EXT 2 TERM 2 2005

QUESTION 1 (15 Marks) **Marks**

(a) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the $x-$ axis and the line $x = \frac{\pi}{2}$. Cross-sections of this solid perpendicular to the $x-$ axis are equilateral triangles with one side lying in the base of the solid. Find the exact volume of the solid. 6

(b) A particle of mass m is projected upwards in a resistive medium where the force against the motion is inversely proportional to v, where v is the velocity of the particle, ie; $m\ddot{x} = -mg - \frac{mK}{v}$, where K is a constant and g the acceleration due to gravity. 9

(i) If the initial velocity of projection is U m/s, show that the time taken by the particle as a function of its velocity is given by an equation of the form $t = A + B \ \ln C$, and find expressions for A, B and C.

(ii) Derive an expression for the time taken to reach its maximum height in this medium.

QUESTION 2 (15 Marks)

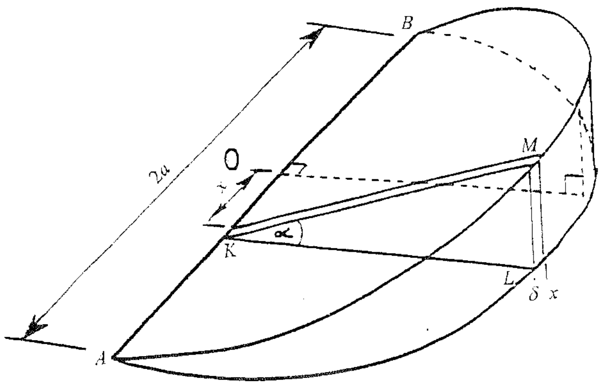
(a) A hole of diameter a centimetres is bored through the centre of a solid sphere of diameter $2a$ centimetres. Use the "method of cylindrical shells" to find the exact volume of the remaining solid. 6

(b) The solid wedge as shown was made by slicing a right cylinder of radius a at an angle α through diameter AB of its base. A triangular slice of thickness δx perpendicular to the base and line AB is positioned at distance x from the centre o as in the diagram. 9

(i) Show that ML, the height of the triangular slice is $\sqrt{a^2 - x^2} \ \tan\alpha$.

(ii) Deduce a formula for the volume of the wedge.

(iii) Given that $\alpha = \frac{2\pi}{n}$ and tan α decreases whilst n increases, find the volume of n identical wedges with a common diameter AB.



QUESTION 3 (15 Marks)

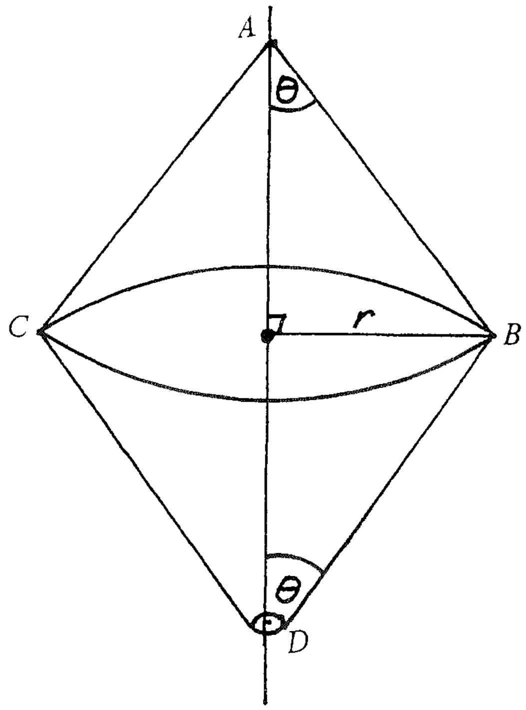
(a) The acceleration of a body of unit mass moving towards earth under gravitational attraction varies inversely as the square of its distance from the centre of the earth, **5**

ie; $\ddot{x} = \dfrac{-k}{x^2}$ (k a constant).If the body starts from rest at a distance a from the centre of the earth, show that its speed at a distance x from the centre of the earth is $v = \sqrt{\dfrac{2k(a-x)}{ax}}$.

(b) Two equal masses are connected to the ends of two rods AB and AC (in the same plane) of equal length which are hinged together at the point A to a vertical shaft. Two supporting rods DB and DC are also hinged together to a ring D which can slide up and down the shaft. The rods $AB = AC = DB = DC = L$, the masses at B and C are M kg and the ring has mass m kg. **10**

(i) Copy the diagram onto your answer sheet showing all the acting forces.

(ii) Show that the semi-vertical angle θ when rotating at a speed of ω radians/second is given by $\sec\theta = \dfrac{ML\omega^2}{(M+m)g}$.



QUESTION 4 (15 Marks)

(a) A square $ABCD$ has sides of length 1 unit. The asymptotes of hyperbola $xy = c^2$ lie on sides AB and AD of the square, and the point $C(1,1)$ is one of the foci. Show that the hyperbola bisects the other two sides. **3**

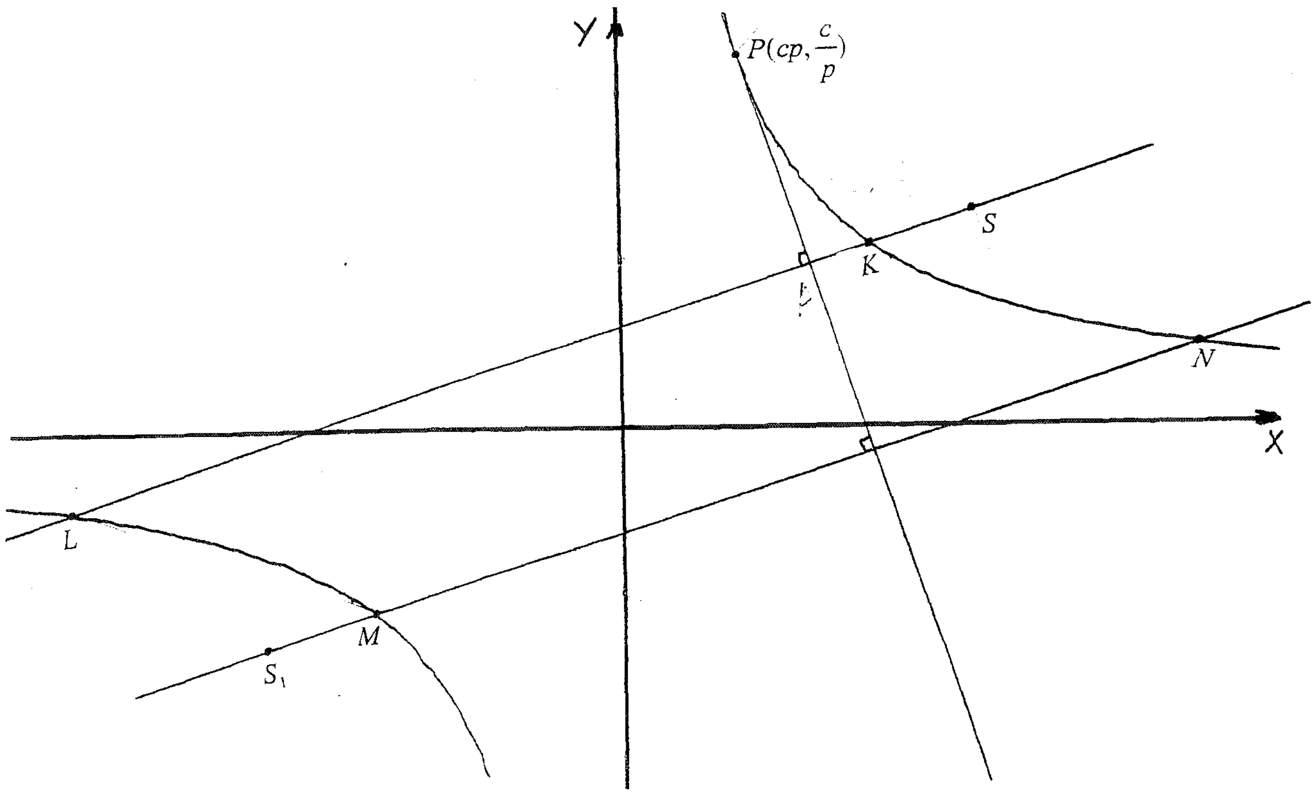
(b) The perpendiculars drawn from the foci S and S_1 of hyperbola $xy = c^2$ to the tangent at a point $P(cp, \frac{c}{p})$ meet the curve at K, L, M, N as shown in the diagram. **12**

(i) Find the equation of the line SKL.

(ii) Find, as a function of p, an expression for the parameter "k" at the point K.

(iii) Hence, or otherwise, prove that $KLMN$ is a parallelogram.

(iv) Show that the sides KN and LM of the parallelogram are perpendicular to the diameter through P.



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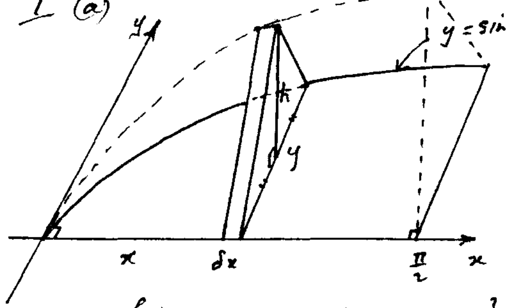
SOLUTIONS TO TERM 2 ASSESSMENT EXTENSION II 2005

1 (a)



Area of triangular section is:

$A = \frac{1}{2} y^2 \sin 60^\circ$

$= \frac{1}{2} \cdot y^2 \cdot \frac{\sqrt{3}}{2}$

$\therefore A = \frac{y^2\sqrt{3}}{4}$

$\therefore dV = \frac{\sqrt{3}}{4} y^2 \, dx$

Since $y = \sin x$ $\therefore y^2 = \sin^2 x$

$\therefore V = \frac{\sqrt{3}}{4}\int_0^{\frac{\pi}{2}} y^2 \, dx = \frac{\sqrt{3}}{4}\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

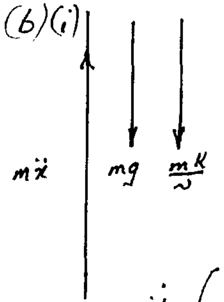
$= \frac{\sqrt{3}}{8}\int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx$

$= \frac{\sqrt{3}}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{\sqrt{3}}{8}\left(\frac{\pi}{2} - 0\right)$

$\therefore V = \frac{\pi\sqrt{3}}{16} \; u^3$

(b)(i)



$m\ddot{x} = -mg - \frac{mK}{v}$

$\therefore \ddot{x} = -g - \frac{K}{v}$

$\therefore \frac{dv}{dt} = -\left(\frac{gv + K}{v}\right)$

$\therefore \frac{v \, dv}{gv + K} = -dt$

$\therefore \int \frac{\frac{1}{g}(gv+K) - \frac{K}{g}}{gv+K} \, dv = -\int dt$ since $\frac{v}{gv+K} = \frac{\frac{1}{g}(gv+K) - \frac{K}{g}}{gv+K}$

$\therefore -\frac{1}{g}\int dv + \frac{K}{g}\int \frac{dv}{gv+K} = \int dt$

$\therefore -\frac{v}{g} + \frac{K}{g^2}\ln(gv+K) = t + c$

When $t = 0$, $v = U$

$\therefore \frac{K}{g^2}\ln(gU+K) - \frac{U}{g} = c$

$\therefore -\frac{v}{g} + \frac{K}{g^2}\ln(gv+K) = t + \frac{K}{g^2}\ln(gU+K) - \frac{U}{g}$

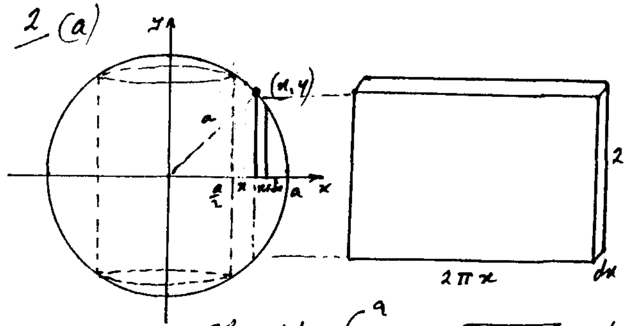
$\therefore t = \left(\frac{U-v}{g}\right) + \frac{K}{g^2}\ln\left(\frac{gv+K}{gU+K}\right)$

$\therefore A = \frac{U-v}{g}$, $B = \frac{K}{g^2}$, $C = \frac{gv+K}{gU+K}$

(ii) Maximum height occurs when $v = 0$

$\therefore t = \frac{U}{g} + \frac{K}{g^2}\ln\left(\frac{K}{gU+K}\right)$

2 (a)



Let the volume of a cylindrical shell be:

$dV = 2\pi x \cdot 2y \, dx$

Since $y = \sqrt{a^2 - x^2}$

$\therefore dV = 4\pi x\sqrt{a^2 - x^2} \, dx$

Now $V = \int_{\frac{a}{2}}^{a} 4\pi x\sqrt{a^2 - x^2} \, dx$

Let $u = a^2 - x^2$

$\therefore du = -2x \, dx$

When $x = \frac{a}{2}$ $u = \frac{3a^2}{4}$

when $x = a$ $u = 0$

and $x = \sqrt{a^2 - u^2}$

$= -2\pi\int_{\frac{3a^2}{4}}^{0} u^{\frac{1}{2}} \, du$

$= 2\pi\int_0^{\frac{3a^2}{4}} u^{\frac{1}{2}} \, du$

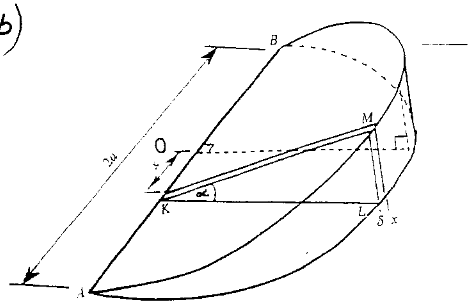
$= 2\pi\left[\frac{2}{3} u^{\frac{3}{2}}\right]_0^{\frac{3a^2}{4}}$

$= \frac{4\pi}{3}\left[\left(\frac{3}{4}a^2\right)^{\frac{3}{2}} - 0\right]$

$= \frac{4\pi}{3} \cdot \frac{3\sqrt{3}}{8} a^3$

$\therefore V = \frac{\sqrt{3}}{2}\pi a^3 \; u^3$

2(b)



(i) From the diagram

$KL^2 = OL^2 - OK^2$ by Pythagoras.

$\therefore KL^2 = a^2 - x^2$

$\therefore KL = \sqrt{a^2 - x^2}$

Now $\tan\alpha = \frac{ML}{KL}$

$\therefore ML = KL \tan\alpha$

$\therefore ML = \sqrt{a^2 - x^2}\,\tan\alpha$

(ii) Area of ΔKLM is $\frac{1}{2} \cdot KL \cdot LM$

$\therefore A = \frac{1}{2} \cdot \sqrt{a^2 - x^2} \cdot \sqrt{a^2 - x^2}\,\tan\alpha$

$\therefore A = \frac{1}{2}(a^2 - x^2)\tan\alpha$

$\therefore \delta V = \frac{1}{2}(a^2 - x^2)\tan\alpha\,\delta x$

$$\therefore V = \frac{\tan\alpha}{2} \times 2\int_0^a (a^2 - x^2)\,dx$$

$$= \tan\alpha\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \tan\alpha\left(a^3 - \frac{a^3}{3}\right)$$

$$\therefore V = \frac{2}{3}a^3\tan\alpha$$

(iii) As $\tan\alpha$ decreases,

$\therefore \tan\alpha \to \alpha$

$\therefore V \to \frac{2a^3}{3}\alpha$

For n identical wedges we have:

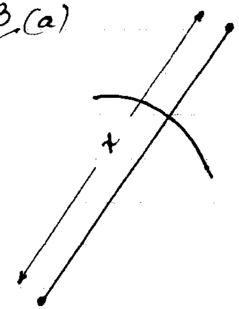
$$V_n = n \cdot \frac{2a^3}{3}\alpha$$

For $\alpha = \frac{2\pi}{n}$

$$\therefore V_n = \cancel{n} \cdot \frac{2a^3}{3} \cdot \frac{2\pi}{\cancel{n}}$$

$$\therefore V_n = \frac{4}{3}\pi a^3$$

3(a)



$\ddot{x} = -\frac{k}{x^2}$ $\therefore v\frac{dv}{dx} = -\frac{k}{x^2}$

$\therefore v\,dv = -\frac{k}{x^2}\,dx$

$\therefore \int v\,dv = -k\int \frac{dx}{x^2}$

$\therefore \frac{1}{2}v^2 = \frac{k}{x} + c$

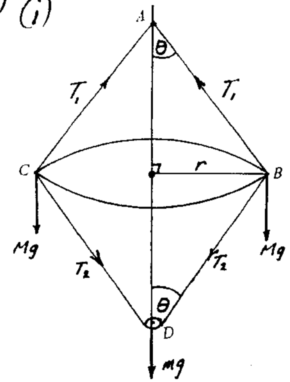
when $x = a$, $v = 0$ $\therefore c = -\frac{k}{a}$

$\therefore \frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{a}$

$\therefore v^2 = 2k\left(\frac{1}{x} - \frac{1}{a}\right) = 2k\left(\frac{a-x}{ax}\right)$

$\therefore v = \sqrt{\frac{2k(a-x)}{ax}}$ for $v > 0$

(b) (i)



(ii) Resolving forces at B:

Vertically: $(T_1 - T_2)\cos\theta = Mg$ — (1)

Horizontally: $(T_1 + T_2)\sin\theta = Mr\omega^2$ — (2)

Forces at D: $2T_2\cos\theta = mg$ — (3)

Now $T_2 = \frac{mg}{2}\sec\theta$ from (3)

Sub. into (1): $\left(T_1 - \frac{mg}{2}\sec\theta\right)\cos\theta = Mg$

$\therefore T_1 = Mg\sec\theta + \frac{mg}{2}\sec\theta$

$= \left(Mg + \frac{mg}{2}\right)\sec\theta$ — (4)

Sub. (4) into (2): $\left[\left(Mg + \frac{mg}{2}\right)\sec\theta + \left(\frac{mg}{2}\sec\theta\right)\right]\sin\theta = Mr\omega^2$

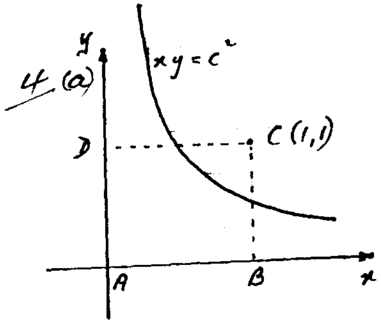
Since $\sin\theta = \frac{L}{r}$ $\therefore r = L\sin\theta$

$\therefore \left[\left(Mg + \frac{mg}{2}\right)\sec\theta + \frac{mg}{2}\sec\theta\right]\cancel{\sin\theta} = M\omega^2 L\cancel{\sin\theta}$

$\therefore (Mg + mg)\sec\theta = M\omega^2 L$

$$\therefore \sec\theta = \frac{M\omega^2 L}{(M+m)g}$$

4 (a)



Now $C(1,1) \equiv C(c\sqrt{2}, c\sqrt{2})$

$\therefore x = y = c\sqrt{2} \qquad \therefore c\sqrt{2} = 1$

$\therefore c^2 = \frac{1}{2}$

Since $xy = c^2$

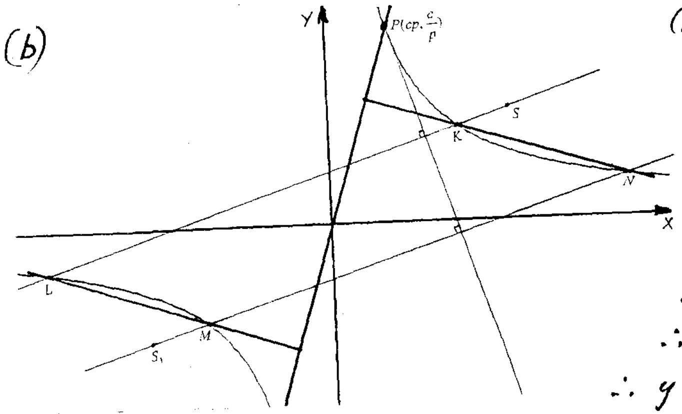
$\therefore xy = \frac{1}{2}$

When $x = 1$, $y = \frac{1}{2}$

Why $y = 1$, $x = \frac{1}{2}$

$\therefore (1, \frac{1}{2})$ is mid-point of BC, and $(\frac{1}{2}, 1)$ mid-point of CD.

(b)



(i) Let equation be

$y - y_1 = m(x - x_1)$

$\therefore y - c\sqrt{2} = m(x - c\sqrt{2})$ at P

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

$= -\frac{1}{p^2}$ at $x = cp$

$\therefore m = p^2$

$\therefore y - c\sqrt{2} = p^2(x - c\sqrt{2})$

$\therefore y - c\sqrt{2} = p^2 x - p^2 c\sqrt{2}$

$\therefore y - p^2 x = c\sqrt{2}\,(1 - p^2)$ —— (1)

(ii) Let $K(x,y) \equiv K(ck, \frac{c}{k})$

Since K lies on (1)

$\therefore \frac{c}{k} - p^2 ck = c\sqrt{2}(1 - p^2)$

$\therefore 1 - p^2 k^2 = \sqrt{2}\,k - \sqrt{2}\,p^2 k$

$\therefore p^2 k^2 + \sqrt{2}\,k - \sqrt{2}\,p^2 k - 1 = 0$

$\therefore p^2 k^2 + \sqrt{2}(1 - p^2)k - 1 = 0$

$\therefore k = \dfrac{\sqrt{2}(p^2 - 1) \pm \sqrt{[\sqrt{2}(1-p^2)]^2 + 4p^2}}{2p^2}$

$= \dfrac{\sqrt{2}(p^2 - 1) \pm \sqrt{2(1 + p^4)}}{2p^2}$

$\therefore k = \dfrac{\sqrt{2}\left[(p^2 - 1) \pm \sqrt{1 + p^4}\right]}{2p^2}$

$\therefore k = \dfrac{\sqrt{2}\left[(p^2 - 1) + \sqrt{1 + p^4}\right]}{2p^2}$ for k in first quadrant.

(iii) Since L lies on SKL, then "l" $= \dfrac{\sqrt{2}\left[(p^2 - 1) - \sqrt{1 + p^4}\right]}{2p^2}$

Similarly, for M and N which lie on the line through $S_1(-c\sqrt{2}, -c\sqrt{2})$, the parameters m and n are $m = \dfrac{-\sqrt{2}\left[(p^2 - 1) + \sqrt{1 + p^4}\right]}{2p^2}$ and $n = \dfrac{-\sqrt{2}\left[(p^2 - 1) - \sqrt{1 + p^4}\right]}{2p^2}$

$\therefore m = -k$ and $l = -n$ ie; diagonals of $KLMN$ bisect each other at origin. $\therefore KLMN$ is a parallelogram.

(iv) The gradient of diameter OP is $m_{OP} = \dfrac{(\frac{c}{p})}{cp} = \dfrac{1}{p^2}$

The gradient of KN is :

$m_{KN} = \dfrac{\left(\frac{c}{k} - \frac{c}{n}\right)}{(ck - cn)}$ for $K(ck, \frac{c}{k})$ and $N(cn, \frac{c}{n})$

$= \dfrac{\cancel{c}(n - k)}{\cancel{c}(k - n)\,kn}$

$= -\dfrac{1}{kn}$

$= -1 \div \left\{ \dfrac{\left[\sqrt{2}(p^2 - 1) + \sqrt{1 + p^4}\right]}{2p^2} \times \dfrac{\left[-\sqrt{2}(p^2 + 1) - \sqrt{1 + p^4}\right]}{2p^2} \right\}$

$= -1 \div \left\{ \dfrac{2}{4p^4}\left[\sqrt{1 + p^4} + (p^2 - 1)\right]\left[\sqrt{1 + p^4} - (p^2 - 1)\right] \right\}$

$= -1 \div \left\{ \dfrac{1}{2p^4}\left[(1 + p^4) - (p^4 - 2p^2 + 1)\right] \right\}$

$= -1 \div \left[\dfrac{1}{2p^4}\left(\cancel{1} + \cancel{p^4} - \cancel{p^4} + 2p^2 - \cancel{1}\right) \right]$

$= -1 \div \dfrac{\cancel{2}p^2}{\cancel{2}p^4}$

$= -p^2$

Now $m_{OP} \times m_{KN} = \dfrac{1}{p^2} \times -p^2$

$= -1$

$\therefore KN \perp OP$

Since $KN \parallel LM$ (opposite sides of parm $KLMN$ parallel)

$\therefore LM \perp OP$