JAMES RUSE AGRICULTURAL HIGH SCHOOL
YEAR 12 MATHEMATICS EXTENSION II ASSESSMENT
TERM II 2004

QUESTION 1 **Marks**

The points $P(5p,\frac{5}{p})$ and $Q(5q,\frac{5}{q})$ lie on a rectangular hyperbola .

(a) Graph the rectangular hyperbola clearly showing location of foci, asymptotes and directrices .Write the corresponding equations on the asymptotes and directrices. 3

(b) Derive the equation of the tangent at P, hence state the equation of the tangent at Q. 3

(c) The tangents from the points P and Q intersect at angle of 45^0 at the point T.

(i) Find the coordinates of the intersection point T. 2

(ii) Show $\left(p^2q^2+1\right)^2=\left(p^2-q^2\right)^2$ 2

(iii) Find expressions in terms of x and y :

(α) $p+q$ 1

(β) pq 1

(γ) $(p-q)^2$ 1

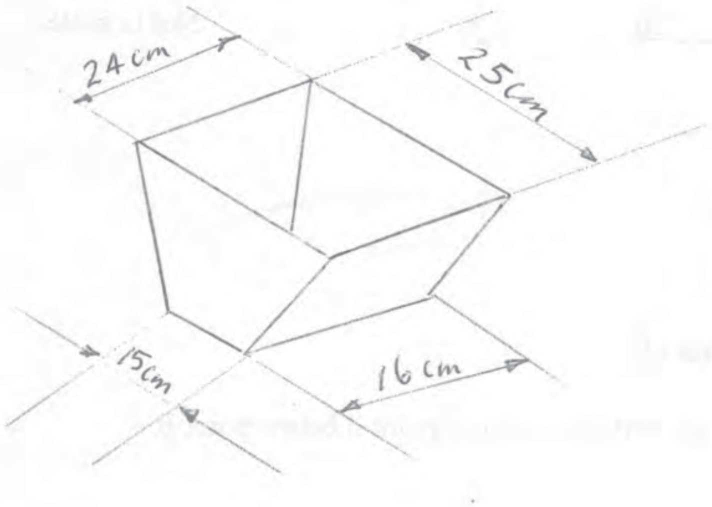
Hence show that equation of the locus of T is given by : 2

$$x^4+y^4+2x^2y^2+400xy-10000=0$$

(Neglect restrictions)

QUESTION 2 (START A NEW PAGE)

(a) Find the volume of a symmetrical tray of height 5 cm if the cross sections parallel to the base are rectangles. 5



Not to scale.

(b) The region bounded by the x axis , the curve $y=xe^x$ and the line x=1 is rotated about the y axis. 5
Using the method of cylindrical shells find the volume of revolution.

(c) The region bounded by the coordinate axes and the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=1$ is rotated about the line $y=-1$. Find the volume of revolution. 5

QUESTION 3 (START A NEW PAGE)

(a)(i) A parachutist of mass 90 kg jumps out of an aeroplane at a height of 100 m. The acceleration due to gravity is $10\ m/s^2$.
Derive the vertical velocity functions *v m/s* in terms of displacement *x* for the parachutist, if the initial velocity is *u m/s* and there is:

(α) **no** resistance force during free fall. 2

(β) a resistance force of $0.27v^2$ Newtons when the parachute is opened. 3

(ii) The parachutist jumps from rest, and opens his parachute 60 m from the ground.

(α) Find the vertical velocity when the parachute is opened. 1

(β) Find the vertical velocity (1 decimal place) on landing. 2

(γ) What percentage is the landing velocity compared to the terminal velocity ? 2

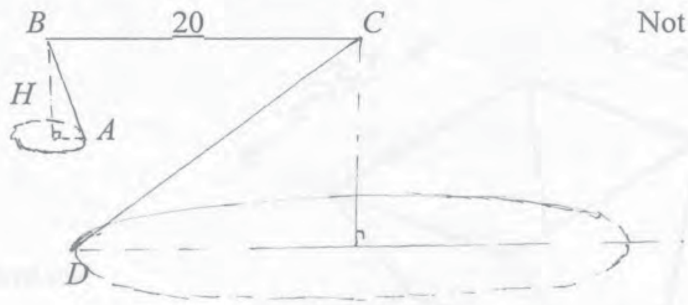
(b) A particle moves in a straight line. 5
The displacement function *x* metres in terms of velocity *v* m/s is given by :

$$x = 5v - 2\ln(v+1)$$

Find the time *t* seconds of the particle as a function of it's velocity *v,* if the particle is initially moving at 1 *m/s.*

QUESTION 4 (START A NEW PAGE)

A light inextensible string *AD* passes over frictionless supports at *B* and *C* 20 cm apart. At the ends of the string a mass *M* is attached to the end *A* 10 cm from *B*, and a mass *m* is attached to the end *D* 25 cm from *C*.
Both mass *M* and mass *m* rotate in the horizontal plane with uniform angular velocities of ω_1 and ω_2 respectively.
The acceleration due to gravity *g* is $9.8\ m/s^2$.



Not to scale.

(a) Show a force diagram for mass *M*. 1

(b) Prove $H\omega_1^2 = g$ where *H* is the vertical height of point *A* below point *B*. 4

(c) Prove $m = \dfrac{2M\omega_1^2}{5\omega_2^2}$ 2

(d) If $M = 200$ grams, $\omega_1 = 15$ rad / s then :

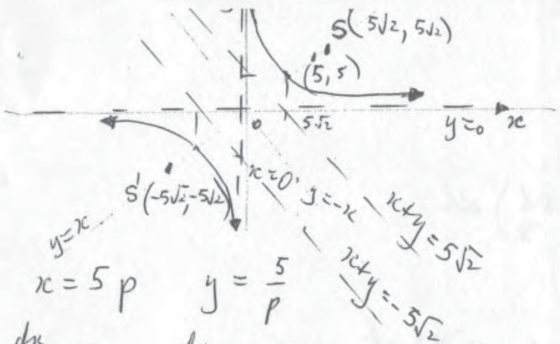
(i) find the vertical height H (nearest mm). 1

(ii) find the tension T Newtons in the string. 1

(iii) find the angular velocity ω_2 and corresponding mass *m* when the rotation of mass *m* comes into contact with the motion of mass *M*. 6

END OF EXAM

1 (a)



Foci $S(5\sqrt{2}, 5\sqrt{2})$
$S'(-5\sqrt{2}, -5\sqrt{2})$

Directrices $x+y=5\sqrt{2}$
$x+y=-5\sqrt{2}$

3.

(b) $x = 5p \quad y = \frac{5}{p}$

$\frac{dx}{dp} = 5 \quad \frac{dy}{dp} = \frac{-5}{p^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dp}}{\frac{dx}{dp}} = \frac{\frac{-5}{p^2}}{5} = -\frac{1}{p^2}$$

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Term 2

Ext 2

Eqn tangent

$y - y_1 = m(x - x_1)$

$y - \frac{5}{p} = -\frac{1}{p^2}(x - 5p)$

$p^2 y - 5p = -x + 5p$

$x + p^2 y = 10p$ 2

At Q $x + q^2 y = 10q$ 1

c (i) $y[p^2 - q^2] = 10[p - q]$

$y = \frac{10(p-q)}{(p-q)(p+q)}$

$= \frac{10}{p+q} \quad p \neq q$

$x = 10p - \frac{10p^2}{p+q}$

$= \frac{10pq}{p+q}$

(1)

$\Rightarrow T\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$ 2

(ii) If intersect at 45°

$\tan\theta = \left|\frac{m_1 - m_2}{m_1 + m_2}\right|$

$\tan 45^\circ = \left|\frac{\frac{-1}{q^2} + \frac{1}{p^2}}{1 + \frac{-1}{q^2}\cdot\frac{-1}{p^2}}\right|$ 1

$1 = \left|\frac{p^2 - q^2}{1 + p^2 q^2}\right|$

Or $1 = \frac{(p^2 - q^2)^2}{(1 + p^2 q^2)^2}$ 1

OR $(p^2 q^2 + 1)^2 = (p^2 - q^2)^2$

(iii) $y = \frac{10}{p+q} \Rightarrow p+q = \frac{10}{y}$ 1

$\frac{x}{y} = \frac{10pq}{p+q} \div \frac{10}{p+q}$

$= pq$

$\Rightarrow pq = \frac{x}{y}$ 1

$(p-q)^2 = (p+q)^2 - 4pq$

$= \left(\frac{10}{y}\right)^2 - \frac{4x}{y}$ 1

$= \frac{100 - 4xy}{y^2}$

(2)

$\therefore \quad (p^2q^2+1)^2 = (p^2-q^2)^2$

$(p^2q^2+1)^2 = (p-q)^2(p+q)^2$ 1

$\left(\frac{x^2}{y^2}+1\right)^2 = \frac{100-4xy}{y^2} \cdot \left(\frac{10}{y}\right)^2$

$(x^2+y^2)^2 = 10000 - 400xy$ 1

$x^4+y^4+2x^2y^2+400xy-10000=0$

3

Q2

(a) $l = 15+2h$ 1

$b = 16 + \frac{8h}{5}$ 1

$V = \int_0^5 (15+2h)\left(16+\frac{8h}{5}\right) dh$ 1

$= \int_0^5 \left(240 + 24h + 32h + \frac{16h^2}{5}\right) dh$

$= \left[240h + 28h^2 + \frac{16}{15}h^3\right]_0^5$ 1

$= 1200 + 700 + 133\frac{1}{3}$

$V = 2033\frac{1}{3}\ cm^3$ 1

b) $\delta V = 2\pi x \cdot xe^x \,\delta x$

$\therefore V = 2\pi \int_0^1 x^2 e^x \, dx$ 1

$= 2\pi \left[\left[x^2e^x\right]_0^1 - \int_0^1 2xe^x \, dx \right]$ 1

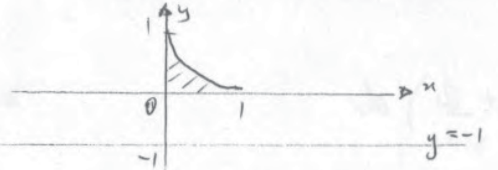
$= 2\pi \left[e - 0 - 2\left(xe^x - e^x\right]_0^1 \right]$ 1

$= 2\pi \left[e - 2\left(e - e - (-1)\right) \right]$ 1

$V = 2\pi\left[e-2\right] \text{ unit}^3$ 1

4

(c)



$y = (1-x^{1/2})^2$

$y = 1 - 2x^{1/2} + x$

$\delta V = \pi\left[(y+1)^2 - 1^2\right]\delta x$

$= \pi\left[y(y+2)\right]\delta x$

$V = \pi\int_0^1 (1-2x^{1/2}+x)(1-2x^{1/2}+x+2)\,dx$

$= \pi\int_0^1 (1-2x^{1/2}+x)(3-2x^{1/2}+x)\,dx$

$= \pi\int_0^1 (3-2x^{1/2}+x-6x^{1/2}+4x-2x^{3/2}+3x-2x^{3/2}+x^2)\,dx$

$= \pi\int_0^1 (3-8x^{1/2}+8x-4x^{3/2}+x^2)\,dx$

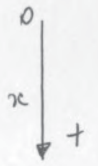
$= \pi\left[3x - \frac{16}{3}x^{3/2} + 4x^2 - \frac{8}{5}x^{5/2} + \frac{x^3}{3}\right]_0^1$

$= \pi\left[3 - 5\tfrac{1}{3} + 4 - 1\tfrac{3}{5} + \tfrac{1}{3}\right]$

$V = \frac{2\pi}{5}$ unit^3

Qu 3

a (i)



(α) $m\ddot{x} = mg$

$\ddot{x} = 10$

$\int_u^v v\,dv = \int_0^x 10\,dx$ 1

$\frac{v^2 - u^2}{2} = 10x$ 1

$v = \sqrt{u^2 + 20x} \quad v > 0$

(β) $m\ddot{x} = mg - 0{\cdot}27v^2$

$\ddot{x} = 10 - \frac{0{\cdot}27}{90}v^2$ 1

$\ddot{x} = 10 - 0{\cdot}003v^2$

$\int_u^v \frac{v\,dv}{10 - 0{\cdot}003v^2} = \int_0^x dx$

$\frac{-1}{0{\cdot}003}\cdot\frac{1}{2}\ln\left(\frac{10-0{\cdot}003v^2}{10-0{\cdot}003u^2}\right) = x$ 1

$\ln\left(\frac{10-0{\cdot}003v^2}{10-0{\cdot}003u^2}\right) = -0{\cdot}006x$

$\frac{10-0{\cdot}003v^2}{10-0{\cdot}003u^2} = e^{-0{\cdot}006x}$

$10 - 0{\cdot}003v^2 = (10 - 0{\cdot}003u^2)e^{-0{\cdot}006x}$ 1

$v = \sqrt{\frac{10 - (10-0{\cdot}003u^2)e^{-0{\cdot}006x}}{0{\cdot}003}} \quad v > 0.$

(ii) (α) $x = 40m$

$v = \sqrt{0^2 + 20 \times 40}$

$= 20\sqrt{2}$ or 28.28 m/s 1

(β)

$v = \sqrt{\dfrac{10 - (10 - 0.003 \times 800)\, e^{-0.003 \times 60}}{0.003}}$

$v = 39.6$ m/s 2

(γ) $v_T = \sqrt{\dfrac{10}{0.003}}$

$= 57.7$ m/s

$\% \; \dfrac{v_T}{v} = \dfrac{39.6}{57.7} \times \dfrac{100}{1}\%$

$= 68.6\%$

(c) $x = 5v - 2\ln(v+1)$

$\dfrac{dx}{dv} = 5 - \dfrac{2}{v+1}$

$= \dfrac{5v+3}{v+1}$ 1

$\ddot{x} = v\dfrac{dv}{dx}$

$\dfrac{dv}{dt} = v.\left(\dfrac{v+1}{5v+3}\right)$ $t=0$ $v=0$ 1

$\therefore \displaystyle\int_1^v \frac{5v+3}{v(v+1)}\,dv = \int_0^t dt$

But $\dfrac{5v+3}{v(v+1)} = \dfrac{a}{v} + \dfrac{b}{v+1} \Longrightarrow 5v+3 = a(v+1) + bv$

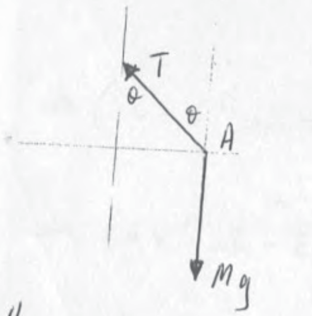
$a = 3$ $b = +2$

$t = \displaystyle\int_1^v \left(\frac{3}{v} + \frac{2}{v+1}\right) dv$ 2

$= \left[3\ln v + 2\ln(v+1)\right]_1^v$

$t = 3\ln v + 2\ln\left(\dfrac{v+1}{2}\right)$ 1

Qu 4



AB = 10 cm

Forces vertically

$T\cos\theta = Mg$ 1

Forces horizontally 1

$T\sin\theta = Mr\omega_1^2$ And $r = 0.1\sin\theta$

$= M.(0.1\sin\theta)\,\omega_1^2$

$T = 0.1\,M\omega_1^2$ 1

$\cos\theta = \dfrac{Mg}{0.1\,M\omega_1^2}$

$\cos\theta = \dfrac{g}{0.1\,\omega_1^2}$ 1

But $H = 0.1\cos\theta$

$= 0.1 \cdot \dfrac{g}{0.1\,\omega_1^2}$

$\Rightarrow H\omega_1^2 = g$.

(c) From (f) $T = 0 \cdot 025\, m w_2^2$ for mass m. 1

$\therefore \quad 0.25\, m w_2^2 = 0.1\, M w_1^2$ 1

$m = \dfrac{0.1\, M w_1^2}{0.25\, w_2^2}$

$= \dfrac{2\, M w_1^2}{5\, w_2^2}$

(d) (i) $H = \dfrac{g}{w_1^2}$

$= \dfrac{9.8}{15^2}$

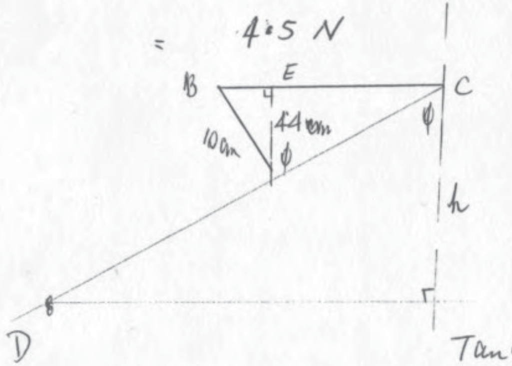
$H = 44$ mm 1

(ii) $T = 0.1\, M w_1^2$

$= 0.1 \times 0.2 \times 15^2$ 1

$= 4.5$ N

(iii)



$BE = \sqrt{10^2 - 4.4^2}$

$BE = 8.98$ cm

$\therefore CE = 20 - 8.98$

$CE = 11.02$ cm

$\tan\phi = \dfrac{11.02}{4.4}$

$\phi = 68^\circ 14'$

$h = 25 \cos 68^\circ 14'$ 2

$= 9.27$ cm.

$\therefore h w_2^2 = g$

$w_2 = \sqrt{\dfrac{10}{0.0927}}$

$= 10 \cdot 38$ rad/s

2

For mass from (c) $m = \dfrac{2 \times 0.2 \times 15^2}{5 \times 10.38^2}$

$= 1.67$ gm

2