 0$, $t = 0$
$\theta = \frac{\pi}{2}$, $t = 1.$

$\therefore I = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{1}{\frac{2+2t^2+1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{2}{3+t^2}\,dt$

$= \frac{2}{\sqrt3}\left[\tan^{-1}\frac{t}{\sqrt3}\right]_0^1$

$= \frac{2}{\sqrt3}\left[\tan^{-1}\frac{1}{\sqrt3} - \tan^{-1}0\right]$ (4)

$= \frac{2}{\sqrt3}\left(\frac{\pi}{6} - 0\right)$

$= \frac{\pi}{3\sqrt3}$ or $\frac{\pi\sqrt3}{9}.$

3. a) $\int \frac{x^2}{1+9x^2}dx = \frac{1}{9}\int \frac{9x^2}{1+9x^2}dx$

$= \frac{1}{9}\int \frac{1+9x^2}{1+9x^2} - \frac{1}{1+9x^2}\,dx$ (3)

$= \frac{1}{9}\int 1 - \frac{1}{9(\frac{1}{9}+x^2)}\,dx$

$= \frac{1}{9}\left\{x - \frac{1}{9}\times\frac{1}{\frac{1}{3}}\tan^{-1}\frac{x}{\frac{1}{3}}\right\} + c$

$= \frac{1}{9}\left\{x - \frac{1}{3}\tan^{-1}3x\right\} + c$

b) $\int_1^{\sqrt3} \frac{dx}{x^2\sqrt{1+x^2}}$

Let $x = \tan\theta$
$dx = \sec^2\theta\,d\theta$

If $x=1$, $\theta = \frac{\pi}{4}$

If $x=\sqrt3$, $\theta = \frac{\pi}{3}$

$$\therefore \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{1+x^2}} = \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta \, d\theta}{\tan^2\theta \sqrt{1+\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta \, d\theta}{\tan^2\theta \cdot \sec\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{d\theta}{\frac{\sin^2\theta}{\cos^2\theta} \times \cos\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

$$\int_{\pi/4}^{\pi/2} \cot\theta \cdot \csc\theta \, d\theta$$

$$= \left[-\csc\theta\right]_{\pi/4}^{\pi/3}$$

$$= -\frac{2}{\sqrt{3}} - -\sqrt{2}$$

$$= \sqrt{2} - \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{2} - \frac{2\sqrt{3}}{3} \quad \text{or} \quad \frac{3\sqrt{2} - 2\sqrt{3}}{3}$$

(4)

c) (i) $I_n = \int_{-1}^{0} x^n \sqrt{1+x} \, dx, \quad n = 0, 1, 2, \ldots$

$u = x^n \quad v' = (1+x)^{1/2}$

$u' = nx^{n-1}, \quad v = \frac{2}{3}(1+x)^{3/2}$

$$I_n = \left[x^n \cdot \frac{2}{3}(1+x)^{3/2}\right]_{-1}^{0} - \int_{-1}^{0} \frac{2}{3}(1+x)^{3/2} \cdot nx^{n-1} \, dx$$

$$= (0-0) - \frac{2n}{3} \int_{-1}^{0} x^{n-1}(1+x)(1+x)^{1/2} \, dx$$

$$= 0 - \frac{2n}{3} \int_{-1}^{0} x^{n-1}\sqrt{1+x} + x^n\sqrt{1+x} \, dx$$

$$I_n = -\frac{2n}{3}(I_{n-1} + I_n)$$

$$I_n + \frac{2n}{3} I_n = -\frac{2n}{3} I_{n-1}$$

$$\frac{3I_n + 2nI_n}{3} = -\frac{2n}{3} I_{n-1}$$

$$\therefore I_n(3+2n) = -2n \, I_{n-1}$$

$$I_n = \frac{-2n}{2n+3} I_{n-1}$$

(3)

(ii) $I_3 = \frac{-6}{9} I_2$

$$= -\frac{6}{9} \times -\frac{4}{7} \times I_1$$

$$= -\frac{6}{9} \times -\frac{4}{7} \times \frac{-2}{5} \times I_0$$

Now $I_0 = \int_{-1}^{0} (1+x)^{1/2} \, dx$

$$= \left[\frac{2}{3}(1+x)^{3/2} \times 1\right]_{-1}^{0}$$

$$= \frac{2}{3} - 0$$

$$= \frac{2}{3}$$

$$\therefore I_3 = -\frac{6}{9} \times -\frac{4}{7} \times -\frac{2}{5} \times \frac{2}{3}$$

$$= \frac{-32}{315}$$

(2)