Question 1 (20 marks)

a) Find i) $\displaystyle\int \frac{x+4}{x^3+4x}\,dx$ **4**

ii) $\displaystyle\int \sec^2 x \sin x\, dx$ **3**

iii) $\displaystyle\int_0^{\sqrt{2}-1} \frac{dx}{\sqrt{3-x^2-2x}}$ **3**

b) Given that $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n \vartheta\, d\vartheta$, show that $I_n = \dfrac{n-1}{n} I_{n-2}$.

Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 \vartheta\, d\vartheta$. **4**

c) Using the method of cylindrical shells, find the volume of the solid of revolution formed when the region bounded by $y = 3x - x^2$ and $y = x$ is rotated **around the y-axis**. **3**

d) The base of a certain solid is the circle $x^2 + y^2 = 9$. Cross-sections cut perpendicular to the y-axis are squares with one side in the base. Find the volume of the solid. **3**

Question 2 on page 2.

Question 2 (20 marks) **Start this question on a new page.**

a) A particle is thrown vertically upward at a velocity of 60m/s. The retardation due to air resistance is proportional to the square of the velocity with the constant of proportionality being 0.01 (i.e. total resistance, $R = 0.01mv^2$).

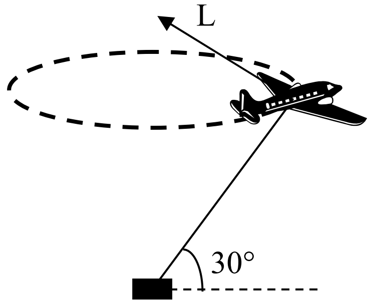
Show that

i) the time taken for the particle to reach its maximum height is given by $t = \frac{10}{\sqrt{g}} \tan^{-1}(\frac{6}{\sqrt{g}})$ **3**

ii) the maximum height reached is given by

$$h = 200 \ln(\frac{g+36}{g})$$ **3**

b) A model plane of mass 5kg attached to the end of an inelastic wire, of length 2m, whose other end is fixed, flies in a horizontal circle of elevation 30°. (Take $g = 9.8ms^2$)



If the lift force, L, acts at right angles to the wire and L is twice the weight of the plane,

i) Draw a diagram clearly showing the forces acting on the plane. **1**
ii) Find the tension in the wire in Newtons, correct to 2 decimal places. **2**
iii) Find the speed of the plane in m/s, correct to 1 decimal place. **2**

c) A car of mass 1tonne travels in a level curve of radius 160m at a constant speed of 70km/h.

i) Calculate the friction between the wheels and the road surface, correct to the nearest Newton. **2**
ii) Assuming the road is banked at an angle of 10°, calculate the frictional force between the road surface and the wheels, correct to the nearest Newton (take $g = 9.8ms^2$). **4**

iii) If the road were banked at 10°, find the speed at which no frictional force is experienced, correct to the nearest m/s. **3**

END OF PAPER

EXT 2 ASSESS TERM 3 Solutions

QUESTION 1

a) i/ $\int \frac{x+4}{x^3+4x}\,dx = \int \frac{x+4}{x(x^2+4)}\,dx$

$$\frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$x+4 = A(x^2+4) + x(Bx+C)$$

$$= Ax^2 + Bx^2 + Cx + 4A$$

1- Partial fractions

$\therefore A+B=0$

$C=1$

$4A=4$

$A=1 \quad \therefore B=-1$

$$\therefore \int \frac{x+4}{x^3+4x}\,dx = \int \frac{1}{x} - \frac{x-1}{x^2+4}\,dx$$

$$= \ln x - \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

1- split second numerator

1- correct ln primitive

1- correct $\tan^{-1}$ primitive

$$= \ln x - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

/4

ii/ $I = \int \sec^2 x \sin x\,dx$

Let $u = \sin x \quad v' = \sec^2 x$

$u' = \cos x \quad v = \tan x$

1- separation into parts

$\therefore I = \sin x \tan x - \int \cos x \tan x\,dx$

1- uses correct formula

$= \sin x \tan x - \int \sin x\,dx$

$= \sin x \tan x + \cos x + c$

1- correct answer

OR

$= \int \sec x \tan x\,dx$

$= \sec x + c$

OR

$\int \sec^2 x \sin x\,dx = \int \frac{\sin x}{\cos^2 x}\,dx$ Let $v = \cos x$, $v' = -\sin x$

$= \frac{1}{\cos x} + c$

OR
1- rearrange
1- balance
1- correct answer

/3

iii/ $\int_0^{\sqrt{2}-1} \frac{dx}{\sqrt{3-x^2-2x}} = \int_0^{\sqrt{2}-1} \frac{dx}{\sqrt{4-(x+1)^2}}$

1- complete square in denom

$$= \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{2}-1}$$

1- correct $\sin^{-1}$ primitive

$$= \left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}\frac{1}{2}\right)$$

1- correct evaluation

$$= \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}$$

/3

b) $I_n = \int_0^{\pi/2} \sin x \sin^{n-1} x \, dx$

Let $u = \sin^{n-1} x \quad v' = \sin x$

$u' = (n-1)\sin^{n-2} x \cos x \quad v = -\cos x$

1- Integ by parts

$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx$

$= 0 + (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x \, dx$

1 – simplification to given form

$= (n-1) I_{n-2} - (n-1) I_n$

$\therefore n I_n = (n-1) I_{n-2}$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$

Hence,

$\int_0^{\pi/2} {}^5 x \, dx = I_5$

$= \frac{4}{5} I_3$

1- uses reduction formula

$= \frac{4}{5} \cdot \frac{2}{3} I_1$

$= \frac{8}{15}\int_0^{\pi/2} \sin x \, dx$

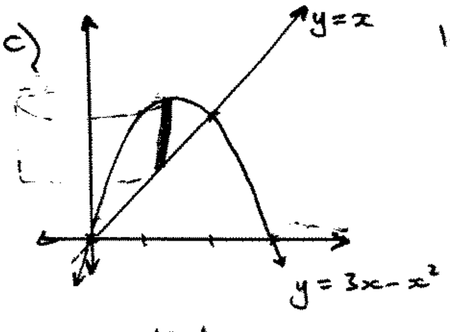
1– correct answer.

$= \frac{8}{15}\left[\cos x\right]_0^{\pi/2}$

$= \frac{8}{15}$

/4

c)



Intersection

$x = 3x - x^2$

1- finds intersection

$0 = 2x - x^2$

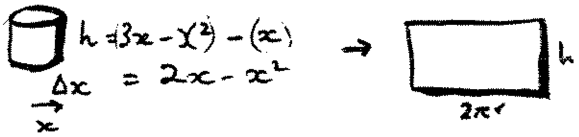
$0 = x(2-x)$

1– ΔV

$\therefore x = 0, x = 2$

1– correct volume.

One Cylinder



$h = (3x - x^2) - (x)$
$\Delta x \quad = 2x - x^2$
x

$\Delta V = 2\pi x \,(2x - x^2)\,\Delta x$

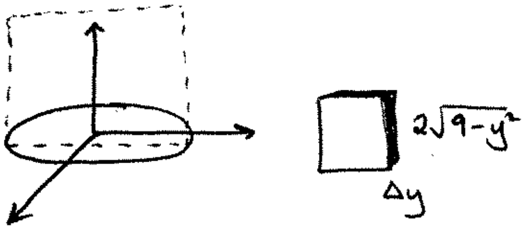
$\therefore V = 2\pi\int_0^2 2x^2 - x^3 \, dx$

$= \pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$

$= \frac{8\pi}{3} u^3$

/3

d)



$$\Delta V = \left(2\sqrt{9-y^2}\right)^2 \Delta y$$

$$V = 4\int_{-3}^{3} 9 - y^2 \, dy$$

$$= 8\left[9y - \frac{y^3}{3}\right]_0^3$$

$$= \underline{144 u^3}$$

/3

1- side length

1- ΔV w.r.t. y.

1- correct volume.

2 (a) $\ddot{x} = \dfrac{-v^2 - 100g}{100}$ ① a correct expression for $\ddot{x}$

$$\frac{dt}{dv} = \frac{-100}{v^2+100g}$$

$$\int_0^T dt = \int_{60}^{0} \frac{-100\,dv}{v^2+100g}$$

$$T = 100\int_0^{60} \frac{dv}{v^2+100g}$$ ① Time = a correct definite integral

$$= \frac{100}{10\sqrt{g}}\left[\tan^{-1}\frac{v}{10\sqrt{g}}\right]_0^{60}$$ ① Correct integration leading to the answer

$$= \frac{10}{\sqrt{g}}\tan^{-1}\frac{6}{\sqrt{g}}$$

Note: If done as an indefinite integral 2nd mark is for evaluating the constant.

(b) $v\dfrac{dv}{dx} = \dfrac{-v^2-100g}{100}$ ①

$$\frac{dx}{dv} = \frac{-100v}{v^2+100g}$$

$$\int_0^H dx = \int_{60}^{0} \frac{-100v}{v^2+100g}\,dv$$ ①

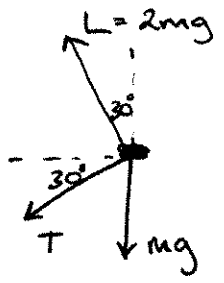
$$H = 50\int_0^{60} \frac{2v}{v^2+100g}\,dv$$

$$= 50\left[\log(v^2+100g)\right]_0^{60}$$ ①

$$= 50\log\left(\frac{36+g}{g}\right)$$

no penalty.

b) i/



/1

1 - all forces labelled.

ii/ Vertically: $L\cos 30° - mg - T\sin 30° = 0$

$$2mg \cdot \frac{\sqrt{3}}{2} - mg = \frac{1}{2}T$$

$$T = 2mg(\sqrt{3}-1)$$

$$= 2 \times 5 \times 9.8(\sqrt{3}-1)$$

$$= \underline{71.74\text{N}}$$

/2

1 - vertical components

1 - answer

iii/ Horizontally: $\frac{mv^2}{r} = L\sin 30° + T\cos 30°$

$r = 2\cos 30°$

$= \sqrt{3}$

$$v^2 = \frac{r}{m}\left(2mg \cdot \frac{1}{2} + T \cdot \frac{\sqrt{3}}{2}\right)$$

$$v^2 = 38.496$$

$$\underline{v = 6.2\text{m/s}}$$

/2

1 - horizontal components

1 - answer

c) i/ $m = 1000$

$v = \frac{70}{3.6}$ m/s

$r = 160$m

$$F = \frac{mv^2}{r}$$

$$= \frac{1000 \times \left(\frac{70}{3.6}\right)^2}{160}$$

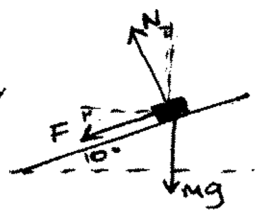
$$= \underline{2363\text{N}}$$

/2

1 - formula

1 - answer

ii/



Horizontally: $\frac{mv^2}{r} = F\cos 10° + N\sin 10°$ ①

Vertically: $N\cos 10° - F\sin 10° - mg = 0$ ②

$$\therefore \quad N = \frac{F\sin 10° + mg}{\cos 10°}$$

Into ① $\frac{mv^2}{r} = F\cos 10° + \tan 10°(F\sin 10° + mg)$

$$F(\cos 10° + \tan 10° \sin 10°) = \frac{mv^2}{r} - mg\tan 10°$$

$$F = \frac{\frac{mv^2}{r} - mg\tan 10°}{\cos 10° + \tan 10° \sin 10°}$$

$$= \underline{625\text{ N}}$$

/4

1 - horiz. components of force

1 - vert. components

1 - correct rearrangement

1 - answer

iii/ $F = 0$

Vertically: $N\cos 10° - mg = 0$

$$N = \frac{mg}{\cos 10°}$$

Horizontally: $\frac{mv^2}{r} = N\sin 10°$

$$\therefore \frac{mv^2}{r} = mg\tan 10°$$

$$\therefore v^2 = rg\tan 10°$$

$$\underline{v = 32 \text{ m/s}}$$

/3

1 - state $F = 0$

1 - resolution of components

1 - answer.

QUESTION 2

a) $\overset{+}{\uparrow} \quad \underset{mg}{\downarrow} \quad \underset{R}{\downarrow}$

$$F = -mg - R$$
$$ma = -mg - 0.01mv^2$$
$$a = -g - 0.01v^2$$

i) $a(v) \to v(t)$ (max ht when $v = 0$)

$$a = \frac{dv}{dt}$$

1 – use $a = \frac{dv}{dt}$

$$\frac{dv}{dt} = -g - 0.01v^2$$
$$\frac{dt}{dv} = \frac{-1}{g + 0.01v^2}$$

$$t = -\int_{60}^{0} \frac{1}{g + 0.01v^2}\,dv$$

1 – correct integral

$$= \left[\frac{1}{0.01} \cdot \frac{1}{\sqrt{g/0.01}} \tan^{-1}\left(\frac{v}{\sqrt{g/0.01}}\right)\right]_0^{60}$$

$$= \frac{\sqrt{0.01}}{\sqrt{g}} \cdot \frac{1}{0.01} \tan^{-1}\left(\frac{60 \times 0.1}{\sqrt{g}}\right)$$

1 – rearrange

$$= \frac{10}{\sqrt{g}} \tan^{-1}\left(\frac{6}{\sqrt{g}}\right)$$

/3

ii) $a(v) \to v(x)$

$$a = v\frac{dv}{dx}$$

1 – use $a = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -g - 0.01v^2$$
$$\frac{dv}{dx} = \frac{-g - 0.01v^2}{v}$$

1 – correct integral

$$\frac{dx}{dv} = \frac{-v}{g + 0.01v^2}$$

$$x = -200\int_{60}^{0} \frac{0.02v}{g + 0.01v^2}\,dv$$

$$= 200\left[\ln(g + 0.01v^2)\right]_0^{60}$$

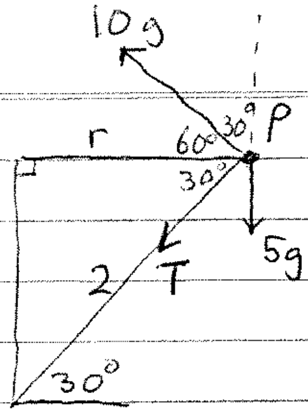
1 – rearrange

$$= 200 \ln\left(\frac{g + 36}{g}\right)$$

/3

(b) (i)

$r = 2\cos 30^\circ$
$= \sqrt{3}$



① forces

(ii) Vertical forces at P

$$10g\cos 30^\circ = T\sin 30^\circ + 5g$$ ① 3 vertical forces

$$T = 10g(\sqrt{3}-1)$$

$$\approx 71{\cdot}74 \text{ N}$$ ①

(iii) Horizontal forces at P

$$10g\sin 30^\circ + T\cos 30^\circ = \frac{5v^2}{r}$$ ① Σ 2 forces $= \frac{mv^2}{r}$

$$5g + \frac{T\sqrt{3}}{2} = \frac{5v^2}{\sqrt{3}}$$

$$v^2 = \frac{\sqrt{3}}{5}\left(5g + \frac{T\sqrt{3}}{2}\right)$$

$$v \approx 6{\cdot}2 \text{ ms}^{-1}$$ ①

(c) $m = 1000\text{ kg}$

$r = 160\text{ m}$

$v = \dfrac{70 \times 1000}{60 \times 60}$

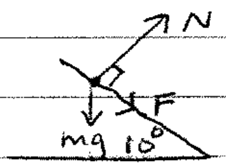
$= \dfrac{175}{9}\text{ ms}^{-1}$

(i) $F = \dfrac{mv^2}{r}$ 1

$= \dfrac{1000 \times \left(\frac{175}{9}\right)^2}{160}$

$\approx 2363\text{ N}$ 1

(ii)



Horizontal

$N\sin 10^\circ + F\cos 10^\circ = \dfrac{mv^2}{r}$ — 1 1

Vertical

$N\cos 10^\circ - F\sin 10^\circ = mg$ — 2 1

$1 \times \cos 10^\circ - 2 \times \sin 10^\circ$ 1

$F(\cos^2 10^\circ + \sin^2 10^\circ) = \dfrac{mv^2}{r}\cos 10^\circ - mg\sin 10^\circ$

$F \approx 625\text{ N}$ 1

(iii) $F = 0$

$\therefore N\sin 10^\circ = \dfrac{mv^2}{r}$ — 3

$N\cos 10^\circ = mg$ — 4

$3 \div 4$ $\quad \tan 10^\circ = \dfrac{v^2}{gr}$ 1

$v^2 = gr\tan 10^\circ$, $v \approx 17\text{ ms}^{-1}$ 1