Name .... Number..

HSC Mathematics Extension 2
Task 3 - 2014

Time Allowed - 1 hour + 5minutes reading

Instructions: Calculators may be used in any parts of the task
A Standard Integrals sheet is on the last page and may be detached if desired.
Necessary working MUST be shown to receive full marks.
Questions 1 - 4 must be answered on the multiple choice answer sheet provided.
Questions 5 - 7 must be answered on your own paper, beginning each question on a new sheet of paper.

Multiple Choice	/4
Question 5	/12
Question 6	/12
Question 7	/12
Total	/40

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Answer this section on the multiple choice answer sheet provided

1. Which expression is equal to $\int \tan x \ dx$

(A) $sec^2 x + c$

(B) $\ln(sec\, x) + c$

(C) $\dfrac{tan^2 x}{2} + c$

(D) $\ln(sec\, x + tan\, x) + c$

2. Which expression is equal to

$$\int \frac{x^3 - 1}{(x^4 - 4x)^{\frac{2}{3}}} dx$$

(A) $\dfrac{3}{4\sqrt[3]{x^4 - 4x}} + C$

(B) $\dfrac{3}{4\left(x^4 - 4x\right)} + C$

(C) $\dfrac{3\sqrt[3]{x^4 - 4x}}{4} + C$

(D) $\dfrac{3\left(x^4 - 4x\right)}{4} + C$

3. The equation of a hyperbola is given by $9x^2 - 4y^2 = 36$. The foci and the directrices of this hyperbola are:

(A) $\left(\pm\sqrt{13}, 0\right)$ and $x = \pm\dfrac{4\sqrt{13}}{13}$.

(B) $\left(0, \pm\sqrt{13}\right)$ and $x = \pm\dfrac{4\sqrt{13}}{13}$.

(C) $\left(\pm\sqrt{13}, 0\right)$ and $y = \pm\dfrac{4\sqrt{13}}{13}$.

(D) $\left(0, \pm\sqrt{13}\right)$ and $y = \pm\dfrac{4\sqrt{13}}{13}$.

4. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\emptyset, b\sin\emptyset)$ lie on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the chord PQ subtends a right angle at the origin.

Which of the following is the correct expression?

(A) $\tan\theta\tan\emptyset = -\dfrac{b^2}{a^2}$

(B) $\tan\theta\tan\emptyset = -\dfrac{a^2}{b^2}$

(C) $\tan\theta\tan\emptyset = \dfrac{b^2}{a^2}$

(D) $\tan\theta\tan\emptyset = \dfrac{a^2}{b^2}$

Question 5 **(12 marks)** **Start a separate sheet of paper** **Marks**

a) i) If $\dfrac{x}{(x-2)^2(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x-1}$, find the values of A, B and C. **2**

ii) Hence evaluate $\displaystyle\int \frac{x}{(x-2)^2(x-1)}\, dx$. **2**

b) i) Derive the reduction formula

$$\int x^n e^x\, dx = x^n e^x - n\int x^{n-1} e^x\, dx.$$ **1**

ii) Hence or otherwise, find $\displaystyle\int_0^2 x^3 e^x\, dx$ **2**

c) Find

i) $\displaystyle\int \frac{dx}{\left(25+x^2\right)^{\frac{3}{2}}}$. using the trigonometric substitution $x = 5\tan\theta$. **3**

ii) $\displaystyle\int \frac{dx}{\sqrt{4+2x-x^2}}$. **2**

Question 6 (12 marks) Start a separate sheet of paper **Marks**

a) An ellipse has equation $\frac{x^2}{25}+\frac{y^2}{16}=1$

(i) Find the eccentricity, coordinates of the foci and equations of the directrices. **3**

(ii) Sketch the curve showing this information and all intercepts. **2**

(iii) Point $P(x_1, y_1)$ lies on this ellipse. Show that the sum of the distances from P to $A(3,0)$ and $B(-3,0)$ is 10 units. **2**

(iv) Using calculus, derive the equation of the tangent to this ellipse at $P(x_1, y_1)$, showing that its equation is

$$\frac{xx_1}{25}+\frac{yy_1}{16}=1$$ **3**

(v) Derive the equation of the chord of contact of tangents from an external point $T\ (x_0, y_0)$ to this ellipse. **2**

Question 7 (12 Marks) Start a new sheet of paper. **Marks**

a) i) Show that the point P $(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ **1**

ii) Show that the gradient of the tangent to the hyperbola at P is $\dfrac{b\sec\theta}{a\tan\theta}$ **2**

iii) The normal at P meets the x axis at G and PN is perpendicular to the x axis.

Prove that $\dfrac{OG}{ON} = e^2$ where O is the origin and e is the eccentricity. **3**

b) i) Show, by first rationalising the numerator, that $\sqrt{x^2 - a^2}\;.\;x^{n-2}$ can be written as

$$\frac{x^n}{\sqrt{x^2 - a^2}} - \frac{a^2\, x^{n-2}}{\sqrt{x^2 - a^2}}$$ **1**

ii) Using integration by parts, with $u = x^{n-1}$, derive the recurrence relation (reduction formula) for

$$I_n = \int \frac{x^n}{\sqrt{x^2 - a^2}}\,dx \quad , \qquad \text{that is:}$$

$$I_n = \frac{x^{n-1}}{n}\sqrt{x^2 - a^2} + \frac{(n-1)}{n}a^2\, I_{n-2}$$ **3**

iii) Hence find a formula for

$$\int_a^{2a} \frac{x^n}{\sqrt{x^2 - a^2}}\,dx$$ 2

END OF TEST

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Mathematics Extension 2 Task 3 2014

Student Name/Number:____________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2+4=$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

Solutions to Extension 2 Task 3 – 2014

1. $\ln(\sec x) + c$ B

2. $\frac{d}{dx}(x^4 - 4x) = 4x^3 - 4$

$= 4(x^3 - 1)$

$\frac{1}{4}\int (4x^3 - 4)(x^4 - 4x)^{-2/3}\, dx$

$= \frac{1}{4}\ (x^4 - 4x)^{1/3} \cdot 3 = \frac{3}{4}\sqrt[3]{x^4 - 4x}$ C

3. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$a = 2$
$b = 3$

$b^2 = a^2(e^2 - 1)$ $e^2 = \frac{13}{4}$

$9 = 4(e^2 - 1)$

$\frac{9}{4} = e^2 - 1$ $e = \frac{\sqrt{13}}{2}$ A

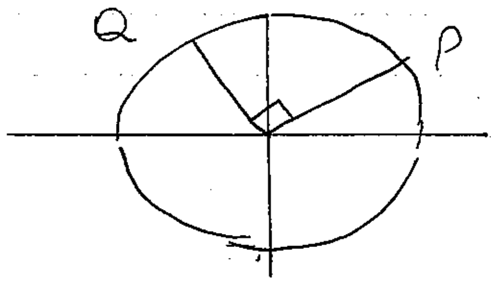
$\therefore$ Foci $(\pm ae, 0) = (\pm\sqrt{13}, 0)$

Directrices $x = \pm\frac{a}{e} = \frac{2}{\sqrt{13}} \cdot 2$

$x = \frac{4}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} = \frac{4\sqrt{13}}{13}$

4. $\frac{b\sin\theta}{a\cos\theta} \times \frac{b\sin\phi}{a\cos\phi} = -1$ B

$\tan\theta\tan\phi = -\frac{a^2}{b^2}$



5 a)

$$\frac{x}{(x-2)^2(x-1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-1}$$

$$x = A(x-2)(x-1) + B(x-1) + C(x-2)^2$$

$x=1 \quad 1 = 0 + 0 + C \Rightarrow C = 1$

$x=2 \quad 2 = 0 + B + 0 \Rightarrow B = 2$

$x=0 \quad 0 = 2A - B + 4C$

$0 = 2A - 2 + 4 \Rightarrow A = -1$

$$\therefore \int \frac{x \, dx}{(x-2)^2(x-1)} = \int \frac{-1}{x-2} + \frac{2}{(x-2)^2} + \frac{1}{x-1} \, dx$$

$$= -\ln(x-2) - \frac{2}{x-2} + \ln(x-1)$$

$$= \ln \frac{x-1}{x-2} - \frac{2}{x-2} + C$$

b) i) $\int x^n e^x \, dx \qquad u = x^n \qquad dv = e^x dx$

$du = nx^{n-1} dx \qquad v = e^x$

$$\int u \, dv = uv - \int v \, du$$

$$\therefore \int x^n e^x dx = x^n e^x - \int e^x . nx^{n-1} dx$$

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

ii) $\int_0^2 x^3 e^x dx = [x^3 e^x]_0^2 - 3 \int_0^2 x^2 e^x dx$

$I_3 = x^3 e^x - 3 I_2$

$= x^3 e^x - 3 \{x^2 e^x - 2 I_1\}$

$= x^3 e^x - 3x^2 e^x + 6 \{x e^x - I_0\}$

$= [x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x]_0^2$

$= (8e^2 - 3\times4 e^2 + 6\times2 e^2 - 6e^2) - (0-0+0-6)$

$= 8e^2 - 12e^2 + 12e^2 - 6e^2 + 6$

$= 2e^2 + 6$

c) i) $\displaystyle\int \frac{dx}{(25+x^2)^{3/2}}$ $\qquad x = 5\tan\theta$

$\qquad dx = 5\sec^2\theta\, d\theta$

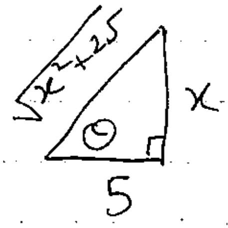
$= \displaystyle\int \frac{5\sec^2\theta\, d\theta}{(25+25\tan^2\theta)^{3/2}}$ $\qquad 25(1+\tan^2\theta)$

$\qquad = 25\sec^2\theta$

$= \displaystyle\int \frac{5\sec^2\theta}{(5\sec\theta)^3}\, d\theta$

$= \displaystyle\int \frac{\sec^2\theta}{25\sec^3\theta}\, d\theta$

$= \displaystyle\int \frac{d\theta}{25\sec\theta} = \frac{1}{25}\int \cos\theta\, d\theta$



$= \dfrac{\sin\theta}{25}$

$= \dfrac{x}{25\sqrt{x^2+25}} + C$

ii) $\displaystyle\int \frac{dx}{\sqrt{4+2x-x^2}}$ $\qquad 4+2x-x^2$

$\qquad 4-(x^2-2x)$

$\qquad 4-(x^2-2x+1)+1$

$\qquad 5-(x-1)^2$

$= \displaystyle\int \frac{dx}{\sqrt{5-(x-1)^2}}$

$= \sin^{-1}\dfrac{(x-1)}{\sqrt{5}} + C$

6 a) $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ $\quad a = 5$, $b = 4$

i) $b^2 = a^2(1 - e^2)$

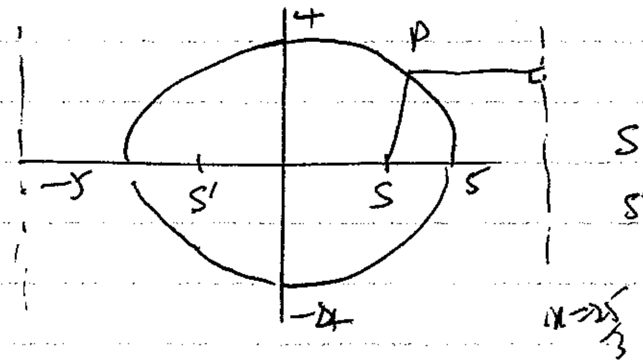
$16 = 25(1 - e^2)$

$\dfrac{16}{25} = 1 - e^2 \Rightarrow e^2 = \dfrac{9}{25}$

$e = \dfrac{3}{5}$

$\therefore$ S $(\pm ae, 0) = (\pm 3, 0)$

directrices $x = \pm \dfrac{a}{e} = \pm \dfrac{25}{3}$



S (3, 0)

S' (-3, 0)

iii $PA + PB = 10$

$PA = e\,PD = e\cdot\left(\dfrac{25}{3} - x_1\right)$

$= \dfrac{3}{5}\left(\dfrac{25}{3} - x_1\right)$

$PB = e\,PD' = e\left(\dfrac{25}{3} + x_1\right)$

$= \dfrac{3}{5}\left(\dfrac{25}{3} + x_1\right)$

$PA + PB = \dfrac{3}{5}\left(\dfrac{25}{3} - x_1 + \dfrac{25}{3} + x_1\right)$

$= \dfrac{3}{5}\cdot\dfrac{50}{3} = 10$

iv) $\dfrac{2x}{25} + \dfrac{2y}{16}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2x}{25}\cdot\dfrac{16}{2y}$

$= -\dfrac{16x}{25y}$

iv) At $P(x_1, y_1)$

$$m = \frac{-16x_1}{25y_1}$$

$$\therefore y - y_1 = \frac{-16x_1}{25y_1}(x - x_1)$$

$$25yy_1 - 25y_1^2 = -16xx_1 + 16x_1^2$$

$$16xx_1 + 25yy_1 = 16x_1^2 + 25y_1^2$$

$\div 16 \times 25$
$$\frac{xx_1}{25} + \frac{yy_1}{16} = \frac{x_1^2}{25} + \frac{y_1^2}{16}$$

But $P(x_1 y_1)$ lies on ellipse $\therefore \frac{x_1^2}{25} + \frac{y_1^2}{16} = 1$

$$\therefore \frac{xx_1}{25} + \frac{yy_1}{16} = 1$$

v) Let chord of contact join $P(x_1 y_1)$ + $Q(x_2 y_2)$ on ellipse.

$\therefore$ Tangent at $P(x_1 y_1)$ is

$$\frac{xx_1}{25} + \frac{yy_1}{16} = 1$$

$(x_0 y_0)$ lies on this

$$\therefore \frac{x_0 x_1}{25} + \frac{y_0 y_1}{16} = 1$$

This means $(x_1 y_1)$ lies on $\frac{xx_0}{25} + \frac{yy_0}{16} = 1$

Tangent at $Q(x_2 y_2)$ is

$$\frac{xx_2}{25} + \frac{yy_2}{16} = 1$$

$(x_0 y_0)$ lies on this

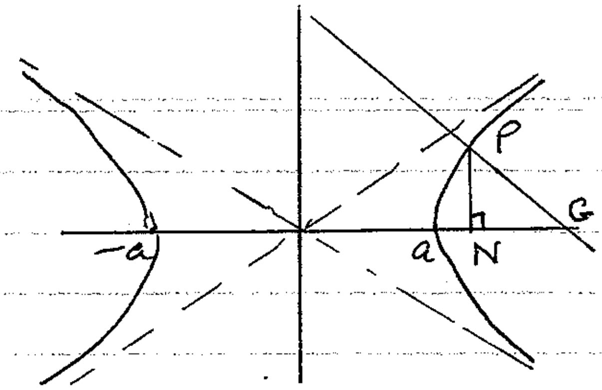
$$\therefore \frac{x_0 x_2}{25} + \frac{y_0 y_2}{16} = 1$$

This means $(x_2 y_2)$ lies on $\frac{xx_0}{25} + \frac{yy_0}{16} = 1$

P + Q both lie on this.

$\therefore$ This is the chord PQ ie $\frac{xx_0}{25} + \frac{yy_0}{16} = 1$

Q7 i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



$LHS = \dfrac{a^2\sec^2\theta}{a^2} - \dfrac{b^2\tan^2\theta}{b^2}$

$= \sec^2\theta - \tan^2\theta$

$= 1 = RHS$

$\therefore P(a\sec\theta, b\tan\theta)$ lies on the hyperbola

ii) $x = a\sec\theta$

$y = b\tan\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

iii) $\therefore$ Gradient of normal is $-\dfrac{a\tan\theta}{b\sec\theta}$

$\therefore$ Normal is $y - b\tan\theta = -\dfrac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

Put $y=0$ for G $\quad -b^2\sec\theta\tan\theta = -a\tan\theta(x - a\sec\theta)$

$-b^2\sec\theta\tan\theta = -a\tan\theta x + a^2\sec\theta\tan\theta$

$a\tan\theta x = (a^2+b^2)\sec\theta\tan\theta$

$x = \dfrac{a^2+b^2}{a}\sec\theta$

N is $(a\sec\theta, 0)$

$\therefore \dfrac{OG}{ON} = \dfrac{a^2+b^2}{a}\cdot\sec\theta\cdot\dfrac{1}{a\sec\theta}$

$= \dfrac{a^2+b^2}{a^2}$

But $a^2+b^2 = a^2e^2$ $\quad \therefore \dfrac{OG}{ON} = \dfrac{a^2e^2}{a^2} = e^2$

7b) i)

$$\sqrt{x^2-a^2}\,.\,x^{n-2} \times \frac{\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}$$

$$= \frac{(x^2-a^2)\,.\,x^{n-2}}{\sqrt{x^2-a^2}}$$

$$= \frac{x^2\,x^{n-2}}{\sqrt{x^2-a^2}} - \frac{a^2\,x^{n-2}}{\sqrt{x^2-a^2}}$$

$$= \frac{x^n}{\sqrt{x^2-a^2}} - \frac{a^2\,x^{n-2}}{\sqrt{x^2-a^2}}$$

ii) $\int \frac{x^n}{\sqrt{x^2-a^2}}\,dx$ put $u = x^{n-1}$, $du = (n-1)x^{n-2}\,dx$; $dv = \frac{x\,dx}{\sqrt{x^2-a^2}}$, $v = \sqrt{x^2-a^2}$

$$\therefore I_n = x^{n-1}\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,.\,(n-1)x^{n-2}\,dx$$

$$= x^{n-1}\sqrt{x^2-a^2} - (n-1)\int \sqrt{x^2-a^2}\,.\,x^{n-2}\,dx$$

$$= x^{n-1}\sqrt{x^2-a^2} - (n-1)\int \frac{x^n}{\sqrt{x^2-a^2}} - \frac{a^2\,x^{n-2}}{\sqrt{x^2-a^2}}\,dx$$

$$I_n = x^{n-1}\sqrt{x^2-a^2} - (n-1)\,I_n + (n-1)\,a^2\,I_{n-2}$$

$$\underbrace{(1+n-1)\,I_n}_{n\,I_n} = x^{n-1}\sqrt{x^2-a^2} + (n-1)\,a^2\,I_{n-2}$$

$$\therefore I_n = \frac{x^{n-1}}{n}\sqrt{x^2-a^2} + \frac{(n-1)}{n}\,a^2\,I_{n-2}$$

$$\therefore \int_a^{2a} I_n = \left[\frac{x^{n-1}}{n}\sqrt{x^2-a^2}\right]_a^{2a} + \frac{n-1}{n}\,a^2\,I_{n-2}$$

$$= \frac{(2a)^{n-1}}{n}\sqrt{4a^2-a^2} - 0 + \frac{n-1}{n}\,a^2\,I_{n-2}$$

$$= \frac{(2a)^{n-1}}{n}\sqrt{3}\,a + \frac{n-1}{n}\,a^2\,I_{n-2}$$