	Year 12 Mathematics (Extension 2) **HSC ASSESSMENT TASK 3** **TERM 2, Week 6, 2006**	Name: ____________________ Teacher: ___________________

RM

Monday 5th June 2006

- Attempt **ALL** questions. Marks may be deducted for careless, insufficient, or illegible work. Calculators may be used. Total possible mark is **50**.
- Begin each question on a new sheet of paper.
- **TIME ALLOWED:** 95 minutes

Question 1: **(18 marks)**

(a) Find the indefinite integral for :-

$$\int \frac{\ln x}{x}\, dx$$ **[1]**

(b) By making an appropriate substitution find:- **[2]**

$$\int x\sqrt{3x-1}\, dx$$

(c)

i) Find real numbers a and b such that for all values of t, **[2]**

$$\frac{1}{(2-t)(1+2t)} = \frac{a}{2-t} + \frac{b}{1+2t}$$

ii) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ and the identity in part (i) to find **[4]**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta}$$

(d) Find the indefinite integral for :-

$$\int \frac{x+3}{\sqrt{x^2-2x+5}}\, dx$$ **[4]**

(Hint : you may need to use the table of standard integrals to complete your answer)

Question 1 continued over page

(e)

i) Find the indefinite integral for :- **[2]**

$$\int \frac{x^3}{x^2+1}\,dx$$

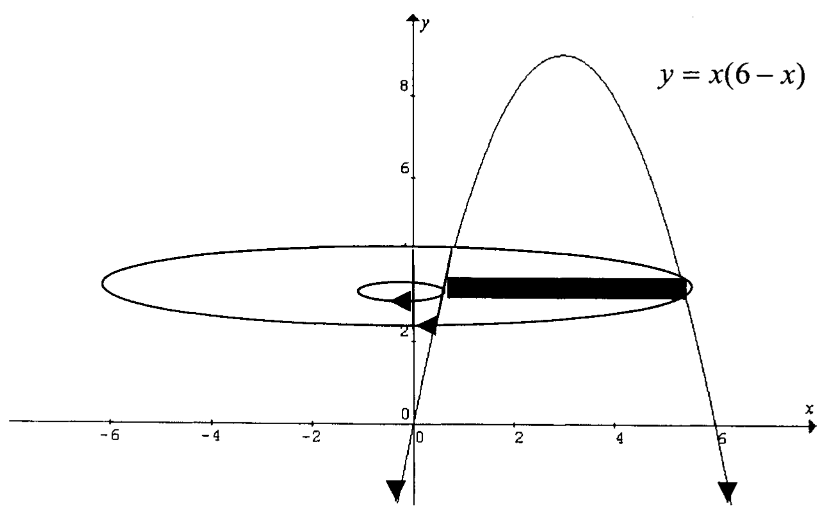
(Hint: degree of numerator is greater than degree of denominator)

ii) By first integrating using parts and then using the result from part i) above evaluate the definite integral:- **[3]**

$$\int_0^1 x^2 \tan^{-1} x\,dx$$

Question 2: **(START NEW PAGE)** **(18 Marks)**

(a) The area bounded by the curve $y = x(6-x)$ and the x-axis is rotated about the y-axis.

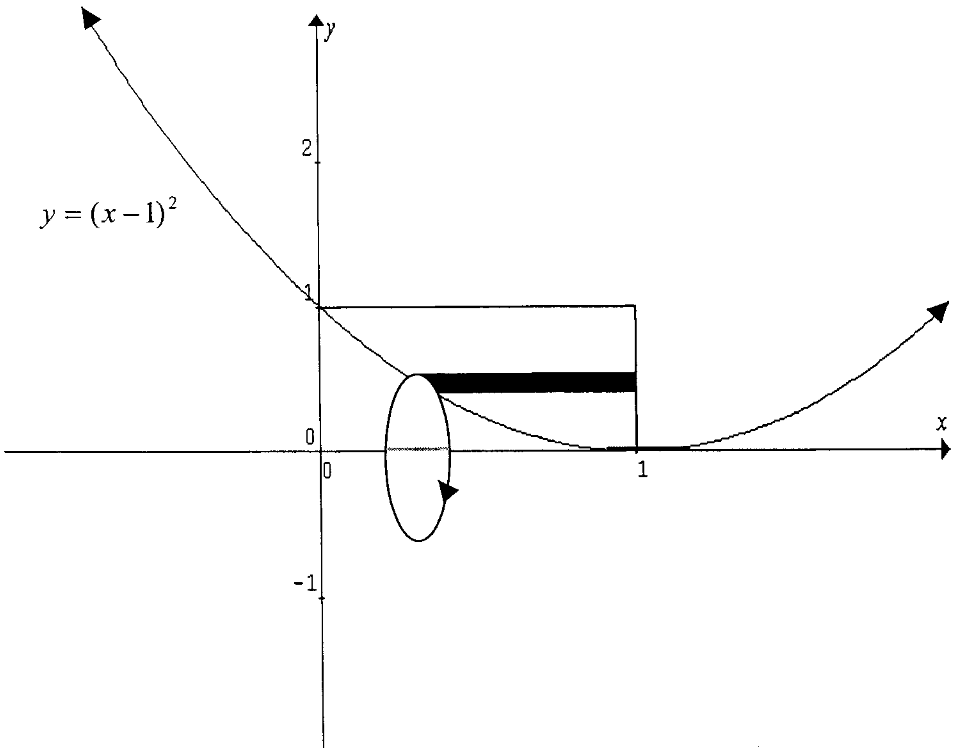


i) Use strips perpendicular to the axis of rotation and show that the x coordinates of the end points of these strips are $3-\sqrt{9-y}$ and $3+\sqrt{9-y}$. **[2]**

ii) Hence find the volume of the solid of revolution in terms of π. **[4]**

Question 2 continued over page

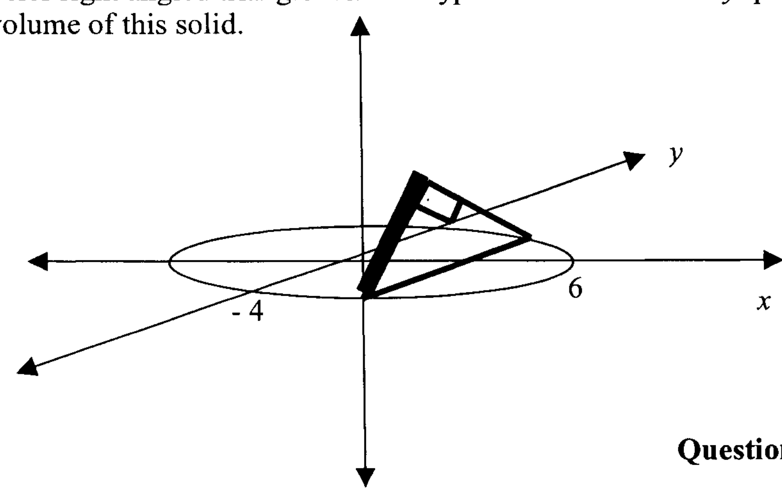
b) The region bounded by the curve $y=(x-1)^2$, $x=1$, and $y=1$ is rotated about the x-axis.



i) Use the method of cylindrical shells to find the volume of one shell δV. **[2]**

ii) Hence find the volume (in terms of π) of the solid described above. **[4]**

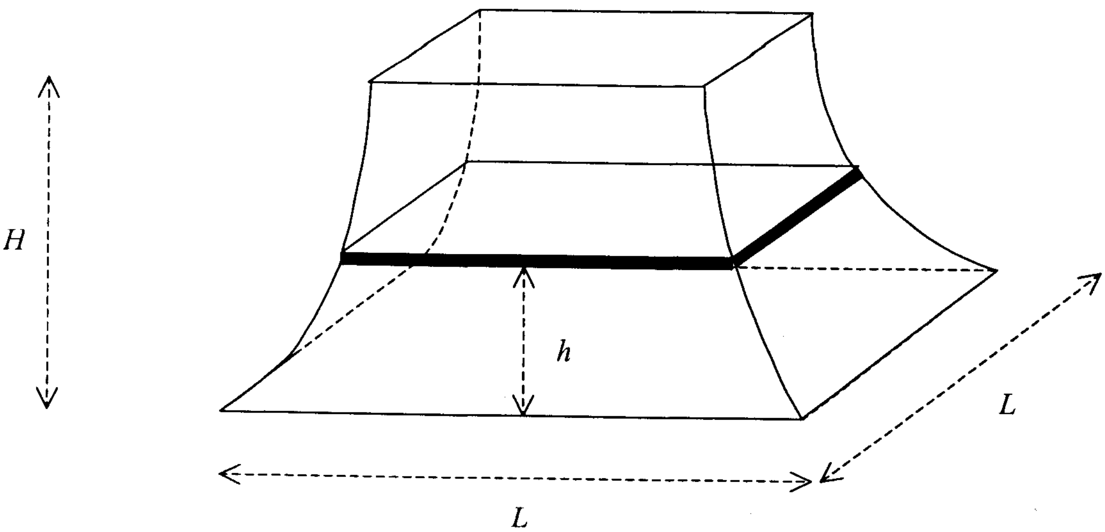
c) The base of a solid is the area of a region bounded by the ellipse whose equation is $4x^2+9y^2=144$ (note: its cuts the x-axis at 6 and –6 and the y-axis at 4 and –4). Each cross-section of the solid formed by a plane perpendicular to the x and y plane is an isosceles right angled triangle with its hypotenuse in the x and y plane. Find the volume of this solid. **[6]**



Question 3 over page

Question 3 **(START NEW PAGE)** **(14 Marks)**

a)



A wooden block of height H cm has the shape of a flat-topped square "pyramid" With curved sides as shown above. The cross-section h cm above the base is a square with the sides parallel to the sides of the base and of length $l(h) = \dfrac{L}{\sqrt{h+1}}$.

Find the volume (correct to the nearest cm) of the wooden block given that $H = L = 30\,cm$. **[6]**

b)

i) If $I_n = \int \cos^n x \, dx$ show that $I_n = \dfrac{1}{n}\cos^{n-1} x \sin x + \dfrac{n-1}{n} I_{n-2}$. **[6]**

ii) Using the substitution $x = \cos\vartheta$ and result from part i) above evaluate **[4]**

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

END OF TASK

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \,, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q1 (a) $\int \frac{\ln x}{x}\,dx$ let $u = \ln x$ $du = \frac{1}{x}dx$ $I = \int u\,du = \frac{u^2}{2} + c$ $= \frac{(\ln x)^2}{2} + c$ ✓	Can go straight to answer without working	$I = \int_0^1 \frac{1}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2\,dt}{(t^2+1)}$ $= \int_0^1 \frac{dt}{2+3t-2t^2}$ ✓ $= \frac{1}{5}\int_0^1 \frac{1}{(2-t)} + \frac{2}{(1+2t)}\,dt$ $= \frac{1}{5}\left[\ln\left\|\frac{1+2t}{2-t}\right\|\right]_0^1$ ✓ $= \frac{1}{5}\left(\ln 3 - \ln\frac{1}{2}\right)$ $= \frac{1}{5}\ln 6$ ✓	
(b) let $u = 3x-1$ $du = 3\,dx$ $\therefore I = \frac{1}{3}\int \frac{(u+1)}{3}\sqrt{u}\,du$ ✓ $= \frac{1}{9}\int u^{3/2} + u^{1/2}\,du$ $= \frac{2}{9}\left(\frac{(3x-1)^{5/2}}{5} + \frac{(3x-1)^{3/2}}{3}\right) + c$ ✓		(d) $I = \int \frac{x+3}{\sqrt{x^2-2x+5}}\,dx$ $I = \frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+5}} + 4\int \frac{dx}{\sqrt{x^2-2x+5}}$ ✓ ✓ $= \sqrt{x^2-2x+5} + 4\int \frac{dx}{\sqrt{(x-1)^2+4}}$ ✓ $= \sqrt{x^2-2x+5} + 4\log\left\|(x-1)+\sqrt{x^2-2x+5}\right\| + c$ ✓	
(c) (i) $1 = a(1+2t) + b(2-t)$ let $t=2 \Rightarrow 1 = 5a \Rightarrow a = \frac{1}{5}$ ✓ $t = -\frac{1}{2} \Rightarrow 1 = \frac{5}{2}b \Rightarrow b = \frac{2}{5}$ ✓ ii.) $t = \tan\frac{\theta}{2}$ $dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$ $= \frac{1}{2}(\tan^2\frac{\theta}{2}+1)\,d\theta$ $= \frac{1}{2}(t^2+1)\,d\theta$ $d\theta = \frac{2\,dt}{t^2+1}$ ✓ $\theta = \frac{\pi}{2} \Rightarrow t = 1$ $\theta = 0 \Rightarrow t = 0$			

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(e)

(i) $\int \frac{x^3}{x^2+1}$

$= \int x - \frac{x}{x^2+1}\,dx$ ✓

$= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + c$ ✓

(ii) $I = \int_0^1 x^2 \tan^{-1}x\,dx$

$u = \tan^{-1}x \quad du = \frac{1}{1+x^2}$

$dv = x^2 \quad v = \frac{1}{3}x^3$

$\therefore I = \left[\frac{1}{3}x^3\tan^{-1}x\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx$ ✓

$= \left[\frac{1}{3}x^3\tan^{-1}x\right] - \frac{1}{3}\int_0^1 x - \frac{x}{x^2+1}\,dx$

$= \left[\frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1)\right]_0^1$ ✓

$= \frac{1}{12}\left(\pi - 2 + 2\ln 2\right)$ ✓

Q2 (a) (i) $y = 6x - x^2$

$= -(x^2-6x+9)+9$ ✓

$= -(x-3)^2+9$

$\therefore (x-3)^2 = 9-y$ ✓

$x = 3 \pm \sqrt{9-y}$

(OR)

$x^2 - 6x - y = 0$ ✓

$x = \frac{6 \pm \sqrt{36-4y}}{2}$ ✓

$= 3 \pm \sqrt{9-y}$

ii.) $\delta V = \pi(R^2 - r^2)\,\delta y$

$= \pi(x_2^2 - x_1^2)\,\delta y$

$= \pi\left[(3+\sqrt{9-y})^2 - (3-\sqrt{9-y})^2\right]\delta y$ ✓

$V = \pi\int_0^9 (3+\sqrt{9-y})^2 - (3-\sqrt{9-y})^2\,dy$ ✓

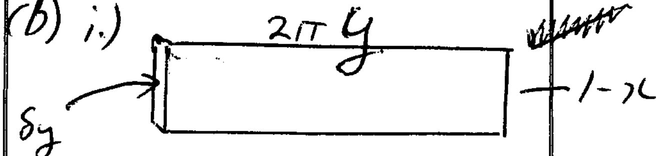
$= \pi\int_0^9 9+(9-y)+6\sqrt{9-y} - 9 + 6\sqrt{9-y} - (9-y)\,dy$

$= \pi\int_0^9 12\sqrt{9-y}$ ✓

$= 12\pi\left[-\frac{2(9-y)^{3/2}}{3}\right]_0^9$

$= 216\pi$ ✓

(b) i.)



$\delta V = 2\pi y(1-x)\,\delta y$ ✓

NOTE: $y = (x-1)^2$

$(x-1) = \pm\sqrt{y}$

$x = 1 \pm \sqrt{y}$ ✓

$\Rightarrow x = 1 - \sqrt{y}$

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ii).

$$V = 2\pi \int_0^1 y(1-x)\,dy$$

$$= 2\pi \int_0^1 y\left(1-(1-\sqrt{y})\right)dy$$

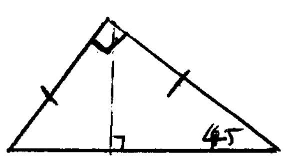
$$= 2\pi \int_0^1 y\sqrt{y}\,dy$$

$$= 2\pi \int_0^1 y^{3/2}\,dy$$

$$= 2\pi \left[\frac{2y^{5/2}}{5}\right]_0^1$$

$$= \frac{4\pi}{5} \text{ units}^3$$

(c)



perp. height $= y$

$\therefore$ Area $= \frac{1}{2}(2y)y = y^2$

$$\delta V = y^2\,\delta x$$

$$\therefore V = 2\int_0^6 y^2\,dx$$

$$= \frac{8}{9}\int_0^6 36 - x^2\,dx$$

$$= \frac{8}{9}\left[36x - \frac{x^3}{3}\right]_0^6$$

$$= \frac{8}{9} \times 216\left[1 - \frac{1}{3}\right]$$

$$= \frac{8}{9} \times 216 \times \frac{2}{3}$$

$$= 128\, u^3$$

Q3

(a) Area of cross-section

$$= s^2 = \left(\frac{L}{\sqrt{h+1}}\right)^2$$

$$= \frac{L^2}{h+1}$$

$$\therefore \delta V = \frac{L^2}{h+1}\,\delta h$$

$$V = \int_0^H \frac{L^2}{h+1}\,dh$$

$$= L^2\left[\ln(h+1)\right]_0^H$$

$$= L^2 \ln(H+1)$$

$$L = H = 30$$

$$\therefore V = 30^2 \ln 31$$

$$\doteqdot 3091 \text{ cm}^3$$

NOTE: $9y^2 = 144 - 4x^2$

$$y^2 = \frac{1}{9}(144 - 4x^2) = \frac{4}{9}(36 - x^2)$$

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Q3 b)

i) $I_n = \int \cos^n x \, dx$

$= \int \cos^{n-1} x \cos x \, dx$

$u = \cos^{n-1} x \quad du = (n-1)\cos^{n-2} x \sin x$

$dv = \cos x \quad v = \sin x$

$\therefore I_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$

(ii) $x = \cos\theta \quad dx = -\sin\theta \, d\theta$

$x = 1 \Rightarrow \theta = 0$

$x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore \int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx = -\int_{\frac{\pi}{2}}^{0} \frac{\cos^3\theta}{\sqrt{1-\cos^2\theta}} \times \sin\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta = I_3$

$I_3 = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} I_1$

$I_1 = \int \cos x \, dx = \sin x$

$\therefore I_3 = \left[\frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x \right]_0^{\frac{\pi}{2}}$

$= \frac{2}{3}$