YEAR 12 Mathematics Ext 2
HSC Course
Assessment Task 3
2015

1. There are 3 sections
2. Multiple choice questions are 1 mark each. Marks allocated to each free response question are indicated.
3. Answer the multiple choice section on the response sheet provided.
4. Start each section on a new page. You are to hand in the integration and conics free response sections separately.
5. Calculators may be used
6. Time allowed - **60 minutes plus 5 minutes reading**

Topic	Mark
1. Integration section including Q1 and Q2 from multiple choice	/22
2. Integration section including Q3 and Q4 from multiple choice Conics	/20

TOTAL **/42**

Multiple Choice Section (1 mark each)

Answer each question on the multiple choice answer sheet provided.

1. The definite integral

$$\int_{e^3}^{e^4} \frac{1}{xlnx}\,dx$$

can be written in the form

$$\int_a^b \frac{1}{u}\,du$$

where

(A) $u = ln(x),\ a = ln3,\ b = ln4$

(B) $u = ln(x),\ a = 3,\ b = 4$

(C) $u = ln(x),\ a = e^3, b = e^4$

(D) $u = \frac{1}{x},\ a = e^3,\ b = e^4$

2) What is the value of

$$\int_0^1 \frac{e^x}{1+e^x}\,dx$$

(A) $ln\,(1+e)$

(B) 1

(C) $ln\,(\frac{1+e}{2})$

(D) $ln\left(\frac{e}{2}\right) - 2$

3) P is any point on the hyperbola with equation $x^2 - \frac{y^2}{4} = 1$.

If m is the gradient of the hyperbola at P, then m could be

(A) $any\ real\ number\ x$

(B) $-4 < x < 4,\ where\ x\ is\ real$

(C) $-2 < x < 2,\ where\ x\ is\ real$

(D) $x > 2\ or\ x < -2,\ where\ x\ is\ real$

4) In the equation $Ax^2 + Cy^2 = F\ if\ AC > 0,\ A < C\ and\ F > 0$, then the curve is

(A) an ellipse with a longer axis along the x axis

(B) an ellipse with a longer axis along the y axis

(C) a hyperbola intersecting the x axis

(D) a hyperbola intersecting the y axis

(END OF MULTIPLE CHOICE QUESTIONS)

MARKS

Integration Section (Start a new page)

1) Find

$$\int \frac{x}{\sqrt{9 - 16x^2}}\, dx$$ 2

2) Find

$$\int \frac{x^2}{x + 1}\, dx$$ 2

3) Evaluate

$$\int_0^{ln3} xe^x dx$$ 3

4) (a) Find real numbers A,B and C such that $\frac{2}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ 3

(b) Hence evaluate

$$\int_0^1 \frac{2}{(x+1)(x^2+1)}\, dx$$ 3

(c) By using the substitution $t = tan\,\frac{x}{2}$

evaluate $$\int_0^{\frac{\pi}{2}} \frac{sinx}{1 + sinx - cosx}\, dx$$ 3

5) (a) Given $I_n = \int \sin^n x\, dx,$ show that the expression for I_n in terms of I_{n-2} is given by $I_n = -\frac{1}{n} cosx\, sin^{n-1}x + \frac{n-1}{n} I_{n-2}$ 4

Conics Section (Start a new page)

1) Find the distance between the foci of the ellipse $2x^2 + 3y^2 = 6$ 2

2) Consider the Cartesian Equation $4x^2 - 9y^2 = 36$. Find
   (a) the eccentricity 1
   (b) the foci coordinates 1
   (c) the equations of the directrices 1

3) Consider the hyperbola H with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$
   (a) Show that the hyperbola H is satisfied by the parameters 1
   $x = 4sec\theta$ and $y = 3tan\theta$

   (b) Show that the equation of the tangent to the hyperbola H at the point $(4sec\theta, 3tan\theta)$ is $\frac{xsec\theta}{4} - \frac{ytan\theta}{3} = 1$. 3

4) $P\left(cp.\frac{c}{p}\right)$ and $Q\left(cq.\frac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$.
Tangents to the rectangular hyperbola at ***P*** and ***Q*** intersect at ***R (X,Y)***.

(a) Show that the tangent to the rectangular hyperbola at $(ct.\frac{c}{t})$ is given by $x + t^2y = 2ct$ **2**

(b) Show that $X = \frac{2cpq}{p+q}$, $Y = \frac{2c}{p+q}$ **3**

(c) If ***P*** and ***Q*** are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2,$ show that the equation of the locus of ***R*** is $y^2 + xy = 2c^2$ **4**

END OF EXAMINATION

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Extension two task 3 2015

(Solutions)

Multiple Choice

1/ $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

when $x = e^4$

$u = \ln e^4$

$u = 4$ (b)

when $x = e^3$

$u = 3$ (a)

$\therefore$ (B)

2/ $\left[\ln(1+e^x)\right]_0^1 = \ln(1+e) - \ln 2$

$= \ln\left(\frac{1+e}{2}\right)$

$\therefore$ (C)

3/ Asymptotes are $y = \pm\frac{bx}{a}$

$= \pm 2x$

$\therefore$ (D)

4/ Let $A = 1$, $C = 2$, $F = 2$

$x^2 + 2y^2 = 2$

$\frac{x^2}{2} + y^2 = 1$

$\therefore$ (A)

Integration Section

1/ $\int x(9-16x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{32}\int -32x(9-16x^2)^{-\frac{1}{2}}\,dx$

$= -\frac{1}{32} \times (9-16x^2)^{\frac{1}{2}} \times \frac{2}{1} + c$

$= -\frac{1}{16}\sqrt{9-16x^2} + c$

2/
$$\begin{array}{r} x - 1 \\ x+1 \enclose{longdiv}{x^2 + 0x + 0} \\ \underline{x^2 + x} \\ -x + 0 \\ \underline{-x - 1} \\ 1 \end{array}$$

$\therefore \int \frac{x^2}{x+1}\,dx = \int x - 1 + \frac{1}{x+1}\,dx$

$= \frac{x^2}{2} - x + \ln|x+1| + c$

3/ $u = x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$ $\quad v = e^x$

$\int_0^{\ln 3} x e^x\,dx = \left[x e^x\right]_0^{\ln 3} - \int_0^{\ln 3} e^x\,dx$

$= 3\ln 3 - \left[3 - 1\right]$

$= 3\ln 3 - 2$

4 (a) $\frac{2}{(x+1)(x^2+1)} = \frac{A(x^2+1)}{(x+1)(x^2+1)} + \frac{(Bx+C)(x+1)}{(x+1)(x^2+1)}$

$\therefore 2 \equiv A(x^2+1) + (Bx+C)(x+1)$

Let $x=-1$ | Let $x=0$ | Let $x=1$
$2=2A$ | $2=1+C$ | $2=2+2B+2$
$1=A$ | $C=1$ | $-2=2B$
| | $B=-1$

(b) $\int_0^1 \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= \left[\ln|x+1|\right]_0^1 - \frac{1}{2}\left[\ln|x^2+1|\right]_0^1 + \left[\tan^{-1}x\right]_0^1$

$= \ln 2 - 0 - \left[\frac{1}{2}\ln 2 - 0\right] + \left[\frac{\pi}{4} - 0\right]$

$= \ln 2 - \frac{1}{2}\ln 2 + \frac{\pi}{4}$

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$

(c) $t = \tan\frac{x}{2}$ $\quad dx = \frac{2\,dt}{1+t^2}$ $\quad$ when $x=0$, $t=0$; $x=\frac{\pi}{2}$, $t=1$

$\therefore = \int_0^1 \frac{\frac{2t}{1+t^2} \times \frac{2\,dt}{1+t^2}}{1+\frac{2t}{1+t^2} - \left[\frac{1-t^2}{1+t^2}\right]}$

$= \int_0^1 \frac{\frac{4t}{(1+t^2)^2}\,dt}{\frac{1+t^2+2t-1+t^2}{(1+t^2)}}$

$= \int_0^1 \frac{4t}{(1+t^2)} \times \frac{1}{2t^2+2t}\,dt$

$= \int_0^1 \frac{4t}{(1+t^2)} \times \frac{1}{2t(t+1)}\,dt$

$= \int_0^1 \frac{2}{(1+t^2)(t+1)}$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$ (From (b))

5) $I_n = \int \sin x \sin^{n-1} x \, dx$

$u = \sin^{n-1} x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$

$V = -\cos x$

$I_n = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$

$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - \sin^n x \, dx$

$I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$I_n + nI_n - I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$nI_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$\therefore I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n} I_{n-2}$

Conics Section

1/ $\frac{x^2}{3} + \frac{y^2}{2} = 1$

$2 = 3(1-e^2)$

$2 = 3 - 3e^2$

$3e^2 = 1$

$e = \frac{1}{\sqrt{3}}$

$\therefore S$ is $(1,0)$

S' is $(-1,0)$

$\therefore$ Distance between foci is 2 units

2/ $\frac{x^2}{9} - \frac{y^2}{4} = 1$

(a) $\frac{4}{9} = e^2 - 1$

$\frac{13}{9} = e^2$

$e = \frac{\sqrt{13}}{3}$

(b) foci are $(\pm ae, 0)$

$= (\pm\sqrt{13}, 0)$

(c) $x = \pm\frac{9}{\sqrt{13}}$

3/ (a) $\frac{16\sec^2\theta}{16} - \frac{9\tan^2\theta}{9} = LHS$

$1 + \tan^2\theta - \tan^2\theta = LHS$

$\therefore LHS = RHS$

(b) $\frac{2x}{16} - \frac{dy}{dx}\frac{2y}{9} = 0$

$\frac{2x}{16} \times \frac{9}{2y} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{9x}{16y}$ at $x = 4\sec\theta$ $y = 3\tan\theta$

$m_T = \frac{\overset{3}{\cancel{9}} \times \overset{1}{\cancel{4}}\sec\theta}{\underset{4}{\cancel{16}} \times \underset{1}{\cancel{3}}\tan\theta}$

$= \frac{3\sec\theta}{4\tan\theta}$

$$y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}(x - 4\sec\theta)$$

$$4y\tan\theta - 12\tan^2\theta = 3x\sec\theta - 12\sec^2\theta$$

$$12\sec^2\theta - 12\tan^2\theta = 3x\sec\theta - 4y\tan\theta$$

$$12(1+\tan^2\theta - \tan^2\theta) = 3x\sec\theta - 4y\tan\theta$$

$$12 = 3x\sec\theta - 4y\tan\theta$$

$$1 = \frac{x\sec\theta}{4} - \frac{y\tan\theta}{3}$$

(as required)

4/ $y = cx^{-1}$

$$\frac{dy}{dx} = \frac{-c^2}{x^2} \quad \text{at } x = ct$$

$$m_T = \frac{-c^2}{c^2t^2}$$

$$= \frac{-1}{t^2}$$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = -x + ct$$

$$t^2y + x = 2ct \quad \text{(as required)}$$

(b) Tangent at P $\quad p^2y + x = 2cp$

Tangent at Q $\quad q^2y + x = 2cq$

$$\therefore 2cp - p^2y = 2cq - q^2y$$

$$2cp - 2cq = p^2y - q^2y$$

$$2c(\cancel{p-q}) = y(\cancel{p-q})(p+q)$$

$$\frac{2c}{p+q} = y$$

$$\therefore Y = \frac{2c}{p+q}$$

$$x = 2cp - \frac{p^2(2c)}{p+q}$$

$$x = \frac{2cp(p+q) - 2cp^2}{p+q}$$

$$x = \frac{2cp^2 + 2cpq - 2cp^2}{p+q} \quad \therefore X = \frac{2cpq}{p+q}$$

(c) $p^2+q^2=2$

$$(p+q)^2-2pq=2$$

$$\boxed{(p+q)^2=2+2pq} \quad (1)$$

$$\frac{x}{y}=\frac{2\cancel{c}pq}{\cancel{(p+q)}}\times\frac{\cancel{(p+q)}}{2\cancel{c}}$$

$$\boxed{\frac{x}{y}=pq} \quad (2)$$

$$\boxed{p+q=\frac{2c}{y}} \quad (3)$$

Sub (3) & (2) into (1)

$$\frac{4c^2}{y^2}=2+2\frac{x}{y}$$

$$4c^2=2y^2+2xy$$

$$\therefore y^2+xy=2c^2$$