NEWINGTON COLLEGE

IN FIDE SCIENTIAM

Assessment 3 2008

HSC Extension 2 MATHEMATICS

Time allowed - 2 Hours

DIRECTIONS TO CANDIDATES:

- All questions are NOT of equal value.
- All questions may be attempted.
- In every question, show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Approved non-progammable calculators may be used.
- A table of standard integrals is provided for your convenience.
- The answers to the five questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2, etc.
- Each bundle must show the candidate's computer number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Unless otherwise stated candidates should leave their answers in simplest exact form.

Outcomes to be Assessed:

E3 Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 Uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

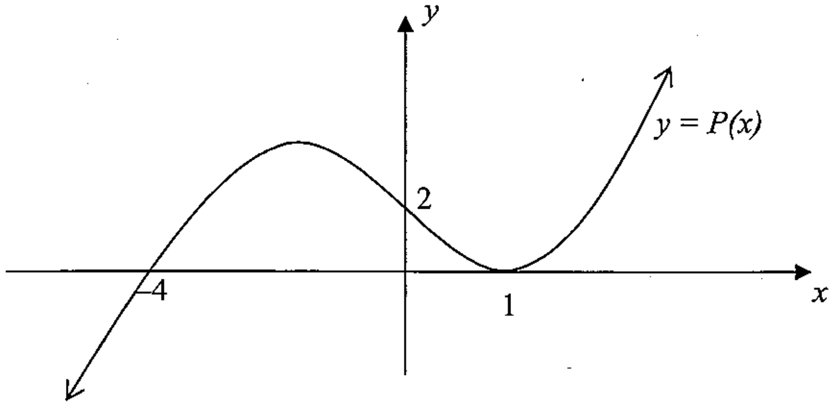
E6 Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 Uses the techniques of slicing and cylindrical shells to determine volumes.

E8 Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

Question 1 *(20 marks)* **Marks**

a) For the polynomial $y = P(x)$ **10**



On separate diagrams, sketch the graphs of, showing the intercepts on the coordinate axes and the equations of any asymptotes.

(i) $y = |P(x)|$

(ii) $y = P(|x|)$

(iii) $y = \dfrac{1}{P(x)}$

(iv) $y = \sqrt{P(x)}$

(v) $y^2 = P(x)$

b) Sketch $y = \dfrac{x^2 - 1}{(x+2)(x-3)}$, showing clearly the points of intersections with the coordinate axes, coordinates of any turning points and equations of any asymptotes **10**

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100

Question 2 *(18 marks)* Start this question on a new page.

a) Find: $\displaystyle\int \operatorname{cosec} x \, dx$ by using the substitution $t = \tan\dfrac{x}{2}$. **3**

(0,1)

Question 2 continued. **Marks**

b) If $I_n = \int_0^1 x^n e^x dx$,where n is a positive integer, show that **4**

$I_{n+1} = e-(n+1)I_n$.

Hence evaluate $\int_0^1 t^3 e^t dt$, leaving your answer in terms of e.

c) Use partial fractions to find the $\int \frac{x^2}{(x+1)(x+2)}dx$. **4**

d) Use integration by parts to evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}dx$. **3**

e) Using the substitution $x = 4\sin^2\theta$ or otherwise show that **4**

$\int_0^2 \sqrt{x(4-x)}dx = \pi$.

Question 3 *(25 marks)* Start this question on a new page.

a) For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ **5**

Find:

(i) The eccentricity

(ii) The coordinates of the foci

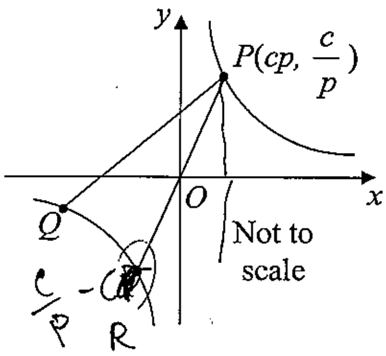
(iii) The equations of the directrices.

b) For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, (where $a > b > 0$). Show that the tangent at the point $P(a\sec\theta, b\tan\theta)$ has equation $bx\sec\theta - ay\tan\theta = ab$. **3**

Question 3 continued. **Marks**

c) Let S and S' be the foci of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ and P be an arbitrary point on the hyperbola. Prove that $|PS'-PS|=2a$. **3**

d) The point $P(cp,\dfrac{c}{p})$, where $p \neq \pm 1$, is a point on the hyperbola $xy=c^2$, and the normal to the hyperbola at P intersects the second branch at Q. The line through P and the origin O intersects the second branch at R. **9**



(i) Show that the equation of the normal at P is $py-c=p^3(x-cp)$.

(ii) Show that $x^2-c(p-\dfrac{1}{p^3})x-\dfrac{c^2}{p^2}=0$ is the equation to be used to find the x coordinates of the points P and Q.

(iii) Find the coordinates of Q and R, and deduce that the $\angle QRP$ is a right angle.

e) The line $y=mx+c$ is a tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. **5**

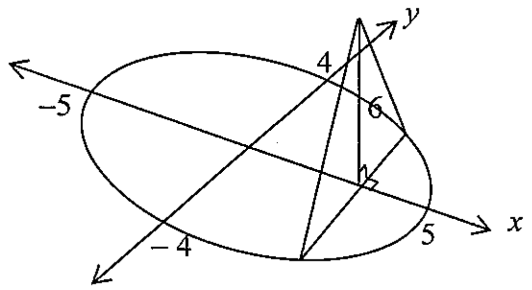
(i) Show that $c^2=a^2m^2+b^2$.

(ii) Show that the pair of tangents drawn from the point $(3, 4)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ are at right angles to one another.

Question 4 *(15 marks)* Start this question on a new page. **Marks**

a) A solid is formed by rotating the region bounded by the curve $y=-x^2+2x$ and the line $y=0$ about the $y-$axis. Use the method of cylindrical shells to find the volume of this solid. **3**

b) A solid has a base in the form of an ellipse with major axis 10 units and minor axis 8 units. Every section perpendicular to the major axis is an isosceles triangle with altitude *6* units. **4**



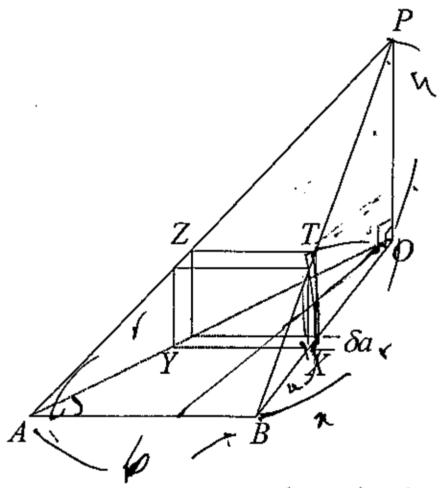
(i) Show that the area of a perpendicular section is $A=\frac{24}{5}\sqrt{25-x^2}$, where x is the distance from the centre of the ellipse along the major axis.

(ii) Show that the volume of the solid is 60π cubic units.

Question 4 continued. **Marks**

c) Let ABO be an isosceles triangle, $AO = BO = r$, $AB = b$. Let $PABO$ be a triangular pyramid with height $OP = h$ and OP perpendicular to the plane of ABO as in the diagram. Consider a slice S of the pyramid of width δa as in the diagram. The slice s is perpendicular to the plane of ABO at XY with $YX \parallel AB$ and $XB = a$. Note that $XT \parallel OP$. **4**

$r - \sqrt{r^2 - \left(\frac{1}{2}b\right)^2}$



$h -$

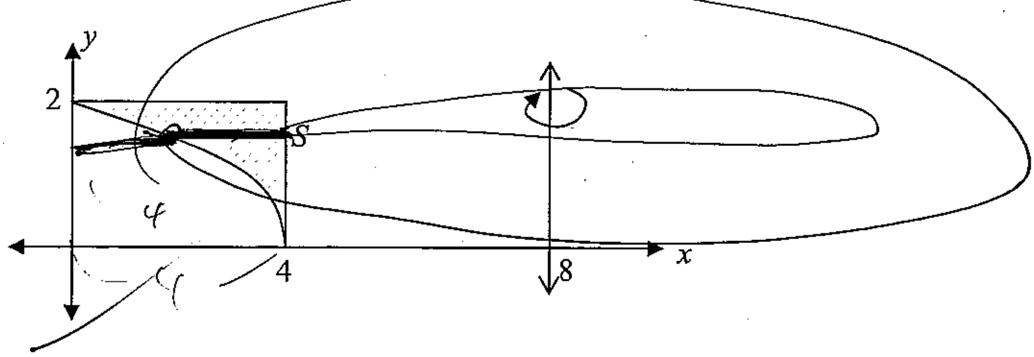
$r^2 - b^2$

$\sqrt{r^2 - \left(\frac{1}{2}b\right)^2}$

(i) Show that the volume of S is $\left(\frac{r-a}{r}\right)b\left(\frac{ah}{r}\right)\delta a$, when δa is small. (You may assume that the slice is approximately a rectangular prism of base $XYZT$ and height δa.)

(ii) Hence show that the pyramid $PABO$ has volume $\frac{hbr}{6}$ units^3.

d) The area bounded by $y = \sqrt{4-x}$, $x = 4$ and $y = 2$ is rotated about $x = 8$. **4**
When the region is rotated, the line ~~segment S sweeps out~~ an annulus.



(i) Show that the area of the annulus is given by:
$\pi\left(y^4 + 8y^2\right)$ square units.

(ii) Hence, find the volume of the solid.

Question 5 *(17 marks)* Start this question on a new page. **Marks**

a) (i) Show that $z = i$ is a root of the equation **3**

$(2 - i)z^2 - (1 + i)z + 1 = 0.$

(ii) Find the other root of the equation in the form $z = a + ib$, where a and b are real numbers.

b) (i) Use De Moivre's Theorem to show that **7**

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) Deduce $8x^3 - 6x - 1 = 0$ has solution $x = \cos\theta$ where $\cos 3\theta = \dfrac{1}{2}$.

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$.

(iv) Hence evaluate $\cos\dfrac{\pi}{9}\cos\dfrac{2\pi}{9}\cos\dfrac{4\pi}{9}$.

c) The hyperbola $xy = c^2$ touches the circle $(x - 1)^2 + y^2 = 1$ at the point Q. **7**

(i) Show this information on a sketch.

(ii) Show that if β is a repeated root of any polynomial equation $P(x) = 0$ then β is also a root of $P'(x) = 0$.

(iii) Show that the point of intersection of the circle and the hyperbola is found by using the polynomial, $P(x) = x^2(x - 1)^2 - x^2 + c^4$.

(iv) Explain why $P(x) = x^2(x - 1)^2 - x^2 + c^4$ has a repeated real root $\beta > 0$, and two non-real complex roots.

(v) Find the value of β, c^2 and the remaining roots of the polynomial $P(x) = x^2(x - 1)^2 - x^2 + c^4$.

End of Paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$