NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 2 ASSESSMENT

2006

MATHEMATICS

EXTENSION COURSE 2

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 32% of the HSC Assessment Mark

<u>Question 1</u> (21 Marks)

a) Find :

i. $\int (\cos x + \sin x) e^{(\cos x - \sin x)}\, dx$ **2**

ii. $\int \tan^3 x \ dx$ **3**

iii. $\int \cos^{-1} x \, dx$ **3**

b)

i. Find real constants a, b and c such that

$$\frac{1}{(x^2+1)(x+1)} \equiv \frac{ax+b}{x^2+1} + \frac{c}{x+1}$$ **3**

ii. Hence evaluate $\int_0^1 \frac{dx}{(x^2+1)(x+1)}$ **4**

c) If $I_n = \int \sin^n x \ dx$ for $n \geq 0$,

i. Show that $I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$ for $n \geq 2$. **4**

ii. Hence evaluate $\int_{\pi}^{3\pi} \sin^4 x \, dx$ **2**

<u>Question 2</u> (17 Marks)

a)

i. Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xa^2}{x_1} + \frac{yb^2}{y_1} = a^2 + b^2$. **3**

ii. The normal to the hyperbola $\frac{x^2}{2} - y^2 = 1$ at $P\left(\sqrt{3}, \frac{1}{\sqrt{2}}\right)$ cuts the *y*-axis at *A* and the *x*-axis at *B*. Show that $PA{:}PB = 2:1$. **4**

b) The chord of contact from an external point *R* to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the ellipse at *P* and *Q*.

If *R* lies on the directrix $x = \frac{a}{e}$, show that *PQ* is a focal chord. **3**

c) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has centre *O*. *P* is the point on this ellipse with parameter θ.

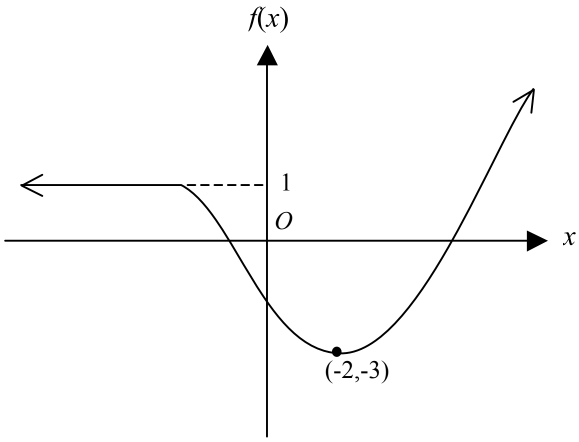
i. Find the gradient of the tangent at *P*. **1**

ii. A line drawn through *O*, parallel to the tangent to the ellipse at *P*, meets the ellipse at *Q* and *R*. Find the coordinates of *Q* and *R*. **3**

iii. Prove that the area of triangle *PQR* is independent of the position of *P*. **3**

<u>Question 3</u> (22 Marks)

a) Here is a graph of $y = f(x)$:



i. For what value(s) of k does the equation $f(x) = k$ have no solutions? **1**

ii. For what value(s) of k does the equation $f(x) = k$ have an infinite number of solutions? **1**

b) Consider the relation $x^2 + y^2 + bxy = 1$

Suppose that $P(x_1, y_1)$ is a point on the graph of this relation where the tangent to the graph is vertical.

i. Show that $y_1 = \dfrac{-bx_1}{2}$ **3**

ii. Hence find the values of b for which the graph of $x^2 + y^2 + bxy = 1$ contains points at which the tangent to the graph is vertical. **4**

Question 3 continued

c) The equation of the tangent at $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$ is $x + t^2 y = 2ct$.

i. Find the equation of the perpendicular to this tangent passing through the origin. **1**

ii. Let T be the point where the line in part (i) intersects the tangent.

Find the equation of the locus of T. **5**

d)

i. Show that $\int_a^b f(x)\,dx = \int_a^b f(b - x + a)\,dx$ **3**

ii. Prove that $\int_L^{L+\frac{\pi}{2}} \frac{A\sin(x-L) + A + B\cos(x-L) + B}{\sin(x-L) + \cos(x-L) + 2}\,dx = \frac{\pi}{4}(A+B)$ **4**

where L, A and B are constants

End of paper

Solutions

(1) (a) (i) $$\int (\cos x + \sin x)e^{(\cos x - \sin x)}\ dx = -\int (-\sin x - \cos x)e^{(\cos x - \sin x)}\ dx$$
$$= -e^{(\cos x - \sin x)} + c$$

(ii) $$\int \tan^3 x\ dx = \int \tan x . \tan^2 x\ dx$$
$$= \int \tan x(\sec^2 x - 1)\,dx$$
$$= \int \tan x \sec^2 x\ dx - \int \tan x\ dx$$
$$= \frac{\tan^2 x}{2} - \int \frac{\sin x dx}{\cos x}$$
$$= \frac{\tan^2 x}{2} + \int \frac{-\sin x dx}{\cos x}$$
$$= \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

(iii) $$\int \cos^{-1} x\, dx = \int 1 \times \cos^{-1} x\, dx$$
$$= \int \frac{d}{dx}(x) \times \cos^{-1} x\, dx$$
$$= x\cos^{-1} x - \int x \times \left(-\frac{1}{\sqrt{1-x^2}}\right) dx$$
$$= x\cos^{-1} x - \frac{1}{2}\int \left(\frac{-2x dx}{\sqrt{1-x^2}}\right)$$
$$= x\cos^{-1} x - \frac{1}{2}\int \left(\frac{1}{\sqrt{u}}\right) du$$
$$= x\cos^{-1} x - \frac{1}{2}\left[2\sqrt{u}\right] + c$$
$$= x\cos^{-1} x - \sqrt{1-x^2} + c$$

(b) (i) $$\frac{1}{(x^2+1)(x+1)} \equiv \frac{ax+b}{x^2+1} + \frac{c}{x+1}$$
$$\therefore 1 = (x+1)(ax+b) + c(x^2+1)$$
$x = -1:$ $\quad 1 = 2c \Rightarrow c = \frac{1}{2}$

$x = 0:$ $\quad 1 = b + c \Rightarrow b = \frac{1}{2}$

$a + c = 0$ $\quad$ [equating coefficient of x^2]

$\therefore a = -\frac{1}{2}$

$\therefore a = -\frac{1}{2}, b = \frac{1}{2}, c = \frac{1}{2}$

(ii) $$\int_0^1 \frac{dx}{(x^2+1)(x+1)} = \frac{1}{2}\int \left(\frac{-x+1}{x^2+1} + \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{-x}{x^2+1}\right)dx + \frac{1}{2}\int \left(\frac{1}{x^2+1}\right)dx + \frac{1}{2}\int \left(\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{4}\int \left(\frac{2x}{x^2+1}\right)dx + \frac{1}{2}\tan^{-1} x + \frac{1}{2}\ln|x+1| + c$$

$$= -\frac{1}{4}\ln\left(x^2+1\right) + \frac{1}{2}\tan^{-1} x + \frac{1}{2}\ln|x+1| + c$$

(c) (i) $$I_n = \int \sin^n x\, dx$$

$$= \int \sin^{n-1} x \times \sin x\, dx$$

$$= \int \sin^{n-1} x \times \frac{d}{dx}(-\cos x)\, dx$$

$$= -\cos x \sin^{n-1} x - \int -\cos x \times \frac{d}{dx}\left(\sin^{n-1} x\right)dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \left(1-\sin^2 x\right)\sin^{n-2} x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore I_n + (n-1)I_n = nI_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$$

$$\therefore I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \left(\frac{n-1}{n}\right)I_{n-2}$$

(ii) $$\int_\pi^{3\pi} \sin^4 x\, dx = I_4$$

$$= \left[-\frac{1}{4}\cos x \sin^3 x\right]_\pi^{3\pi} + \frac{3}{4}I_2$$

$$= \frac{3}{4}I_2$$

$$= \frac{3}{4}\left[\left[-\frac{1}{2}\cos x \sin x\right]_\pi^{3\pi} + \frac{1}{2}I_0\right]$$

$$= \frac{3}{8}I_0$$

$$= \frac{3}{8}\int_\pi^{3\pi} (\sin x)^0\, dx$$

$$= \frac{3}{8}[3\pi - \pi]$$

$$= \frac{3\pi}{4}$$

(2) (a) (i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\frac{b^2x}{a^2y}$

At $(x_1,y_1), \frac{dy}{dx}=\frac{b^2x_1}{a^2y_1}$

$\therefore$ Normal has gradient $-\frac{a^2y_1}{b^2x_1}$

$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1)$

$\therefore b^2x_1y-b^2x_1y_1=-a^2y_1x+a^2y_1x_1$

$\therefore a^2y_1x+b^2x_1y=a^2y_1x_1+b^2x_1y_1=x_1y_1(a^2+b^2)$

$\therefore \frac{xa^2}{x_1}+\frac{yb^2}{y_1}=a^2+b^2 \qquad [\div x_1y_1]$

(ii) $\frac{x^2}{2}-y^2=1 \Rightarrow a^2=2, b^2=1$

$(x_1,y_1)=\left(\sqrt{3},\frac{1}{\sqrt{2}}\right)$

So the normal has equation $\frac{2x}{\sqrt{3}}+\sqrt{2}y=3$

$A: \quad \left(0,\frac{3}{\sqrt{2}}\right)$

$B: \quad \left(\frac{3\sqrt{3}}{2},0\right)$

$PA^2=\left(\sqrt{3}\right)^2+\left(\frac{1}{\sqrt{2}}-\frac{3}{\sqrt{2}}\right)^2=3+\left(\frac{2}{\sqrt{2}}\right)^2=5$

$PB^2=\left(\sqrt{3}-\frac{3\sqrt{3}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\left(\frac{2\sqrt{3}-3\sqrt{3}}{2}\right)^2+\frac{1}{2}=\frac{3+2}{4}=\frac{5}{4}$

$\therefore PA^2:PB^2=5:\frac{5}{4}=4:1$

$\therefore PA:PB=2:1\left(=a^2:b^2\right)$

(b) If $R(x_1,y_1)$ then the chord of contact has equation $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

$x_1=\frac{a}{e}:\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \Rightarrow \frac{x}{ae}+\frac{yy_1}{b^2}=1$

The point $(ae,0)$ clearly lies on this chord (check $y=0$), so it is a focal chord!

(c) (i) $P(a\cos\theta, b\sin\theta)$

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\therefore y'=-\frac{b^2x}{a^2y}$$

$$\therefore y'_P=-\frac{b^2a\cos\theta}{a^2b\sin\theta}=-\frac{b\cos\theta}{a\sin\theta}$$

(ii) $y=-\dfrac{b\cos\theta}{a\sin\theta}x$ is the line parallel to the tangent passing through O

$$y=-\frac{b\cos\theta}{a\sin\theta}x \Rightarrow \frac{y}{b}=-\frac{\cos\theta}{a\sin\theta}x=-\frac{1}{a}\cot\theta$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{a^2}+\left(-\frac{x}{a}\cot\theta\right)^2=1$$

$$\therefore \frac{x^2}{a^2}\left(1+\cot^2\theta\right)=1$$

$$\therefore \frac{x^2}{a^2}\left(\operatorname{cosec}^2\theta\right)=1 \Rightarrow x=\pm a\sin\theta$$

$$\therefore y=\mp b\cos\theta$$

$$Q(a\sin\theta, -b\cos\theta), R(-a\sin\theta, b\cos\theta)$$

(iii) $y=-\dfrac{b\cos\theta}{a\sin\theta}x \Rightarrow xb\cot\theta+ay=0$

Let d be the distance of P from this line

$$d=\frac{\left|a\cos\theta\times b\cot\theta+a(b\sin\theta)\right|}{\sqrt{b^2\cot^2\theta+a^2}}=\frac{\dfrac{ab\cos^2\theta+ab\sin^2\theta}{|\sin\theta|}}{\sqrt{b^2\cot^2\theta+a^2}}$$

$$=\frac{ab}{|\sin\theta|\sqrt{b^2\cot^2\theta+a^2}}=\frac{ab}{\sqrt{b^2\sin^2\theta\cot^2\theta+a^2\sin^2\theta}}$$

$$=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

$$QR^2=(2a\sin\theta)^2+(-2b\cos\theta)^2=4\left(a^2\sin^2\theta+b^2\cos^2\theta\right)$$

$$\therefore QR=2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$\text{Area }\Delta PQR=\frac{d}{2}\times QR=ab$$

This area is independent of the position of P.

(3) (a) (i) $k<-3$

(ii) $k=1$

(b) (i) $x^2+y^2+bxy=1$

$\therefore 2x+2yy'+b(xy'+y)=0$

$\therefore 2yy'+bxy'=-(2x+by)$

$\therefore (2y+bx)y'=-(2x+by)$

$\therefore y'=-\dfrac{2x+by}{2y+bx}$

Vertical tangents are when y' is undefined ie $2y+bx=0$

$\therefore 2y_1+bx_1=0$

$\therefore y_1=-\dfrac{bx_1}{2}$

(ii) $x_1^{\,2}+y_1^{\,2}+bx_1y_1=1$

$y_1=-\dfrac{bx_1}{2}$

$\therefore x_1^{\,2}+\left(-\dfrac{bx_1}{2}\right)^2+bx_1\left(-\dfrac{bx_1}{2}\right)=1$

$\therefore 4x_1^{\,2}+b^2x_1^{\,2}-2b^2x_1^{\,2}=4$

$\therefore 4x_1^{\,2}-b^2x_1^{\,2}=4$

$\therefore (4-b^2)x_1^{\,2}=4$

This will have a solution as long as $4-b^2>0 \Rightarrow -2<b<2$

(c) (i) $Ax+By+C=0$ has perpendicular $Bx-Ay+D=0$

$\therefore x+t^2y=2ct$ has a perpendicular of $t^2x-y=0$ passing through the origin

(ii)

$$\left.\begin{aligned} x+t^2y&=2ct \\ t^2x-y&=0 \end{aligned}\right\} \Rightarrow x+t^2\left(t^2x\right)=2ct$$

$\therefore (1+t^4)x=2ct \Rightarrow x=\dfrac{2ct}{1+t^4}$

$\therefore y=t^2\times\dfrac{2ct}{1+t^4}=\dfrac{2ct^3}{1+t^4}$

$$T\left(\underbrace{\frac{2ct}{1+t^4}}_{x},\underbrace{\frac{2ct^3}{1+t^4}}_{y}\right)$$

$\therefore \dfrac{y}{x}=t^2$

$$y^2 = \left(\frac{2ct^3}{1+t^4}\right)^2 = \frac{4c^2\left(t^2\right)^3}{\left[1+\left(t^2\right)^2\right]^2} = \frac{4c^2\left(\frac{y}{x}\right)^3}{\left[1+\left(\frac{y}{x}\right)^2\right]^2} = \frac{\frac{4c^2y^3}{x^3}}{\left(\frac{x^2+y^2}{x^2}\right)^2} = \frac{4c^2y^3x}{\left(x^2+y^2\right)^2}$$

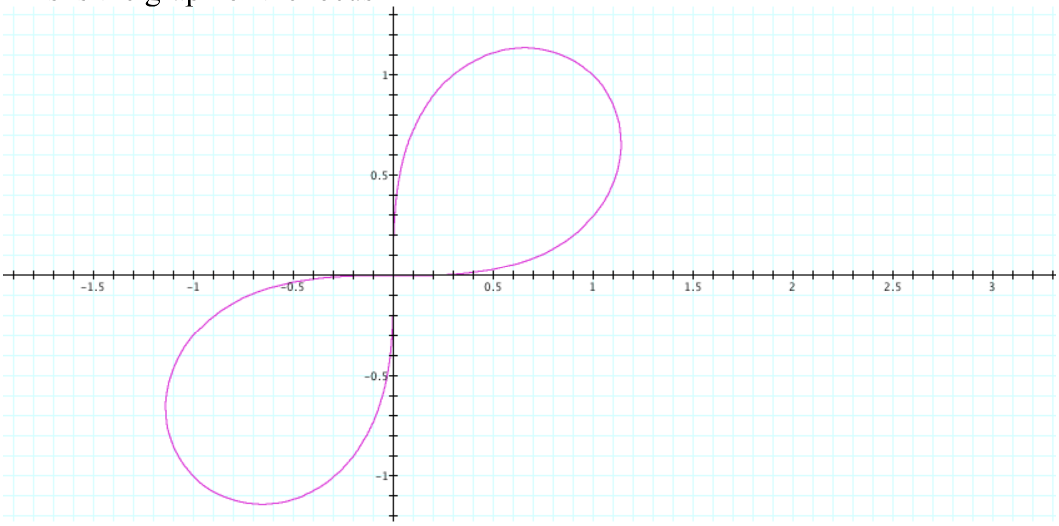
$$\therefore y^2 = \frac{4c^2y^3x}{\left(x^2+y^2\right)^2} \Rightarrow y^2 - \frac{4c^2y^3}{\left(x^2+y^2\right)^2} = 0$$

$$\therefore \frac{y^2}{\left(x^2+y^2\right)^2}\left[\left(x^2+y^2\right)^2 - 4c^2xy\right] = 0$$

$\therefore y = 0$ or $\left(x^2+y^2\right)^2 = 4c^2xy$

But $y = 0 \Rightarrow t = 0 \Rightarrow (0,0)$

This is the graph of the locus



(d) (i) NTP $\int_a^b f(x)\,dx = \int_a^b f(b-x+a)\,dx = \int_a^b f(b+a-x)\,dx$

LHS $= \int_a^b f(x)\,dx$ $\quad\left[\begin{array}{l} u = b+a-x, du = -dx \\ x = a, u = b; x = b, u = a \end{array}\right]$

$= -\int_b^a f(b+a-u)\,du$

$= \int_a^b f(b+a-u)\,du$ $\quad\left[\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \int_a^b f(b+a-x)\,dx$ $\quad$ [Definite integrals are independent of pronumeral]

$= \int_a^b f(b-x+a)\,dx$

$=$ RHS

(ii) $$I=\int_{L=a}^{L+\frac{\pi}{2}=b}\frac{A\sin(x-L)+A+B\cos(x-L)+B}{\sin(x-L)+\cos(x-L)+2}\,dx$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{A\sin(u)+A+B\cos(u)+B}{\sin(u)+\cos(u)+2}\,du \qquad [u=x-L,du=dx]$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{A\sin\left(\frac{\pi}{2}-u\right)+A+B\cos\left(\frac{\pi}{2}-u\right)+B}{\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)+2}\,du \qquad [\text{From (i)}]$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{A\cos(u)+A+B\sin(u)+B}{\cos(u)+\sin(u)+2}\,du$$

$$2I=\int_{0}^{\frac{\pi}{2}}\frac{A\sin(u)+A+B\cos(u)+B}{\sin(u)+\cos(u)+2}\,du+\int_{0}^{\frac{\pi}{2}}\frac{A\cos(u)+A+B\sin(u)+B}{\cos(u)+\sin(u)+2}\,du$$

$$=\int_{0}^{\frac{\pi}{2}}\left[\frac{A\sin(u)+A+B\cos(u)+B}{\sin(u)+\cos(u)+2}+\frac{A\cos(u)+A+B\sin(u)+B}{\cos(u)+\sin(u)+2}\right]du$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{A\sin(u)+A+B\cos(u)+B+A\cos(u)+A+B\sin(u)+B}{\sin(u)+\cos(u)+2}\,du$$

$$=\int_{0}^{\frac{\pi}{2}}\frac{(A+B)\left[\sin(u)+\cos(u)+2\right]}{\sin(u)+\cos(u)+2}\,du$$

$$=\int_{0}^{\frac{\pi}{2}}(A+B)du$$

$$=(A+B)\int_{0}^{\frac{\pi}{2}}1du$$

$$=(A+B)\left[\frac{\pi}{2}-0\right]$$

$$=(A+B)\frac{\pi}{2}$$

$$\therefore I=(A+B)\frac{\pi}{4}$$