Name________________________ Teacher____________________

GOSFORD HIGH SCHOOL

2016

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 3

MATHEMATICS – EXTENSION 2

Duration- 60 minutes plus 5 minutes reading time

Integration Section	Only write on **one** side of the paper for your solutions. Staple this section together at the end of the exam.	/26
Conics Section	Start a **new page** for this question. Only write on one side of the paper. Staple this section together at the end of the exam	/18
TOTAL		/44

Integration Section

Marks

1) Find

$$\int \frac{e^x}{1+e^{2x}}\,dx \quad \text{using the substitution } u = e^x$$ **2**

$\tan^{-1} e^x$

2) Evaluate

$$\int_0^{\frac{\pi}{2}} x\cos x\ dx$$ **2**

$\frac{\pi}{2} - 1$ ✓

3) Evaluate

$$\int_3^4 \frac{x^2+x-4}{x-2}\,dx$$ **3**

$\frac{13}{2} + 2\ln 2$

4) (a) Find real numbers A,B and C such that

$$\frac{3x+7}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$ **3**

$a = 2$

$B = -1$

$C = -1$

(b) Hence evaluate

$$\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$$ **3**

$\ln 2$

5) By using the substitution $t = \tan\frac{x}{2}$

evaluate $$\int_0^{\frac{\pi}{3}} \frac{dx}{1+\cos x - \sin x}$$ **4**

6) (a) Simplify $sin(a+b)+sin(a-b)$ **1**

(b) Hence evaluate

$$\int_0^{\frac{\pi}{4}} sin5x\ cos3x\ dx$$ **2**

7)

Given $$I_n = \int_0^1 x^n e^{-x} dx$$

(a) Show that $\boldsymbol{I_n = -\frac{1}{e} + nI_{n-1}}$ **3**

(b) Hence find the exact value of **3**

$$I_n = \int_0^1 x^3 e^{-x} dx$$

End of Integration section.

Go on to Conics section

Start a new page for the Conics section

Conics Section

Marks

1) Find the equation of the locus of a point ***P(x,y)*** such that the distance from the point ***S(1,0)*** is <u>**half**</u> its distance from the line ***x=4.*** **2**

2) Find the coordinates of the foci and the equation of the directrices of the rectangular hyperbola $xy = 12$ **2**

3) (a) Show that if $y = px + q$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then $p^2a^2 - b^2 = q^2$ **3**

(b) Hence find the equations of the tangents from the point (1,3) to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$ **3**

4) An ellipse ***E*** can be described as the locus of a point moving such that the **sum** of its distances from two fixed points ***(foci)*** is constant.
Hint- Draw a diagram

(a) If the two fixed points are $\boldsymbol{S(4,0)}$ and $\boldsymbol{S'(-4,0)}$ and the sum of the distance of $P(x,y)$ from these points is **10 units**, show that the equation of ***E*** is given by $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **2**

(b) Verify that $x = 5cos\theta$ and $y = 3sin\theta$ are the parametric equations of ***E.*** **1**

(c) Find the equation of the normal to E at the point where $\theta = \frac{\pi}{6}$ **3**

(d) Determine the eccentricity of E and hence the equations of the directrices. **2**

End of Examination

Integration Solutions

1) $\int \frac{e^x}{1+(e^x)^2}\,dx$

Let $u = e^x$

$\frac{du}{dx} = e^x$

$\frac{du}{e^x} = dx$ (1)

$\int \frac{1}{1+u^2}\,du$

$= \tan^{-1} u + C$

$= \tan^{-1}(e^x) + C$ (1)

2) Let $u = x$ $\quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ $\quad v = \sin x$

$$\int_0^{\frac{\pi}{2}} x\cos x\,dx = \Big[x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx \quad (1)$$

$$= \left[\frac{\pi}{2}\times 1 - 0\right] + \Big[\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \left[0 - 1\right]$$

$$= \frac{\pi}{2} - 1 \quad (1)$$

3/

$$
\begin{array}{r}
x+3 \\
x-2 \enclose{longdiv}{x^2+x-4} \\
x^2-2x \\
\hline
3x-4 \\
3x-6 \\
\hline
2
\end{array}
$$

$$= \int_3^4 x+3+\frac{2}{x-2}\,dx \qquad (1)$$

$$= \left[\frac{x^2}{2}+3x+2\ln|x-2|\right]_3^4 \qquad (1)$$

$$= \frac{13}{2}+2\ln 2 \qquad (1)$$

4/ (a) $3x+7 = A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$

Let $x=-2$

$\therefore\ 1 = B(-1)(1) \quad \therefore\ B=-1$ (1)

Let $x=-1$

$\therefore\ 4 = A(1)(2) \quad \therefore\ A=2$ (1)

Let $x=-3$

$\therefore\ -2 = C(-2)(-1) \quad \therefore\ C=-1$ (1)

$$\therefore\ \frac{3x+7}{(x+1)(x+2)(x+3)} = \frac{2}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}$$

(b) $= \Big[2\ln(x+1)-\ln(x+2)-\ln(x+3)\Big]^1$ (1)

$= 2\ln 2-\ln 3+\ln 2-\ln 4+\ln 3^0$ (1)

$= 3\ln 2-2\ln 2$ (1)

$= \ln 2$

5) $t = \tan\frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$

when $x=0$, $t=0$ (1)

when $x=\frac{\pi}{3}$, $t=\frac{1}{\sqrt{3}}$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt \quad (1)$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,dt}{1+t^2+1-t^2-2t}$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{\cancel{2}}{\cancel{2}(1-t)}\,dt$$

$$= \Big[-\ln|1-t|\Big]_0^{\frac{1}{\sqrt{3}}} \quad (1)$$

$$= -\ln\left(1-\frac{1}{\sqrt{3}}\right) + \ln 1$$

$$= -\ln\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) \quad (1)$$

$$= \ln\left|\frac{\sqrt{3}}{\sqrt{3}-1}\right|$$

6/ (a) $\sin a \cos b + \cos a \sin b + \sin a \cos b - \cos a \sin b$

$= 2 \sin a \cos b$ (1)

(b) From (a)

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\therefore \int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx = \frac{1}{2}\int \sin 8x + \sin 2x \, dx$$ (1)

as $A = 5x$, $B = 3x$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 2\pi + \frac{1}{8} - \frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{8} - 0 + \frac{1}{8} + \frac{1}{2}\right]$$

$$= \frac{1}{4}$$ (1)

7/(a) $u = x^n$ $\qquad \frac{dV}{dx} = e^{-x}$

$\frac{du}{dx} = nx^{n-1}$ $\qquad V = -e^{-x}$ (1)

$$I_n = \left[-e^{-x} \cdot x^n\right]_0^1 + \int_0^1 nx^{n-1} \cdot e^{-x}\,dx \quad (1)$$

$$= \left[-e^{-1} \cdot 1 - 0\right] + nI_{n-1}$$

$$= -\frac{1}{e} + nI_{n-1} \quad (1)$$

(b) $I_3 = -\frac{1}{e} + 3I_2$

$I_2 = -\frac{1}{e} + 2I_1$

$I_1 = -\frac{1}{e} + I_0$ (1)

$$I_0 = \int_0^1 e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^1$$

$$= 1 - \frac{1}{e} \quad (1)$$

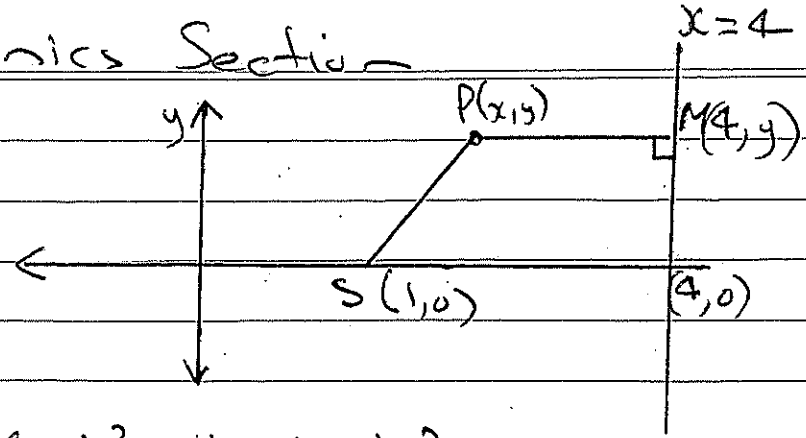
$\therefore I_1 = 1 - \frac{2}{e}$

$I_2 = 2 - \frac{5}{e}$

hence $I_3 = 6 - \frac{16}{e}$ (1)

Conics Section

1)



$$(PS)^2 = \frac{1}{4}(PM)^2$$

$$(x-1)^2 + y^2 = \frac{1}{4}(x-4)^2 \qquad (1)$$

$$x^2 - 2x + 1 + y^2 = \frac{1}{4}(x^2 - 8x + 16)$$

$$4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16$$

$$3x^2 + 4y^2 = 12 \qquad (1)$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

2) $xy = 12$ $\quad c^2 = 12$

$c = \sqrt{12}$ $\quad e = \sqrt{2}$

Foci $\pm(\sqrt{2}c, \sqrt{2}c) = \pm(\sqrt{24}, \sqrt{24})$ (1)

$= \pm(2\sqrt{6}, 2\sqrt{6})$

Directrices $x + y = \pm\sqrt{2c}$

$= \pm\sqrt{24}$ (1)

$= \pm 2\sqrt{6}$

3/(a) $\dfrac{x^2}{a^2} - \dfrac{(px+q)^2}{b^2} = 1$

$bx^2 - a^2(p^2x^2 + 2pqx + q^2) = a^2b^2$ (1)

$(b^2 - a^2p^2)x^2 - 2a^2pqx - (a^2q^2 + a^2b^2) = 0$

tangent $\therefore \Delta = 0$

$4a^4p^2q^2 - 4(b^2 - a^2p^2)x - a^2(q^2 + b^2) = 0$ (1)

Divide by $4a^2$

$a^2p^2q^2 + (b^2 - a^2p^2)(q^2 + b^2) = 0$

$\cancel{a^2p^2q^2} + b^2q^2 + b^4 - \cancel{a^2p^2q^2} - a^2p^2b^2 = 0$

$b^2(q^2 + b^2 - a^2p^2) = 0$

$q^2 + b^2 - a^2p^2 = 0$ as $b \neq 0$

$\therefore q^2 = a^2p^2 - b^2$ (1)

(b) $a^2 = 4$ $b^2 = 15$

tangent through $(1, 3)$

$\therefore 3 = p + q \Rightarrow p = 3 - q$ (1)

Using (a) $q^2 = 4p^2 - 15$

$q^2 = 4(3-q)^2 - 15$

$q^2 = 4(9 - 6q + q^2) - 15$

$0 = 3q^2 - 24q + 21$ (1)

$0 = q^2 - 8q + 7$

$0 = (q-7)(q-1)$

$\therefore q = 7$ and $q = 1$

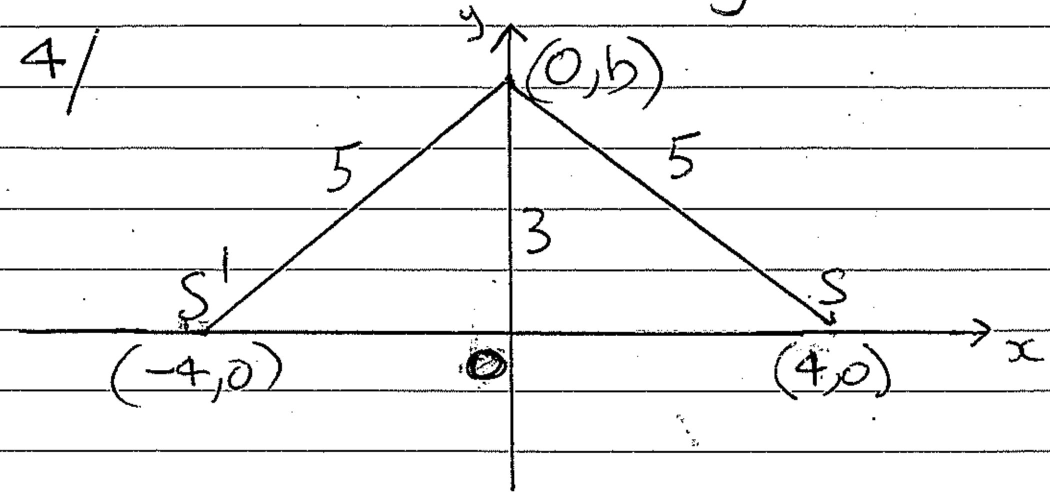
$p = -4$ $p = 2$ (1)

$\therefore$ tangents are $y = -4x + 7$

and

$y = 2x + 1$

4/



When P is at $(0,b)$, $b = 3$ (by Pythag)

When P is at $(a,0)$, $a = 5$ (2)

as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

E is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(ii) $x = 5\cos\theta$ $\quad \cos\theta = \frac{x}{5}$

$y = 3\sin\theta$ $\quad \sin\theta = \frac{y}{3}$ (1)

$\therefore \frac{x^2}{25} + \frac{y^2}{9} = \cos^2\theta + \sin^2\theta$

$= 1$

(c) $\frac{dx}{d\theta} = -5\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{3\cos\theta}{-5\sin\theta}$ when $\theta = \frac{\pi}{6}$

$x = \frac{5\sqrt{3}}{2}$

$m_T = \frac{3 \times \frac{\sqrt{3}}{2}}{-5 \times \frac{1}{2}}$ $\quad y = \frac{3}{2}$ (1)

$= \frac{3\sqrt{3}}{2} \times \frac{2}{-5}$

$= -\frac{3\sqrt{3}}{5}$

$\therefore m_n = \frac{5}{3\sqrt{3}}$ (1)

$y - \frac{3}{2} = \frac{5}{3\sqrt{3}}\left(x - \frac{5\sqrt{3}}{2}\right)$

$2y - 3 = \frac{10}{3\sqrt{3}}\left(x - \frac{5\sqrt{3}}{2}\right)$ (1)

$6\sqrt{3}y - 9\sqrt{3} = 10x - 25\sqrt{3}$

$= 10x - 6\sqrt{3}y - 16\sqrt{3}$

$= 5x - 3\sqrt{3}y - 8\sqrt{3}$

(d) $e = \frac{4}{5}$ (1)

directrices are $x = \pm\frac{25}{4}$ (1)