JRAHS Year 12 Extension 2 Term 2 Assessment 2011

QUESTION 1 (15 Marks) **Marks**

(a) A rectangular hyperbola has equation $xy = a^2$

(i) Find the equation of the tangent at the point $P\left(at, \frac{a}{t}\right)$ on the hyperbola. **2**

(ii) The tangent at P meets the *x* and *y* axes at L and M respectively. O is the origin and POQ is a diameter. The line MQ meets the *x*-axis at T. **4**

Prove that the area of $\Delta\mathrm{QOT} = \frac{a^2}{3}$ units^2

(b) (i) If a particle moving with velocity *v* experiences air resistance equal to kv^2 per unit mass (*k* being constant), prove that, in falling from rest in a vertical line through a distance *s*, it will acquire a velocity of *v* given by $v = V\sqrt{1 - e^{-2ks}}$ **3**

where $V = \sqrt{\frac{g}{k}}$ (the terminal velocity) and *g* is the acceleration due to gravity (assumed constant).

(ii) With the same air resistance as in (i) acting on a particle, show that if projected upwards with velocity *U*, it will reach a height *x*, given by **3**

$$x = \frac{1}{2k} ln\left(\frac{g + kU^2}{g}\right)$$

(iii) Hence prove that under the same conditions, a particle projected upwards with velocity *U* will return to the point of projection with velocity *W* given by **3**

$$W^{-2} = U^{-2} + V^{-2}$$

<u>QUESTION 2</u> **(START A NEW PAGE)** **(15 Marks)** **Marks**

(a) A particle of unit mass is projected vertically upwards against a constant gravitational force g and a resistance $\frac{v}{c}$, where v is the velocity of the particle and c is a constant and s is the distance travelled in time t; at $t = 0$, $s = 0$, and $v = c(h - g)$ where h is a constant.

(i) Write down the equation of motion of the particle. **1**

(ii) Find the time taken by the particle to reach its highest point, and find the height of that point. **4**

(iii) The particle falls to its original position under gravity and under the same law of resistance. Will the time of descent be greater or less than the time of ascent? Give reasons for your answer. **3**

(b) A region R is bounded by the curves $y = 12 - x^3$ and $y = 12 - 4x$.

(i) Find the co-ordinates of the points of intersection of these curves and hence draw a neat sketch showing the resultant regions. **4**

(ii) By taking cylindrical shells of width Δx and radius x, show that the volume generated by revolving R about the y-axis is $\frac{256\pi}{15}$ units^3. **3**

QUESTION 3 (START A NEW PAGE) (15 Marks) **Marks**

(a) The section of a solid cut by any plane perpendicular to the x-axis is a square with the ends of a diagonal lying on the parabolas $y^2 = 9x$ and $x^2 = 9y$.

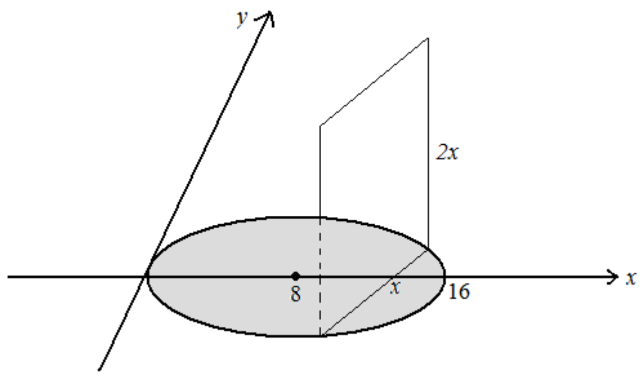
(i) Find the points of intersection of the two parabolas. **2**

(ii) Show that the area $A(x)$ is given by: **4**

$$A(x) = \frac{1}{162}\left[(27)^2 x - 54x^{\frac{5}{2}} + x^4\right]$$

(iii) Find the volume of the solid. **4**

(b) The base of a solid is the circle $x^2 + y^2 = 16x$, and every planar section perpendicular to the x-axis is a rectangle whose height is twice the distance of the plane of the section from the origin as shown in the diagram:



(i) Show that the area $A(x)$ is given by: **2**

$A(x) = 4x\sqrt{64 - (x-8)^2}$

(ii) Find the volume of the solid. **3**

<u>QUESTION 4</u> (START A NEW PAGE) (15 Marks) **Marks**

(a) The tangent at $P(\tan\theta, \sec\theta)$ on the hyperbola $y^2 - x^2 = 1$ cuts the *x* and *y* axes at *X* and *Y* respectively.

(i) Show this hyperbola is rectangular. **1**

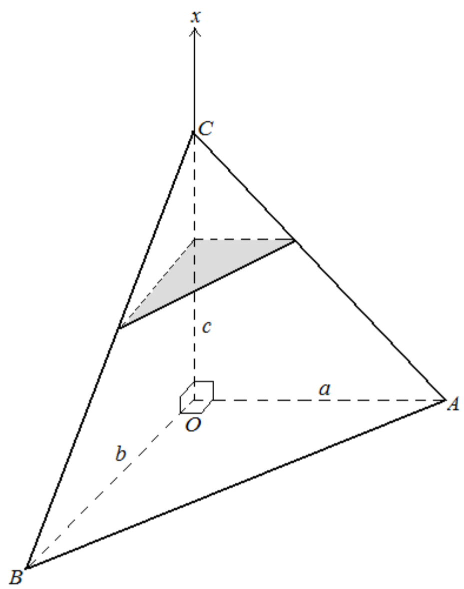
(ii) Find the coordinates of *X* and *Y*. **2**

(iii) Through *X* and *Y* lines are drawn parallel to the co-ordinate axes to intersect at *Q*. Find the coordinates of *Q*. **1**

(iv) Hence, find the cartesian equation of the locus *Q* as *P* varies. **2**

(v) Sketch the graph of the locus. **2**

(b) The tetrahedron formed by three mutually perpendicular edges of lengths *a, b, c* is shown in the diagram below:



Let the origin be the intersection of the edges, and let the *x*-axis lie along the edge of the length *c* as shown in the diagram. A typical cross section is a right triangle with legs of length *d* and *e*, parallel respectively to the edges of lengths *a* and *b*.

(i) Using similar triangles, show that the area of the cross-section $A(x)$ is given by:

$$A(x) = \frac{ab}{2c^2}(c-x)^2$$ **4**

(ii) Find the volume of the solid generated. **3**

Question (1)

(i) $xy = a^2$, $y = a^2/x$

$\therefore \dfrac{dy}{dx} = -a^2/x^2$

$$\boxed{\left.\frac{dy}{dx}\right|_{x=at} = \frac{-a^2}{a^2t^2} = -\frac{1}{t^2}}$$

Gradient of tgt $= -1/t^2$

[1]

$\therefore$ Equation of tgt at P

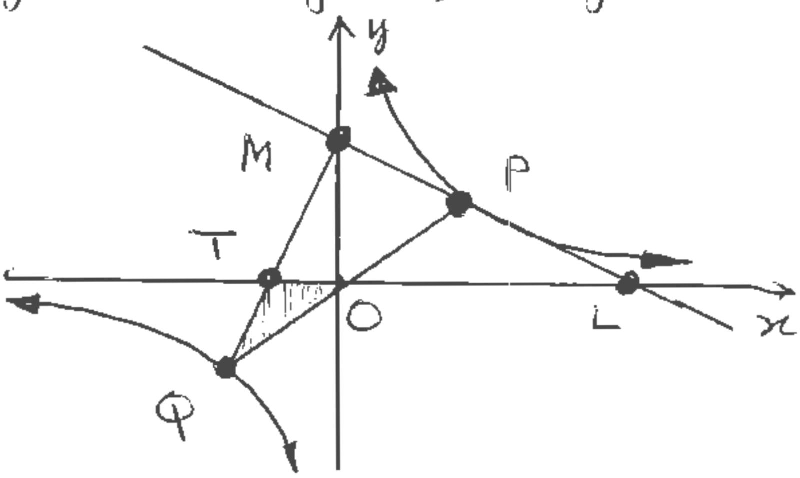
$$y - \frac{a}{t} = -\frac{1}{t^2}(x - at)$$

$$\boxed{\begin{aligned} x + t^2y - 2at &= 0 \\ \left(\frac{1}{t^2}\right)x + y &= \frac{2a}{t} \\ y &= \left(-\frac{1}{t^2}\right)x + \frac{2a}{t} \end{aligned}}$$

Correct equation

[1]

(ii) $Q(-at, -a/t)$ by symmetry of hyperbola.



When $x = 0$, $y = \dfrac{2a}{t}$

$\therefore M\left(0, \dfrac{2a}{t}\right)$

Coordinates of M $\left(0, \dfrac{2a}{t}\right)$

[1]

$$m_{MQ} = \frac{2a/t + a/t}{0 + at} = \frac{3a}{t^2}$$

$$= \frac{3}{t^2}.$$

$\therefore$ Equation of MQ is:

$$y - \frac{2a}{t} = \frac{3}{t^2}(x - 0)$$

$$\boxed{\begin{aligned} &3x - t^2 y + 2at = 0 \\ &y = \left(\frac{3}{t^2}\right)x + \frac{2a}{t} \end{aligned}}$$

correct equation of MQ

[1]

When $y = 0$, MQ meets x-axis at T

$$\therefore -\frac{2a}{t} = \frac{3x}{t^2}.$$

$$\implies x = -\frac{2at}{3}$$

$\therefore$ $T\left(-\frac{2at}{3}, 0\right)$

i.e. $|OT| = \frac{2at}{3}$ unit

Coordinates of T $\left(-\frac{2at}{3}, 0\right)$

[1]

Area of $\triangle$ QOT

$= \frac{1}{2}|OT| \cdot |y \text{ coord of } Q|$

$= \frac{1}{2} \times \frac{2at}{3} \times \frac{a}{t}$

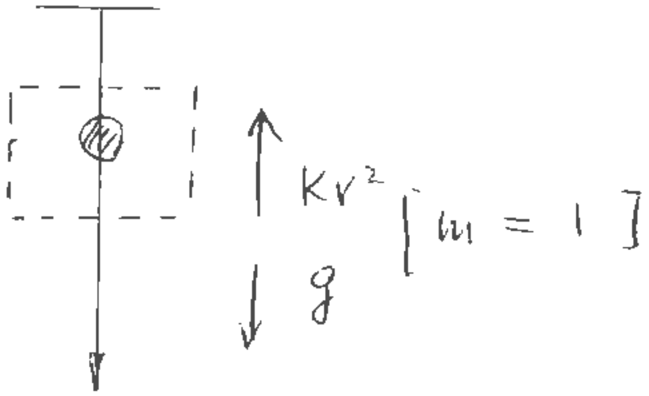
$= \frac{a^2}{3}$

$\boxed{\frac{a^2}{3}}$ [1]

Incorrect expression for |OT|, |y|

[1/2]

(b)



Taking positive to be 'down'

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

Correct form of D.E

[1]

$$\int dx = \int \frac{v\,dv}{g-kv^2}, \quad \int dx = \frac{-1}{2k}\int\frac{-2kv\,dv}{g-kv^2}$$

$$x = -\frac{1}{2k}\ln|g - kv^2| + c.$$

When $x = 0$, $v = 0$

$$\Rightarrow c = \frac{1}{2k}\ln g$$

i.e. $x = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$

When $x = s$

$$s = \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|$$

Correct expression for s with approp. initial condition

[1]

$$e^{2ks} = \frac{g}{g-kv^2}$$

$$g - kv^2 = ge^{-2ks}$$

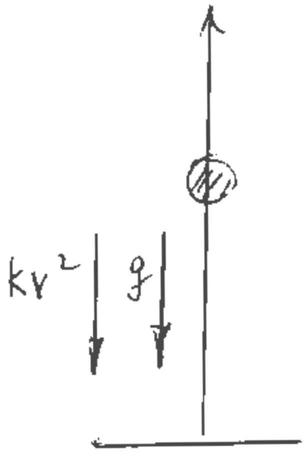
$$\boxed{v^2 = \frac{g}{k}(1 - e^{-2ks})}$$

ie $v = V(1 - e^{-2ks})$ | 1

$(v > 0)$

Correct expression for v

[1]



Taking positive to be 'up'

$$\ddot{x} = -g - kv^2$$

$$v\frac{dv}{dx} = -(g+kv^2)$$

$$\int dx = -\int \frac{v}{g+kv^2}\,dv$$

$$= -\frac{1}{2k}\int \frac{2kv\,dv}{g+kv^2}$$

$$\therefore\ x = -\frac{1}{2k}\ln(g+kv^2) + c$$

When $x = 0$, $V = u$

$$\Longrightarrow\ c = \frac{1}{2k}\ln(g+kU^2)$$

$$\therefore\ \boxed{x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)}$$

At max. height $v=0$

$$x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$$

2

Correct DE and integral $\int dx = -\int \frac{v\,dv}{g+kv^2}$

[1]

[1]

Correct expression for x before reaching maximum height

[1]

Max height
$v = 0$
$\therefore\ x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$

From equation 1 in (i)

$$v^2 = \frac{g}{k}\left(1 - e^{-2ks}\right)$$

Rearr-ing,

$$\boxed{e^{-2ks} = 1 - \frac{kv^2}{g}} \quad \text{— } 3$$

From equation 2 in (ii)

$$s = \frac{1}{2k} \ln\left(\frac{g+kU^2}{g}\right)$$

Substitute s into 3

and $v = W$, we haw

$$e^{-\ln\left(\frac{g+kU^2}{g}\right)} = \frac{g - kW^2}{g}$$

$$\therefore \boxed{\frac{g}{g+kU^2} = \frac{g-kW^2}{g}}$$

$$g^2 = (g + kU^2)(g - kW^2)$$

$$gkU^2 = gkW^2 + k^2U^2W^2$$

$$U^2 = W^2 + \frac{k}{g}U^2W^2$$

$$U^2 = W^2 + \frac{U^2W^2}{V^2}$$

$$V^2U^2 = V^2W^2 + U^2W^2$$

$$W^2 = \frac{V^2U^2}{U^2+V^2}$$

$$W^{-2} = \frac{U^2+V^2}{V^2U^2}$$

$$\therefore \boxed{W^{-2} = V^{-2} + U^{-2}}$$

- Correct expression to start off
- or equivalent merit

[1]

Equivalent expression for e^{-2ks}

i.e $\frac{g}{g+kU^2} = \frac{g-kW^2}{g}$

or equivalent merit

[1]

Algebraic manipulation to get to

$W^{-2} = V^{-2} + U^{-2}$

[1]

2011 Yr 12 T2 **MATHEMATICS Extension 2 : Question** 2

Suggested Solutions	Marks	Marker's Comments
(i) $m\ddot{x} = -\frac{v}{c} - mg$ or $m\ddot{x} = -m\frac{v}{c} - mg$	$\frac{1}{2}$	
$m=1$ $\ddot{x} = -\frac{v}{c} - g$	$\frac{1}{2}$	
(ii) $\int \frac{c\,dv}{v+cg} = -\int dt$	$\frac{1}{2}$	
$c\ln(v+cg) = -t + k_1$	$\frac{1}{2}$	
* $t = c(\ln(ch) - \ln(v+cg))$	$\frac{1}{2}$	
max ht $v=0$, $T_{max} = c\ln\frac{h}{g}$	$\frac{1}{2}$	forgot max ht $v=0$ $-\frac{1}{2}$
$v\frac{dv}{dx} = -\left(\frac{cg+v}{c}\right)$		
$\int_{c(h-g)}^{0} \frac{v}{cg+v}\,dv = -\frac{1}{c}\int_0^S dx$ (max ht, $v=0$)		
$-S = \int_{c(h-g)}^{0} c - \frac{c^2g\,dv}{v+cg}$	1	
$S = c^2(h-g) + \left[c^2g\ln(v+cg)\right]_{c(h-g)}^{0}$	1	
$S_{max} = c^2(h-g) + c^2g\ln\frac{g}{h}$	1	
or		
FROM * $v = e^{-t/c}.ch - cg$		
$\frac{ds}{dt} = e^{-t/c}.ch - cg$		
$S = -c\,ch(e^{-\frac{t}{c}}) - cg + k_2$	1	

2011 Yr 12 T2 **MATHEMATICS Extension 2 : Question** 2

Suggested Solutions	Marks	Marker's Comments
$t=0,\ s=0,\ k_2=c^2h$		Common mistakes.
$s=-c^2he^{-t/c}-cgt+c^2h$	1	$-\int\frac{cv}{cg-v}dv=\int_0^s dx$
at max ht $T=c\ln\frac{h}{g}$		max 2m.
max $s=c^2(h-g)+c^2g\ln\left(\frac{g}{h}\right)$	1	
iii) Time to reach max ht $=c\ln\frac{h}{g}$		
Since $V=c(h-g)>0\ \therefore\ h>g$		
Try to find the downward velocity at $t=c\ln\frac{h}{g}$		$\ddot{x}=g-\frac{v}{c}$
$\ddot{x}=g-\frac{v}{c}$		most students get this $\frac{1}{2}$m only when they do not successfully integrate any of the $\underset{(s)}{x}$, v, or t.
$\int_0^v\frac{dv}{g-\frac{v}{c}}=\int_0^{c\ln\frac{h}{g}}dt$		
$\left[-\cancel{c}\ln(cg-v)\right]_0^v=\cancel{c}\ln\frac{h}{g}$		
$\ln\left(\frac{cg}{cg-v}\right)=\frac{h}{g}\quad\therefore\ \frac{cg}{cg-v}=\frac{h}{g}$		
$v=\frac{cg(h-g)}{h}=cg\left(1-\frac{g}{h}\right)$	1m	
Try to find distance travelled from $t=0$ to $t=c\ln\frac{h}{g}$		
$v\frac{dv}{dx}=\frac{cg-v}{c}\quad\therefore\ \int_0^{cg(1-\frac{g}{h})}\frac{v\,dv}{cg-v}=\int_0^x\frac{dx}{c}$		
$\int_0^{cg(1-g/h)}\left(-1+\frac{cg}{cg-v}\right)dv=\frac{x}{c}$		

$$-cg\left(1-\frac{g}{h}\right)-\Big[cg\ln(cg-v)\Big]_0^{cg(1-g/h)}=\frac{x}{c}$$

$$-cg\left(1-\frac{g}{h}\right)-cg\ln\left(\frac{\cancel{cg}-\cancel{cg}+cg\,g/h}{cg}\right)=\frac{x}{c}$$

$$cg\left(\frac{g}{h}-1\right)-cg\ln\left(\frac{g}{h}\right)=\frac{x}{c}$$

$$x=c^2g\left(\frac{g}{h}-1\right)-c^2g\ln\left(\frac{g}{h}\right) \qquad 1m$$

h) = MAX ht $-x = c^2(h-\cancel{g})-c^2g\ln\frac{h}{g}-c^2g\left(\frac{g}{h}-\cancel{1}\right)+c^2g\ln\frac{g}{h}$

$$f(h)=c^2h-\frac{c^2g^2}{h}-2c^2g\ln\frac{h}{g}$$

$$f'(h)=c^2+\frac{c^2g^2}{h}-2c^2g\left(\frac{g}{h}\times\frac{1}{g}\right)=c^2\left(1-\frac{2g}{h}+\frac{g^2}{h^2}\right)=c^2\left(\frac{g}{h}-1\right)^2>0$$

Since $h>g$, $f(x)>0$ $\therefore$ max ht $>x$

At $t=c\ln\frac{h}{g}$, has not reached pt of projection. must have conclusion

$\therefore$ Time descent is <u>greater</u> than ascend $\frac{1}{2}m$

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many students try to find the Time to reach the pt of projection $\frac{dv}{dt}=\frac{cg-v}{c}$

$$\int_0^v\frac{dv}{cg-v}=\int_0^t\frac{dt}{c} \qquad \ln\left(\frac{cg}{cg-v}\right)=\frac{t}{c} \qquad \therefore\ t=c\ln\left(\frac{cg}{cg-v}\right) \qquad 1m$$

or try to find the velocity reaching pt of projection.

$$\frac{t}{c}=\ln\frac{cg}{cg-v} \qquad v=cg(1-e^{-t/c}) \qquad 1m$$

to relate to the distance $\frac{v\,dv}{dx}=\frac{cg-v}{c}$

$$\int_v^v\frac{v\,dv}{cg-v}=\int_0^s\frac{dx}{c} \qquad \int_0^v\left(-1+\frac{cg}{cg-v}\right)dv=\frac{s}{c}$$

$$s=-\left[cv+c^2g\ln\frac{cg-v}{cg}\right] \qquad 1m$$

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments

Q2(b) (i) $y = 12 - x^3$ — (1)

$y = 12 - 4x$

$\therefore \quad 12 - 4x = 12 - x^3$

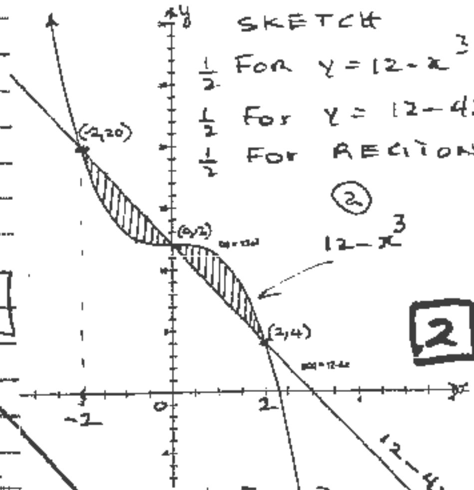
$\therefore \quad x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$\therefore x = 0$ or ± 2

$\therefore$ curve intersect

at $(-2, 20)$; $(0, 12)$ and $(2, 4)$ ✓ $\boxed{2}$



SKETCH

$\frac{1}{2}$ For $y = 12 - x^3$

$\frac{1}{2}$ For $y = 12 - 4x$

$\frac{1}{2}$ For REGION

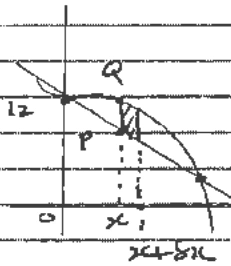
$\frac{1}{2}$ For Points of intersections

[$-\frac{1}{2}$ For not indicating -2 on x-axis]

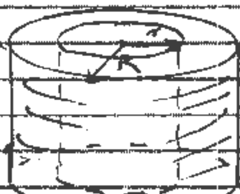
$\boxed{2}$

(ii) $\times$ the area bounded for $-2 \le x \le 0$ rotated about y-axis is identical to the area bounded for $0 \le x \le 2$, rotated about the y-axis

$\therefore$ Volume $= 2 \times$ Volume $0 \le x \le 2$



Taking a slice of thickness δx parallel to the y-axis at x is rotated about the y-axis forming a cylindrical shell



APPROACH I

$h = PQ = y_2 - y_1 = 12 - x^3 - (12 - 4x)$
$= 4x - x^3$.

$r = x$
$R = x + \delta x$

$\boxed{3}$

$A(x) = \pi[R^2 - r^2] = \pi(R + r)(R - r)$

$= \pi(2x + \delta x)(\delta x)$ ✓ 1

$= \pi(2x\,\delta x + (\delta x)^2)$

$\therefore A(x) \doteq 2\pi x\,\delta x$, neglect $(\delta x)^2$ as $(\delta x)^2 \doteq 0$

volume of shell $\quad \delta V \doteq 2\pi x\,\delta x \times h$

$\delta V \doteq 2\pi x(4x - x^3)\,\delta x$ 1

volume of solid $\quad V \doteq 2 \times \sum_{1}^{n} 2\pi x(4x - x^3)\,\delta x$ ✓ (as $V_{left} = V_{right}$)

$V = 4\pi \lim_{\substack{n \to \infty \\ \delta x \to 0}} \sum_{0}^{2} x(4x - x^3)\,\delta x$

[max if do $\pi \int_{-2}^{0} x^4 - 4x^2\,dx = \frac{-128}{5}\pi$

$V = 4\pi \int_0^2 4x^2 - x^4\,dx$

$= 4\pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$

$V = 4\pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{256\pi}{15}$ $\quad$ 1 For $\frac{256\pi}{15}$

$\therefore$ VOLUME is $\frac{256\pi}{15}\,u^3$,

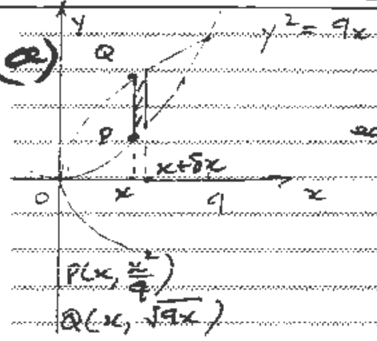
OR $V = V_1 + V_2 \rightsquigarrow 2\pi\int_{-2}^{0} x(y_U - y_L)\,dx + 2\pi\int_0 x(y_U - y_L)\,dx$

MATH EXT 2, ASSESSMENT TEST 3
TERM 2, 2011

MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments

$x^2 = 9y$



TOP VIEW

(a) (i) $x^2 = 9y$ — (1)

$y^2 = 9x$ — (2)

eqn (1) $\Rightarrow$ $y = \frac{x^2}{9}$ — (1a)

$\therefore y^2 = \frac{x^4}{81} = 9x$

$\therefore x^4 = 81.9x = 729x$

so $x(x^3 - 729) = 0$

$\therefore$ $x = 0$ or $x = 9$

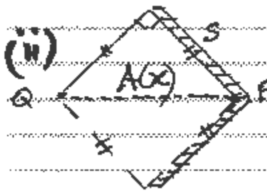
$y = 0$ $\quad$ $y = 9$

$\therefore$ Points of intersection $(0,0)$ and $(9,9)$

Marker's Comments:
$y^2 = 9x$
$y^4 = 81x^2 = 81.9y$
etc

$\frac{1}{2}$ For correct equation
$\frac{1}{2}$ For $(0,0)$
1 For $(1,1)$

$\boxed{2}$



(ii) Diagonal $PQ = y_2 - y_1 = 3\sqrt{x} - \frac{x^2}{9}$ $\quad$ 1

$= \frac{1}{9}(27\sqrt{x} - x^2)$

Area of Square $= s^2$ or $\frac{1}{2} PQ \times PQ$ ($\frac{1}{2}$ product of diagonals) $\quad$ 1

$\therefore A(x) = \frac{1}{2}\left[\frac{1}{9}(27\sqrt{x} - x^2)\right]^2$

$= \frac{1}{2} \times \frac{1}{81}(27^2 x - 54x^{\frac{5}{2}} + x^4)$ $\quad$ 1

$= \frac{1}{162}(27^2 x - 54x^{\frac{5}{2}} + x^4)$

Marker's Comments:
$-\frac{1}{2}$ for $\frac{x^2}{9} - 3\sqrt{x} < 0$

For organising/stating

For getting to $A(x)$ correctly

$\boxed{3}$

(iii) • Volume of slice $\delta V \doteqdot A(x)\,\delta x$

• Volume of solid $V \doteqdot \sum \delta V \left(= \sum_i A(x_i)\,\delta x\right)$ $\quad$ 1

$\therefore V = \lim_{\substack{n \to \infty \\ \delta x \to 0}} \sum_{0}^{9} A(x)\,\delta x$

ie $V = \int_0^9 \frac{1}{162}(27^2 x - 54x^{\frac{5}{2}} + x^4)\,dx$

$V = \frac{1}{162}\left[27^2 \frac{x^2}{2} - 54 \times \frac{2}{7} x^{\frac{7}{2}} + \frac{1}{5}x^5\right]_0^9$ ✓ $\quad$ 1

$= \frac{1}{162}\left[\frac{59049}{2} - \frac{236196}{7} + \frac{59049}{5} - 0\right]$

$\left(= \frac{1}{162}\left[29524.5 - 33742.28\ldots + 11809.8\right]\right)$

$V = \frac{6561}{140} = 46.864\ldots$ ✓ $\quad$ 1

$= 46\frac{121}{140}$

$\therefore$ Volume of solid is $46\frac{121}{140}\ u^3$ $\quad$ 1

Marker's Comments:
For organising/developing the set up to V

For integrating correctly

$\boxed{4}$

$-\frac{1}{2}$ if
$V = 46\frac{121}{140}\ u^3$
Volume $= 46\frac{121}{140}$

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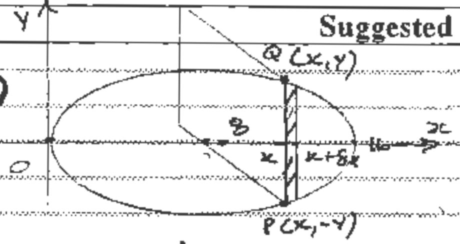
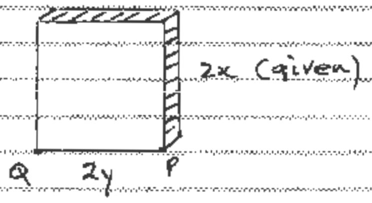
MATHEMATICS Extension 2: Question 3

Suggested Solutions	Marks	Marker's Comments

(b)



A TOP VIEW



(i) $x^2 + y^2 = 16x$

$x^2 - 16x + 64 + y^2 = 64$

$\therefore (x-8)^2 + y^2 = 64$

$y^2 = 64 - (x-8)^2$

upper semi-circle:

$y = \sqrt{64-(x-8)^2}$

$\therefore PQ = y_2 - y_1 = y - (-y) = 2y$ — $\tfrac{1}{2}$

Area $A(x) = 2y \times 2x = 4xy$ — $\tfrac{1}{2}$

ie $A(x) = 4x\sqrt{64-(x-8)^2}$

Marker's comments: OR: Circle centre $(8,0)$ radius 8 is $(x-8)^2 + y^2 = 64$

1 For getting/showing where $64-(x-8)^2$ comes from

$\boxed{2}$

(ii) Volume of slice $\delta V \doteqdot A(x)\,\delta x$

" of solid $V \doteqdot \sum_{1} A(x_i)\,\delta x$

ie $V = \lim_{\substack{n\to\infty \\ \delta x \to 0}} \sum_{0}^{16} A(x)\,\delta x$

$V = \int_0^{16} 4x\sqrt{64-(x-8)^2}\,dx$

Marker's comments: $(\tfrac{1}{2})$ For $\int_0^{16}$ if ...

METHOD I Let $u = x - 8$ — 1

$x = u + 8$

$dx = du$

x	u
16	8
0	−8

$\therefore V = 4\int_{-8}^{8} (u+8)\sqrt{64-u^2}\,du$ — $\tfrac{1}{2}$

$= 4\left\{\int_{-8}^{8} u\sqrt{64-u^2}\,du + 8\int_{-8}^{8}\sqrt{64-u^2}\,du\right\}$

as ODD Fn $\left[-\tfrac{2}{3}(64-u^2)^{\frac{3}{2}}\right]_{-8}^{8}$

$= 4\{\ 0 \quad + 8 \times \tfrac{1}{2}$ area of circle$\}$ — 1

$= 4\{\ 0 \quad + 8 \times \tfrac{1}{2} \times \pi \times 8^2\}$ — 1

$= 4 \times 4 \times \pi \times 64$

$V = 1024\pi$ — $\tfrac{1}{2}$

$\therefore$ Volume of solid is $1024\pi\ u^3$

Marker's comments: METHOD II (harder way)

$x = 8 + 8\sin\theta$

$dx = 8\cos\theta\,d\theta$

x	θ
16	$\pi/2$
0	$-\frac{\pi}{2}$

$V = 4.8^3\int(1+\sin\theta)\cos^2\theta\,d\theta$

Note: $\sqrt{\cos^2\theta} = |\cos\theta| =$

$V = 2048\int \cos^2\theta + \sin\theta\cos^2\theta\,d\theta$

$V = 2048\int \frac{1+\cos 2\theta}{2} + sc^2\,d\theta$

$= 2048\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta - \frac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\pi/2}$

$= 2048\left[\frac{\pi}{4} + 0 - 0 - \left(-\frac{\pi}{4} + 0 - 0\right)\right]$

$V = 2048 \times \frac{\pi}{2} =$

$\boxed{4}$

No penalty unless (a)(iii)

METHOD III $V = 4\int x\sqrt{64-(x-8)^2}\,dx \quad = 4\int x\sqrt{16x - x^2}\,dx$

$V = 4\int (x-8+8)\sqrt{64-(x-8)^2}\,dx \quad = 4\int -\tfrac{1}{2}(16-2x-16)\sqrt{16x-x^2}\,dx$

$= 4\int (x-8)\sqrt{\ldots} + 8\sqrt{\ldots}\,dx \quad = -2\int_0^{16}(16-2x)\sqrt{16x-x^2}\,dx + 32\int_0^{16}\sqrt{16x-x^2}\,dx$

$= 4\left[-\tfrac{1}{3}(64-(x-8)^2)^{3/2}\right]_0^{16} + 32 \times \tfrac{1}{2} \times \pi \times 8^2 \quad = -\tfrac{1}{3}\left[(16x-x^2)^{\frac{3}{2}}\right]_0^{16} + 32 \times \tfrac{1}{2} \times \pi \times 8^2$

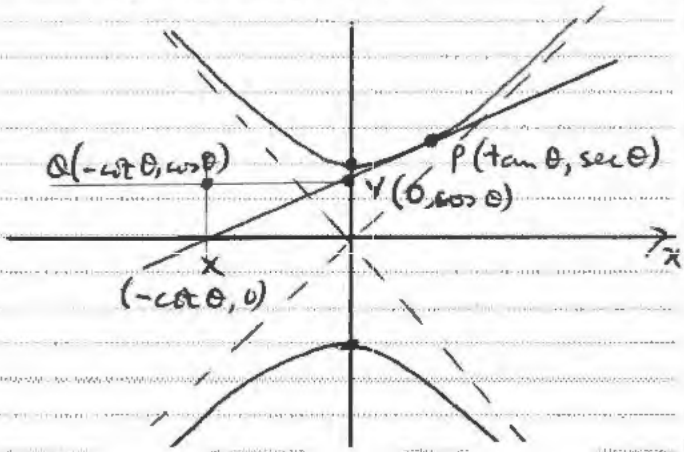
$V = \quad 0 - 0 \quad + 1024\pi \quad = 0 - 0 \quad + 1024\pi$

$\therefore V = 1024\pi$

METHOD IV IBPs ☺ ...

Suggested Solutions	Marks	Marker's Comments

a)



i) Asymptotes given by

$$y^2 - x^2 = 0$$
$$y = \pm x$$

Gradients are ± 1

$\therefore$ Asymptotes are perpendicular so the given hyperbola is rectangular. **1**

Marker's comment: One mark was given if the eccentricity was shown to be $\sqrt{2}$ (tho' this is a consequence rather than the definition)

ii) $2y\dfrac{dy}{dx} - 2x = 0$ $\quad \dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$ at P

$\therefore$ Tangent at P is $y - \sec\theta = \sin\theta(x - \tan\theta)$

$$\underline{y = x\sin\theta + \cos\theta}$$

When $y = 0$, $x = -\cot\theta$ $\quad \underline{\underline{X = (-\cot\theta, 0)}}$

$x = 0$, $y = \cos\theta$ $\quad \underline{\underline{Y = (0, \cos\theta)}}$ **2**

Marker's comment: Reasonable simplif^n was expected. Distressing to see how many left $\frac{1}{\sec\theta}$ in working

iii) From diagram $\underline{\underline{Q = (-\cot\theta, \cos\theta)}}$ **1**

iv) $\left.\begin{array}{l} x = -\cot\theta \\ y = \cos\theta \end{array}\right\}$ since $\sec^2\theta = 1 + \tan^2\theta$

$$\frac{1}{y^2} = 1 + \frac{1}{x^2}$$

$$\underline{\underline{y^2 = \frac{x^2}{x^2+1}}}$$ **2**

Restrictions

$\theta \neq \pi/2, 3\pi/2$ (Pt at ∞)
$\theta \neq 0, \pi, ..$ (Vertices)

$\Rightarrow x \neq 0, y \neq 0$

Marker's comment: Few gave the restrictions (none gave them in terms of θ). ½ off for no restrictions.

Suggested Solutions	Marks	Marker's Comments

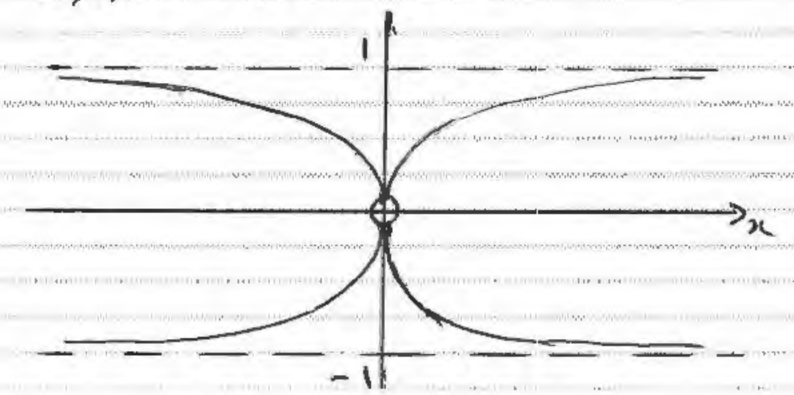
v) Symmetric in x & y

As $x \to \infty$, $y \to \pm 1$ (Horizontal asymptote)

$2y\dfrac{dy}{dx} = -\dfrac{2}{x^3}$ $\therefore \dfrac{dy}{dx} = -\dfrac{1}{x^3 y}$

$\therefore$ As $(x, y) \to (0,0)$, gradient increases without bound (ie vertical) **2**

$(0,0)$ is NOT allowed.



I wanted to mark

½ horizontal asymp

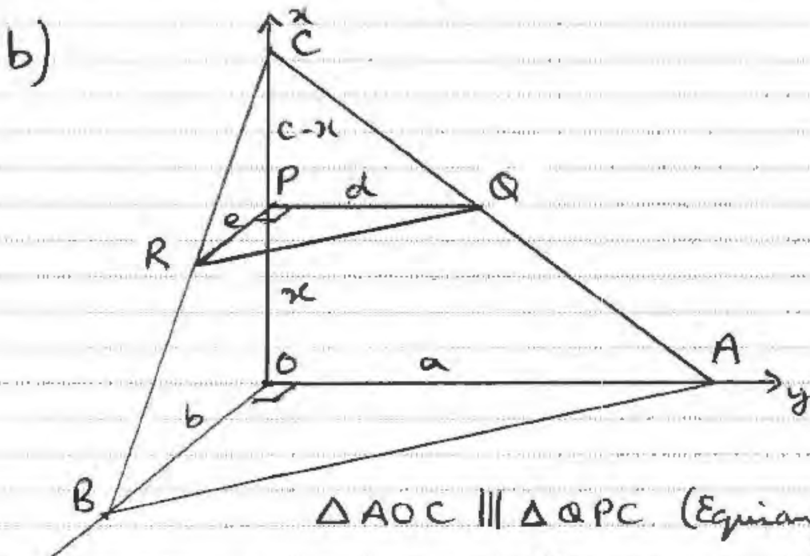
½ 4 way symmetry

½ hole at $(0,0)$

½ vertical near $(0,0)$

but I got a bit lenient with last 2 points.

b)



$\triangle AOC \;|||\; \triangle QPC$ (Equiangular)

$\therefore \dfrac{c}{c-x} = \dfrac{a}{d}$ (Corresponding sides of similar triangles in the same ratio)

$\therefore d = \dfrac{a(c-x)}{c}$ **4**

Similarly, using triangles OBC and CPR,

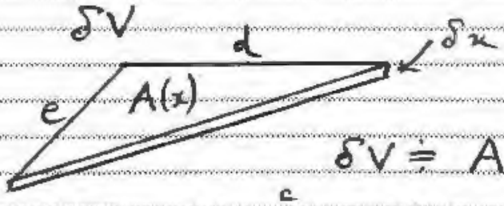
$e = \dfrac{b(c-x)}{c}$ $\therefore A(x) = \dfrac{ed}{2} = \dfrac{ab(c-x)^2}{2c^2}$

Few people drew a decent diagram

1½ marks were given for similar triangle and its trappings (No full proof reqd.)

Diagrams generally very poor and scrappy (maybe time issue)

MATHEMATICS Extension 2: Question........

Suggested Solutions	Marks	Marker's Comments
		
$\delta V \doteqdot A(x)\,\delta x$	½	More diagrams would have been nice.
$V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{c} A(x)\,\delta x$	½	½ mark each for introductory infinitessimals.
$= \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{c} \dfrac{ab(c-x)^2\,\delta x}{2c^2}$		
$= \displaystyle\int_0^c \frac{ab\,(c-x)^2\,dx}{2c^2}$	1	
$= \dfrac{ab}{2c^2}\displaystyle\int_0^c (c-x)^2\,dx$		
$= \dfrac{ab}{2c^2}\left[\dfrac{-(c-x)^3}{3}\right]_0^c$		
$= \dfrac{ab}{2c^2}\left(0-\left(-\dfrac{c^3}{3}\right)\right)$		
$= \dfrac{abc^3}{6c^2} = \dfrac{abc}{6}$		
$\therefore$ Volume is $\dfrac{abc}{6}$ cubic units	1	