Number

JAMES RUSE

YEAR 12

ASSESSMENT TEST 3
TERM 2, 2013

MATHEMATICS
EXTENSION 2

Time Allowed – 90 Minutes
(Plus 5 minutes Reading Time)

- *All* questions may be attempted
- *All* questions are of equal value
- Department of Education approved calculators are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged work
- No grid paper is to be used unless provided with the examination paper
- This is an open book test, any printed or hand written materials are allowed and must be placed on desk.

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 (15 marks) Marks

(a) A dragster racing car accelerates uniformly over a straight line course and completes a 'standing' (that is, starting from rest) 400 metres in eight seconds. At the end of the 400 m course the dragster reaches a speed of 100 ms^{-1}.

At the 400m mark, the dragster stops accelerating. At this instant, the dragster's brakes are applied and, in addition, a small parachute opens at the rear to slow the car down.

The retarding force applied by the brakes (including friction) is 5000 N. The retarding force due to the parachute is $0.5v^2$ N where v ms^{-1} is the velocity of the car x metres beyond the 400 metre mark. The mass of the dragster (car and driver) is 400 kg.

(i) By choosing an appropriate derivative form for acceleration, show that 2

$$\frac{dv}{dx} = -\frac{\left(10^4 + v^2\right)}{800v}.$$

(ii) Hence find, to the nearest metre, the distance the dragster takes to stop from the instant the brakes are applied. 3

(iii) Find the exact time, in seconds, taken to bring the dragster to rest from the 400m mark. 3

(b) $P\left(4p, \frac{4}{p}\right)$ and $Q\left(4q, \frac{4}{q}\right)$ are two variable points on the rectangular hyperbola $xy = 16$. M is the midpoint of PQ.

(i) Show that PQ has equation $x + pqy - 4(p + q) = 0$. 2

(ii) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the chord PQ from the origin $O(0, 0)$ is always 2 units, show that $4(p+q)^2 = 1 + p^2q^2$. 1

iii) Find the equation of the locus of M and state any restrictions on the domain and range of the locus. 4

Question 2 (15 marks) START A NEW PAGE Marks

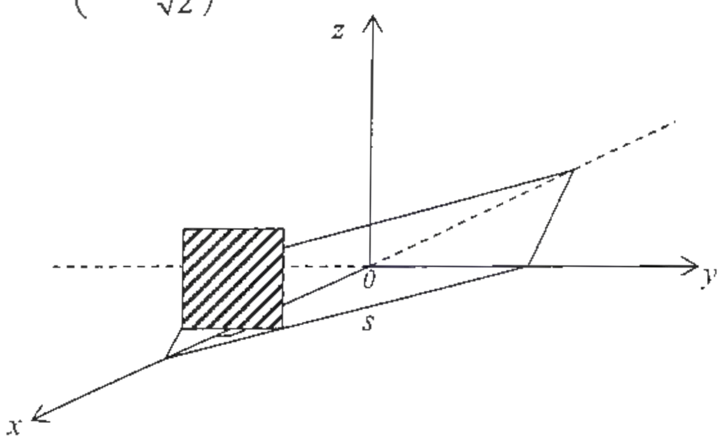
(a) James bought a doughnut. After some careful measuring he concluded that the volume of the shape of his doughnut is the same as the volume generated when the circle $\frac{(x-a)^2}{(2b)^2}+\frac{y^2}{(2b)^2}=1$ (where $4b < a$) is rotated about the y-axis. Accidently, James sat on his doughnut. He then noticed that the cross-sections of the doughnut became congruent ellipses. After some further measuring he found that the previously circular cross-sections have uniformly been halved vertically and doubled horizontally.

(i) Explain briefly, why $y=\pm\frac{1}{4}\sqrt{16b^2-(x-a)^2}$ is the equation of the elliptic cross-section of James' doughnut. 1

(ii) By using the method of cylindrical shells, find the volume of James' elliptic doughnut which can be obtained by rotating the ellipse in (i) about the y-axis. 4

(b) A solid is constructed with a square base, $\Re$, of side length s units. Vertical cross-sections taken perpendicular to one of the diagonals of $\Re$ are squares with one side in the base of the solid.

(i) If the vertices of $\Re$ lie on the coordinate axes, equidistant from the origin, show that the area A of one of the cross-sectional squares, shown in the diagram below, can be given by $A=4\left(-x+\frac{s}{\sqrt{2}}\right)^2$. 2

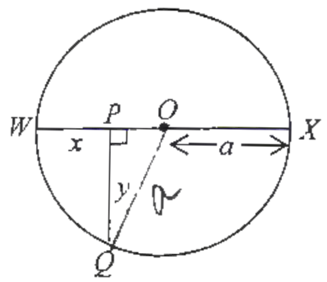


(ii) Find the volume of the solid. 3

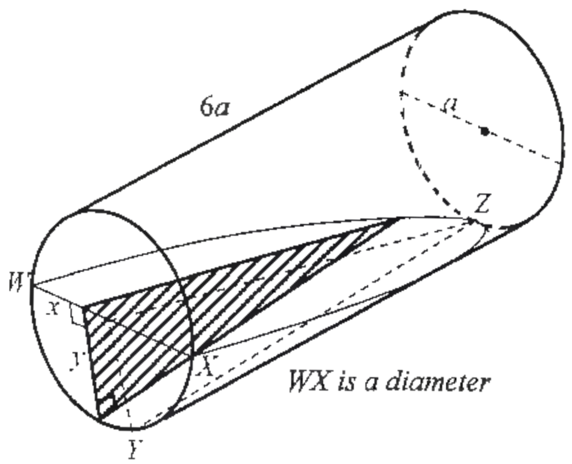
Question 2(c) is continued over the page.

Marks

(c) (i) In the diagram below, WX is a diameter of a circle whose centre is O and whose radius a units. The point P lies on the radius OW and Q is a point on the circle such that QP Let $WP = x$ units and $PQ = y$ units. Show that $y^2 = 2ax - x^2$. 2



(ii) 3



The diagram to the left shows the amount of water remaining in a cylindrical glass of radius a cm and height $6a$ cm. A typical slice of thickness δx is shown. Use the result in part (i) to show that the volume of water remaining in the glass is $4a^3$ cm^3.

Question 3 (15 marks) START A NEW PAGE Marks

(a) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy = c^2$ which move so that the points P, Q and $S(c\sqrt{2}, c\sqrt{2})$ are always collinear. The tangents to the hyperbola at P and Q intersect at the point R.

(i) Show that the tangent to the hyperbola $xy = c^2$ at $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$. 2

(ii) Hence show that R has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. 2

(iii) Show that $p + q = \sqrt{2}(1 + pq)$. 2

(iv) Hence find the equation of the locus of R. 2

(b) A particle is projected vertically downwards under gravity in a medium where resistance is proportional to half the speed of the particle. The terminal velocity of the particle is U ms^{-1}, and the speed of projection is equal to one quarter of this terminal velocity. At time t seconds, the particle has travelled a distance of x metres, has velocity v ms^{-1} and has acceleration $\ddot{x}$ ms^{-2}.

(i) Show $\ddot{x} = \frac{g}{U}(U - v)$, where g ms^{-2} is the acceleration due to gravity. 2

(ii) Show by integration that $-\frac{g}{U}t = \ln\frac{4}{3}\left(1 - \frac{v}{U}\right)$. Hence obtain an expression for $\frac{v}{U}$ in terms of t. 3

(iii) Show that $x = Ut - \frac{U^2}{g}\left(\frac{v}{U} - \frac{1}{4}\right)$. 2

Question 4 is continued over the page.

Question 4 (15 marks) START A NEW PAGE Marks

A particle of mass 1 kg is found to experience a resistive force, in Newtons, of the square of its velocity in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of u metres per second, and reaches a height of h metres before it starts to fall to the ground again. Assume the value of g is 10 ms^{-2}.

(i) Find the time, T, the particle takes to reach a height of h from O. 5

(ii) Show that the height h is $\frac{1}{2}\ln\left[1+\frac{u^2}{10}\right]$ metres. 5

(iii) Show that the particle's velocity w ms^{-1} when it reaches O again is given by 5

$w^2 = \frac{10u^2}{10+u^2}$ where $1-\frac{w^2}{10} \geq 0$.

Ext 2 Yr 12 2013 T2 Q1

a i) $400\ddot{x} = -(5000 + 0.5v^2)$ Newton's 2nd law of motion

$\ddot{x} = \dfrac{-(5000+0.5v^2)}{400}$ 1m

$v\dfrac{dv}{dx} = \dfrac{-(10000+v^2)}{800}$

$\dfrac{dv}{dx} = \dfrac{-(10^4+v^2)}{800v}$ 1m

well done

ii) $\dfrac{dx}{dv} = \dfrac{-800v}{10^4+v^2}$

$x = \displaystyle\int_{100}^{0} \dfrac{-800v\,dv}{10^4+v^2}$ 1m

$= \displaystyle\int_{0}^{100} \dfrac{800v\,dv}{10^4+v^2}$

$= 400\ln(10^4+v^2)\Big]_0^{100}$ 1m

$= 400\ln\left(\dfrac{20000}{10000}\right)$

$= 400\ln 2$

$= 277.2589\ldots$

$= 277$ m. (nearest metres) 1m

$\therefore$ distance dragster takes to stop is 277 m.

Some use C

$x = -400\ln(10^4+v^2) + c$ 1m

$x=0,\ v=100$

$c = 400\ln 20000$

$x = 400\ln\left(\dfrac{20000}{10000}\right)$ 1m

A few answer 677m (include 400 m) 2m.

nearest metres $-\frac{1}{2}$m

iii) $\ddot{x} = \dfrac{dv}{dt} = -\dfrac{(10^4+v^2)}{800}$

$\dfrac{dt}{dv} = \dfrac{-800}{(10^4+v^2)}$

$t = -800\displaystyle\int_{100}^{0} \dfrac{dv}{10^4+v^2}$ 1m

$$t = 800\int_0^{100} \frac{dv}{v^2+10^4}$$

$$= \frac{800}{100}\left[\tan^{-1}\frac{v}{100}\right]_0^{100} \qquad 1m$$

$$= 8\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= 8 \times \frac{\pi}{4}$$

$$= 2\pi \qquad 1m$$

Drangster takes 2π s to stop from the 400 mark.

Some students took $t=8$ $V=100$ so the corresponding answer is $2\pi + 8$ sec. must justify why $t=8$, $v=1$

Some not answer in exact time $-\frac{1}{2}m$

b(i) Gradient of $PQ = \dfrac{\frac{4}{p} - \frac{4}{q}}{4(p-q)}$

$$= \frac{4(\cancel{q-p})^{-1}}{pq} \times \frac{1}{\cancel{4(p-q)}} = \frac{-1}{pq} \qquad 1m$$

Eq of PQ: $y - \frac{4}{p} = -\frac{1}{pq}(x-4p)$

$$pqy - 4q = -x + 4p$$

$$x + pqy - 4(p+q) = 0 \qquad 1m$$

v. well done

(ii) $$2 = \frac{|1\times 0 + pq\times 0 - 4(p+q)|}{\sqrt{1^2+p^2q^2}} \qquad \tfrac{1}{2}m$$

must show $|1\times 0 + pq\times 0 - 4(p+q)|$ because answer is given

$$2\sqrt{1+p^2q^2} = |0+0-4(p+q)|$$

square both sides

$$4(1+p^2q^2) = 16(p+q)^2$$

$$1+p^2q^2 = 4(p+q)^2 \qquad \tfrac{1}{2}m$$

iii) For M

$x = \frac{4p+4q}{2} = 2(p+q)$ $\frac{1}{2}$m

$y = \frac{\frac{4}{p}+\frac{4}{q}}{2} = \frac{4(q+p)}{2pq}$

$y = \frac{2(p+q)}{pq}$ $\frac{1}{2}$m

well done.

$\frac{x^2}{y^2} = p^2q^2$

$= 4(p+q)^2 - 1$ from (ii)

$\frac{x^2}{y^2} = x^2 - 1$

Locus of M: $y^2 = \frac{x^2}{x^2-1}$ # 1m.

Fairly well done.

or $x^2 + y^2 = x^2y^2$ #

$\frac{x^2}{x^2-1} = y^2 \geq 0$

$\therefore x^2 \geq 0$ and $x^2 - 1 \geq 0$

D: $\therefore |x| > 1$ $(x \neq \pm 1)$ 1m

Very few students got this right

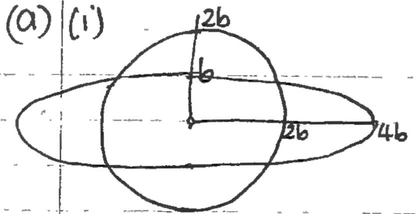
Since $x^2+y^2 = x^2y^2$ which is symmetric in x & y

R: $|y| > 1$ 1m

✓

Question 2

(a) (i)



Equation $\frac{(x-a)^2}{(2b)^2} + \frac{y^2}{(2b)^2} = 1$

becomes $\frac{(x-a)^2}{(4b)^2} + \frac{y^2}{b^2} = 1$

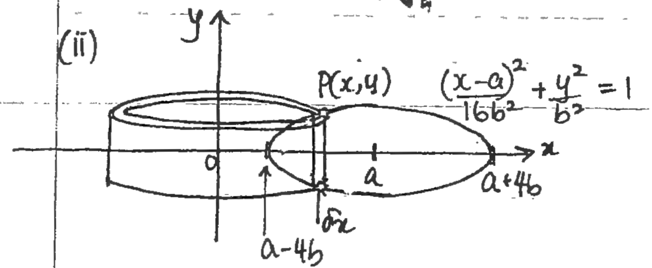
ie. $y^2 = b^2\left[1 - \frac{(x-a)^2}{16b^2}\right]$

$y = \pm \frac{b}{4b}\sqrt{16b^2-(x-a)^2}$

$= \pm \frac{1}{4}\sqrt{16b^2-(x-a)^2}$

1 mark
Students lost $\pm$

(ii)



Volume of cylindrical shell $\delta V \approx 2\pi x \times 2y\, \delta x$

$\approx 4\pi x y \delta x$

$\approx \frac{4\pi x}{4}\sqrt{16b^2-(x-a)^2}\,\delta x$

$\approx \pi x\sqrt{16b^2-(x-a)^2}\,\delta x$

1 mark for setting up
2 methods
students missed $4b < a$

$\therefore V = \pi \lim_{\delta x \to 0} \sum_{x=a-4b}^{a+4b} x\sqrt{16b^2-(x-a)^2}\,\delta x$

1 mark

$= \pi\int_{a-4b}^{a+4b} x\sqrt{16b^2-(x-a)^2}\,dx$ Let $u = x-a$

Note
(This is not an even fn)

$= \pi\int_{-4b}^{4b}(u+a)\sqrt{16b^2-u^2}\,du$

$= \frac{-\pi}{2}\int_{-4b}^{4b} -2u\sqrt{16b^2-u^2}\,du + \pi a\int_{-4b}^{4b}\sqrt{16b^2-u^2}\,du$

$= \frac{-\pi}{2}\times\frac{2}{3}\left[(16b^2-u^2)^{3/2}\right]_{-4b}^{4b} + \pi a \times \underbrace{\tfrac{1}{2}\pi \times 16b^2}_{\text{area of semicircle, } r=4b}$

$= -\pi/3\,[0] + 8\pi^2 ab^2$

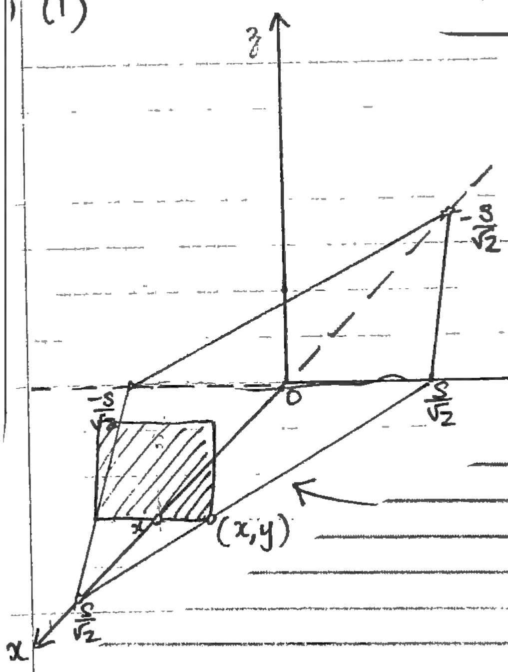
$= 8\pi^2 ab^2$ cubic units is volume of elliptic doughnut.

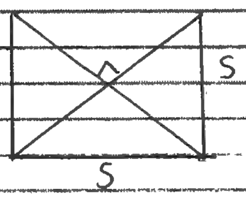
1 for each integral
Integration poorly done.

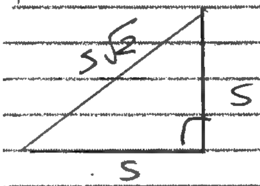
MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments

(i)



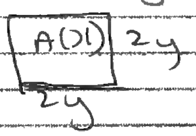




but $S\sqrt{2} = \frac{S}{\sqrt{2}}$ — **1** — a number of students failed to explain this

Method 1: Equation of line through $(s/\sqrt{2}, 0)$, $(0, s/\sqrt{2})$ is $y = -x + \frac{s}{\sqrt{2}}$

OR using similarity



Area $= 2y \times 2y$

$= 4y^2$

$\therefore A = 4\left(-x + \frac{s}{\sqrt{2}}\right)^2$ — **1** — A number of careless errors because a show question must explain fully

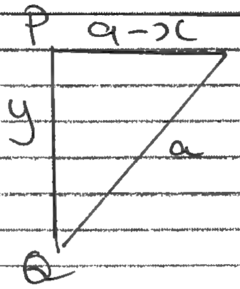
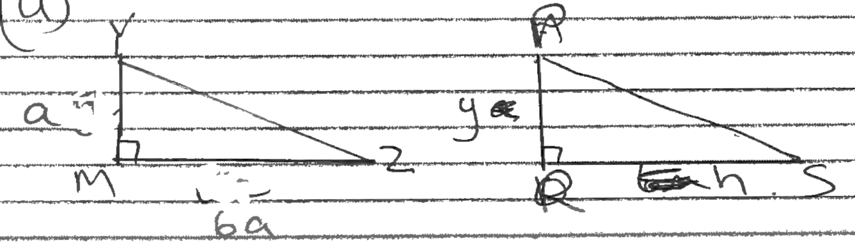
(ii) Volume of a typical slice

$\delta V = 4y^2\,\delta x$

since symmetrical

$$V = \lim_{\delta x \to 0} 2 \sum_{x=0}^{s/\sqrt{2}} 4\left(-x + s/\sqrt{2}\right)^2 dx$$ — **1**

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
$V = 8\int_0^{5/\sqrt{2}} \left(-x+\frac{5}{\sqrt{2}}\right)^2 dx$	1	Too many students assumed $\int_{-\frac{5}{\sqrt{2}}}^{5/\sqrt{2}} \left(-x+\frac{5}{\sqrt{2}}\right)^2 dx$ was even fn.
$= -\frac{8}{3}\left[\left(-x+\frac{5}{\sqrt{2}}\right)^3\right]_0^{5/\sqrt{2}}$ $= -\frac{8}{3}\left[0^3 - \left(\frac{5}{\sqrt{2}}\right)^3\right]$ $= \frac{8\,5^3}{6\sqrt{2}}$ OR $\frac{4\,5^3}{3\sqrt{2}}$	1	maximum 2 $4\int_{-5/\sqrt{2}}^{5/\sqrt{2}} \left(-x+\frac{5}{\sqrt{2}}\right)^2 dx$
Other method needed to split. $V = 4\int_0^{5/\sqrt{2}} \left(-x+\frac{5}{\sqrt{2}}\right)^2 dx + 4\int_{-5/\sqrt{2}}^{5/\sqrt{2}} \left(\cancel{o}x+\frac{5}{\sqrt{2}}\right)^2 dx$		
c(i)  $y^2 + (a-x)^2 = a^2$ (pythagoras theorem) $y^2 + a^2 - 2ax + x^2 = a^2$ $y^2 = 2ax - x^2$	2	well done
(ii)  $\frac{y}{a} = \frac{h}{6a}$ (triangles YMZ and PRS are similar) $\Rightarrow h = 6y$	1	Students had height in wrong position poorly done.

MATHEMATICS Extension 2: Question……..

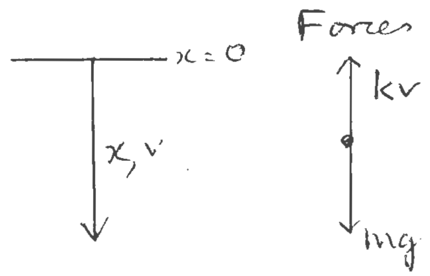
Suggested Solutions	Marks	Marker's Comments
(ii) $\delta V = 3(2ax - x^2)\,\delta x$		
$V = \lim\limits_{\delta x \to 0} 2\sum\limits_{x=0}^{a} 3(2ax - x^2)\,\delta x$		
$V = 2\int_0^a 3(2ax - x^2)\,dx$	1	
$V = 6\int_0^a 2ax - x^2\,dx$		
$= 6\left[ax^2 - \frac{x^3}{3}\right]$		
$= 6\left[a^3 - \frac{a^3}{3}\right]$		
$= \frac{6 \times 2a^3}{3}$		
$= 4a^3$ cm^3	1	

TERM 2 2013 X2 **MATHEMATICS: Question** 3 1/5

Suggested Solutions	Marks	Marker's Comments
a) i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c}$		
$= -1/t^2$	1	
∴ Tangent at $(ct, c/t)$ is		
$(y - c/t) = -\frac{1}{t^2}(x - ct)$		
$t^2y - ct = -x + ct$		Everybody got two marks here
$\underline{\underline{x + t^2y = 2ct}}$	1	
ii) ∴ Tangent at P $x + p^2y = 2cp$ —①		
Tangent at Q $x + q^2y = 2cq$ —②		
Solve for R		
① − ② $y(p^2 - q^2) = 2c(p-q)$		
$y(p+q)(p-q) = 2c(p-q)$		Only danger here was jumping the last step to a given result. (½ mark lost once)
$y(p+q) = 2c$ $(p \neq q)$		
$\therefore \underline{\underline{y = \frac{2c}{p+q}}}$	1	
Substitute back in ①		
$x + p^2\left(\frac{2c}{p+q}\right) = 2cp$		
$x = \frac{2cp(p+q) - 2cp^2}{p+q}$		
$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$		
$= \underline{\underline{\frac{2cpq}{p+q}}}$	1	
∴ R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$		

Suggested Solutions	Marks	Marker's Comments
a) iii) S lies on PQ		
Approach 1 Find eqn of PQ		
$\dfrac{y - c/q}{c/p - c/q} = \dfrac{x - cq}{cp - cq}$		
$\dfrac{pq(y - c/q)}{c(q-p)} = \dfrac{x - cq}{c(p-q)}$		This had to be derived. ½ off if only quoted
$\therefore -pq(y - c/q) = x - cq$		
$\underline{\underline{x + pqy = c(p+q)}}$	1	
$c\sqrt{2}, c\sqrt{2}$ satisfies this		
$\therefore c\sqrt{2} + pq \cdot c\sqrt{2} = c(p+q)$		
$\therefore \sqrt{2} + pq\sqrt{2} = p+q$		
$\underline{\underline{\sqrt{2}(1+pq) = p+q}}$	1	
Approach 2 $m_{SP} = m_{SQ}$ (or equivalent)		
$\dfrac{c\sqrt{2} - c/p}{c\sqrt{2} - cp} = \dfrac{c\sqrt{2} - c/q}{c\sqrt{2} - cq}$	1	
$\dfrac{\sqrt{2} - 1/p}{\sqrt{2} - p} = \dfrac{\sqrt{2} - 1/q}{\sqrt{2} - q}$		
$(\sqrt{2} - 1/p)(\sqrt{2} - q) = (\sqrt{2} - 1/q)(\sqrt{2} - p)$		Some of the algebra got a bit heavy here.
$2 - \dfrac{\sqrt{2}}{p} + \dfrac{q}{p} - \sqrt{2}q = 2 - \dfrac{\sqrt{2}}{q} + \dfrac{p}{q} - \sqrt{2}p$		Most people used first approach.
$\dfrac{q}{p} - \dfrac{p}{q} = \sqrt{2}q - \sqrt{2}p + \dfrac{\sqrt{2}}{p} - \dfrac{\sqrt{2}}{q}$		
$q^2 - p^2 = \sqrt{2}(q-p)pq + \sqrt{2}(q-p)$		
$q + p = \sqrt{2}pq + \sqrt{2}$ (Cancel $p-q$, $p \neq q$)		
$\therefore \underline{\underline{\sqrt{2}(1+pq) = p+q}}$	1	

X2 **MATHEMATICS: Question** 3 3/5

Suggested Solutions	Marks	Marker's Comments
a) iv) R is $\begin{cases} x = \dfrac{2cpq}{p+q} \\ y = \dfrac{2c}{p+q} \end{cases}$		
Divide : $\dfrac{x}{y} = pq$	1	There were various ways of getting through.
also $\dfrac{y}{2c} = \dfrac{1}{p+q}$ $\Rightarrow p+q = \dfrac{2c}{y}$		
Substitute into (iii) $\dfrac{2c}{y} = \sqrt{2}\left(1 + \dfrac{x}{y}\right)$		
$\therefore \underline{\underline{c\sqrt{2} = x + y}}$	1	
b) i) 		N.B. $R \propto V/2 \rightarrow R \propto V$
Newton's 2nd Law $\Rightarrow$ $\underline{\underline{m\ddot{x} = mg - kv}}$ where k is constant of proportionality	1	Surprising number missed out K. (Max 1 mark) Needed either a clear force diagram or quotation of "Newton's 2nd Law" before writing down equation (else ½ off).
But as $v \rightarrow U$, $\ddot{x} \rightarrow 0$ $\therefore mg = kU$ $\therefore k = mg/U$	1	
Substitute back $m\ddot{x} = mg - \dfrac{mgv}{U}$ $\ddot{x} = g - \dfrac{gv}{U}$ $\underline{\underline{= \dfrac{g}{U}(U - v)}}$		

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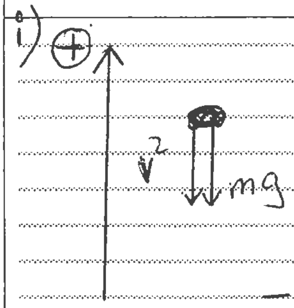
X2 MATHEMATICS: Question 3		4/5
Suggested Solutions	**Marks**	**Marker's Comments**
b) ii) $\ddot{x} = \frac{dv}{dt} = \frac{g}{u}(u-v)$ from (i) $\therefore \int_{u/4}^{v} \frac{dv}{u-v} = \frac{g}{u}\int_0^t dt$ $\left[-\ln(u-v)\right]_{u/4}^{v} = \left[\frac{gt}{u}\right]_0^t$ (Since v is u/4 initially u > v)	1	
$-\ln(u-v) + \ln(u-u/4) = \frac{gt}{u}$ $\therefore \ln\left(\frac{3u/4}{u-v}\right) = \frac{gt}{u}$ $\therefore -\frac{gt}{u} = \ln\left(\frac{u-v}{3u/4}\right)$ $-\frac{gt}{u} = \ln\left(\frac{4(u-v)}{3u}\right)$		If absolute values introduced then they needed to be explained away else ½ lost.
$-\frac{gt}{u} = \ln\left(\frac{4}{3}\left(1-\frac{v}{u}\right)\right)$	1	
$\therefore e^{-gt/u} = \frac{4}{3}\left(1-\frac{v}{u}\right)$ $\frac{3}{4}e^{-gt/u} = 1-\frac{v}{u}$ $\therefore \frac{v}{u} = 1 - \frac{3e^{-gt/u}}{4}$	1	
iii) Shorter approach $\therefore v = \frac{dx}{dt} = u\left(1-\frac{3}{4}e^{-gt/u}\right)$ $\therefore x = ut - \frac{3u}{4}\left(-\frac{u}{g}\right)e^{-gt/u} + k$ $\therefore x = ut + \frac{3u^2}{4g}e^{-gt/u} + k$ But $x=0$ when $t=0$ $\therefore k = -\frac{3u^2}{4g}$		

X2 **MATHEMATICS: Question** 3   5/5

Suggested Solutions	Marks	Marker's Comments
b) iii) (cont)		
$\therefore x = Ut - \dfrac{3U^2}{4g}\left(1 - e^{-gt/U}\right)$	1	There were very large steps to these given answers. Only the worst were docked ½.
But, from (ii), $v/U = 1 - \frac{3}{4}e^{-gt/U}$ $\therefore v/U - 1/4 = \frac{3}{4}\left(1 - e^{-gt/U}\right)$		
$\therefore x = Ut - \dfrac{U^2}{g}\left(\dfrac{v}{U} - \dfrac{1}{4}\right)$	1	
Longer method (discouraged)		GENERAL
$\ddot{x} = v\dfrac{dv}{dx} = \dfrac{g(U-v)}{U}$ (from (i))		*Very hard to distinguish U from v in many scripts.
$\displaystyle\int \frac{-v\,dv}{U-v} = -\int \frac{g\,dx}{U}$		
$\displaystyle\int \frac{U-v}{U-v} - \frac{U}{U-v}\,dv = -\int \frac{g\,dx}{U}$		* dx means nothing by itself
$v + U\ln(U-v) = -\dfrac{gx}{U} + c$ $(U>v)$		$3dx = 4dt$ is WRONG
(When $x=0$, $v = U/4$ $\therefore c = \dfrac{U}{4} + U\ln\dfrac{3U}{4}$		$\int 3dx = \int 4dt$ makes sense.
$\therefore v + U\ln(U-v) = -\dfrac{gx}{U} + \dfrac{U}{4} + U\ln\dfrac{3U}{4}$	1	
But, from part (ii), $-\dfrac{gx}{U} + \dfrac{U}{4} = v - gt$ $\Rightarrow Uv - Ugt = -gx + U^2/4$ $\Rightarrow x = Ut - \dfrac{U}{g}(v - U/4)$		* Try to explain away absolute values while setting up integral. Once used, they must be explained away.
$\Rightarrow x = Ut - \dfrac{U^2}{g}\left(\dfrac{v}{U} - \dfrac{1}{4}\right)$	1	

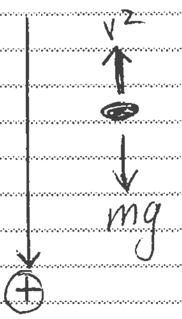
MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments
i)  $m\ddot{x} = -mg - v^2$ By Newton's 2nd Law. $m = 1$ $\ddot{x} = -g - v^2$ $\frac{dv}{d} = -[g + v^2]$ $-\int_u^0 \frac{1}{g+v^2}\,dv = \int_0^T dt$		① Force equation including m sub m = 1 to get $\ddot{x}$ equation ① Form integral equation
when $t = 0$ $v = u$ $t = T$ $v = 0$ $\left[\frac{-1}{\sqrt{g}} \tan^{-1}\frac{v}{\sqrt{g}}\right]_u^0 = \left[t\right]_0^T$ $T = \frac{-1}{\sqrt{g}}\tan^{-1}\left(\frac{0}{\sqrt{g}}\right) - \frac{-1}{\sqrt{g}}\tan^{-1}\left(\frac{u}{\sqrt{g}}\right)$ $T = \frac{1}{\sqrt{10}} \tan^{-1}\left(\frac{u}{\sqrt{10}}\right)$ $g = 10$		① Integrate correctly ① sub values in integrals or find c value. ① Final answer.
ii) $\ddot{x} = -g - v^2$ from above. $v\frac{dv}{dx} = -\left[g + v^2\right]$ $-\int_u^0 \frac{v}{g+v^2}\,dv = \int_0^h dx$ when $v = u$ $x = 0$, $v = 0$ $x = h$ $-\frac{1}{2}\int_u^0 \frac{2v}{g+v^2}\,dv = \left[x\right]_0^h$ $\left[-\frac{1}{2}\ln(g+v^2)\right]_u^0 = h$ $\frac{1}{2}\left[-\ln g + \ln(g+u^2)\right] = h$		① using $v\frac{dv}{dx}$ or $\frac{d(\frac{1}{2}v^2)}{dx}$ ① integral ① integrating ① sub values. * note no need for absolute signs as $g+v^2 > 0$.
$h = \frac{1}{2}\ln\left[\frac{g+u^2}{g}\right]$ $= \frac{1}{2}\ln\left[\frac{10+u^2}{10}\right] = \frac{1}{2}\ln\left[1 + \frac{u^2}{10}\right]$		① Answer from correct working.

MATHEMATICS Extension 2: Question.......

Suggested Solutions	Marks	Marker's Comments

(iii)



$m\ddot{x} = mg - v^2$

By Newton's 2nd Law

$\therefore\ m = 1$

$\ddot{x} = g - v^2$

$\dfrac{v\,dv}{dx} = g - v^2$

$\displaystyle\int_0^w \frac{v}{g - v^2}\,dv = \int_0^h dx$

$x = 0 \quad v = 0, \quad x = h \ v = w$

$\displaystyle -\frac{1}{2}\int_0^w \frac{-2v}{g - v^2}\,dv = \int_0^h dx$

$\left[-\frac{1}{2}\ln(g - v^2)\right]_0^w = \left[x\right]_0^h$

as $\ddot{x} > 0 \quad \therefore\ g - v^2 > 0 \quad g > 0$

$-\frac{1}{2}\ln(g - w^2) + \frac{1}{2}\ln g = h - 0$

$\frac{1}{2}\ln\left(\dfrac{g}{g - w^2}\right) = h$

$h = \frac{1}{2}\ln\left(\dfrac{10}{10 - w^2}\right)$

from (ii) $\quad h = \frac{1}{2}\ln\left[\dfrac{1 + u^2}{10}\right]$

$\therefore\ \dfrac{10}{10 - w^2} = \dfrac{10 + u^2}{10}$

$\dfrac{100}{10 + u^2} = 10 - w^2$

$\therefore\ w^2 = 10 - \dfrac{100}{10 + u^2}$

$= \dfrac{100 + 10u^2 - 100}{10 + u^2}$

$w^2 = \dfrac{10u^2}{10 + u^2}$

Now $\ddot{x} \geq 0$ Terminal velocity occurs when $\ddot{x} = 0$

$g - v^2 \geq 0$ for all v

$\therefore\ 10 - w^2 \geq 0$

$1 - \dfrac{w^2}{10} \geq 0$

Marker's Comments

(1) Force equation including m, sub $m = 1$ to get $\ddot{x}$ equation and must show positive direction

(1) integral

(1) integration and sub values

*note: no absolute signs needed as $g - v^2 > 0$

(1) equate h expressions and simplify to given answer

Proof $1 - \dfrac{w^2}{10} \geq 0$

(1) Proof must refer to $\ddot{x} \geq 0$