NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 2 ASSESSMENT

2007

MATHEMATICS

EXTENSION COURSE 2

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 35% of the HSC Assessment Mark

Question 1 **(20 Marks)**

(a) Use implicit differentiation to find the value of $\frac{dy}{dx}$ at the point (1,-2) on

$(x+y)^3 = -x$ **3**

(b)

i. Express $x^2 - 4x + 6$ in the form $(x-a)^2 + b$ **1**

ii. Consider the implicit function

$2x^2y - 8xy + 12y + 1 = 0$

Express $'y'$ in terms of $'x'$ ie $y = f(x)$ **2**

iii. By writing the denominator of $f(x)$ in part (ii) in the form in part (i), ie $(x-a)^2 + b$, find the domain and range of $y = f(x)$. **2**

(c) Given the equation

$2x^2 + 2y^2 = xy + 30$

i. Show that $\frac{dy}{dx}(4y - x) = y - 4x$ **2**

ii. Find the respective points where the curve has horizontal tangents AND where it has vertical tangents **4**

iii. Use the information from part (ii) to help you sketch the graph of the function, showing its important features. **2**

(d) The equation $\left| z + 4 - 2i \right| - \left| z - 2 - 2i \right| = \pm 1$ corresponds to a hyperbola.

i. Write down the complex number which represents the centre of the hyperbola. **1**

ii. State the lengths of its semi transverse and semi conjugate axes. **3**

Question 2 **(20 Marks)**

(a) The three roots of $x^3 + px + q = 0$ are given by α, β and γ. **3**

Find a cubic equation whose roots are α^2, β^2 and γ^2.

(b) (i) If $(x-r)^2$ is a factor of the polynomial $P(x)$, prove that $(x-r)$ is a factor of $P'(x)$. **3**

(ii) The polynomial $P(x) = x^4 + bx^3 + cx^2 - 24x + 36$ has a double zero at $x = 2$. Determine the values of b and c and hence write $P(x)$ as a product of four linear factors. **6**

(c) (i) Use De Moivres theorem to find the roots of $z^5+1=0$ showing that they are equally spaced on a unit circle in an Argand Diagram. **2**

(ii) Express z^5+1 as a product of irreducible factors with real coefficients. **3**

(iii) Show that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$ **3**

<u>Question 3</u> (20 Marks)

a) i Find the real values of k for which the equation

$$\frac{x^2}{6-k}+\frac{y^2}{4-k}=1$$

defines respectively an ellipse and an hyperbola. **3**

ii Sketch the curve for $k=5$ showing its important features. **5**

iii Explain how the shape of this graph changes as k increases from 5 to 6 and determine if the graph has a limiting position for this increase. **2**

b) $P(a\cos\theta, b\sin\theta)$ where $0<\theta<\frac{\pi}{2}$, is a point which lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad .$$

An interval is drawn from O the origin, parallel to the tangent at P, to meet the ellipse at Q.

i Show that the equation of OQ is given by $y=-\frac{b\cos\theta}{a\sin\theta}x$ **3**

ii Show that there are two possible positions of Q given by $(a\sin\theta, -b\cos\theta)$ and $(-a\sin\theta, b\cos\theta)$. **3**

iii Let M be the foot of the perpendicular from P to OQ.
Hence, or otherwise, prove that the area of ΔOPQ is independent of θ. **4**

End of paper

SOLNS EXT 2 ASSESSMENT TASK; T2 2007, N.S.G.H.S

1a) $(x+y)^3 = -x$

$3(x+y)^2(1+y') = -1$ (1)

$1+y' = -\frac{1}{3(x+y)^2}$

$y' = -\frac{1}{3(x+y)^2} - 1$ (1)

At $x = 1$, $y = -2$

$y' = -\frac{1}{3(-1)^2} - 1$ (1)

$= -\frac{1}{3} - 1$

$= -\frac{4}{3}$

b)(i) $x^2 - 4x + 6 = (x^2 - 4x + 4) + 2$

$= (x-2)^2 + 2$ (1)

(ii) $2x^2y - 8xy + 12y + 1 = 0$

$y(2x^2 - 8x + 12) = -1$

$y = -\frac{1}{2(x^2-4x+6)}$

$y = -\frac{1}{2[(x-2)^2+2]}$ 2

(iii) $D = \{x : x \in R\}$

$R = \{y : -\frac{1}{4} \le y < 0\}$ 1

c) $2x^2 + 2y^2 = xy + 30$

$4x + 4yy' = xy' + y$

$4yy' - xy' = y - 4x$

$y'(4y - x) = y - 4x$

$y' = \frac{y-4x}{4y-x}$ 1

Horizontal tangents at $y - 4x = 0$

$y = 4x$

$2x^2 + 2(4x)^2 = xy + 30$

$34x^2 = 4x^2 + 30$

$30x^2 = 30$ 2

$x^2 = 1 \Rightarrow x = \pm 1$

$(1, 4)(-1, -4)$ have horizontal tangents

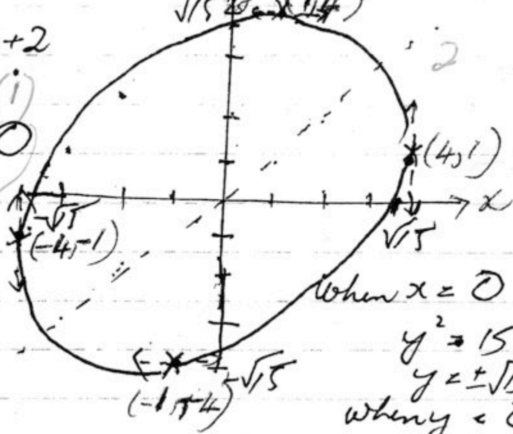
Vertical Tangents at $4y = x$

$2(4y)^2 + 2y^2 = 4y \cdot y + 30$

$32y^2 + 2y^2 = 4y^2 + 30$

$y = \pm 1$

$(4, 1)(-4, -1)$ 2



When $x = 0$

$y^2 = 15$

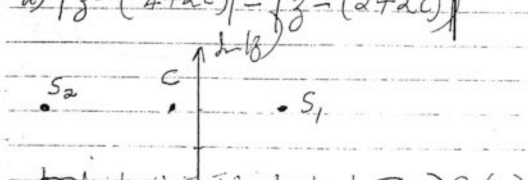
$y = \pm\sqrt{15}$

when $y = 0$

$2x^2 = 30$

$x = \pm\sqrt{15}$

d) $|z-(-4+2i)| - |z-(2+2i)|$



$C(-1, 2)$ ie $-1+2i$

$2a = 1$

ie $a = \frac{1}{2}$ LENGTH OF SEMI TRANSVERSE AXIS

$ae = 3$

$\frac{1}{2}e = 3$

$e = 6$ 1

$b^2 = a^2(e^2 - 1)$

$b^2 = \frac{1}{4}(36 - 1)$

$b^2 = \frac{35}{4}$

$b = \frac{\sqrt{35}}{2}$ ie LENGTH OF SEMI CONJUGATE AXIS

Question 2

(a) $x^3 + px + q = 0 \Rightarrow x(x^2 + p) = -q$

$\therefore x^2(x^2 + p)^2 = q^2$

Let $y = x^2$

$\therefore y(y + p)^2 = q^2$

$\therefore y^3 + 2py^2 + p^2y - q^2 = 0$

Alternatively

Let $y = x^2 \Rightarrow x = \sqrt{y}$

$\therefore x^3 + px + q = 0 \Rightarrow (\sqrt{y})^3 + p\sqrt{y} + q = 0$

$\therefore y\sqrt{y} + p\sqrt{y} = -q$

$\therefore \sqrt{y}(y + p) = -q \Rightarrow [\sqrt{y}(y + p)]^2 = q^2$

$\therefore y(y^2 + 2py + p^2) = q^2$

$\therefore y^3 + 2py^2 + p^2y - q^2 = 0$

(b) (i) $P(x) = (x - r)^2 Q(x),\ \ Q(r) \neq 0$

$\therefore P'(x) = 2(x - r)Q(x) + (x - r)^2 Q'(x)$

$\therefore P(x) = (x - r)[2Q(x) + (x - r)Q'(x)]$

(ii) $P(x) = x^4 + bx^3 + cx^2 - 24x + 36 = (x - 2)^2 Q(x)$

$\therefore P(2) = P'(2) = 0$

$P'(x) = 4x^3 + 3bx^2 + 2cx - 24$

$P(2) = 16 + 8b + 4c - 48 + 36 = 0$

$\therefore 8b + 4c + 4 = 0$

$\therefore 2b + c = -1 \qquad -(1)$

$P'(2) = 32 + 12b + 4c - 24 = 0$

$\therefore 12b + 4c + 8 = 0$

$\therefore 3b + c = -2 \qquad -(2)$

$(2) - (1) \Rightarrow b = -1$

$\therefore c = 1$

$\therefore b = -1, c = 1$

$\therefore P(x) = x^4 - x^3 + x^2 - 24x + 36 = (x - 2)^2 (x^2 + Dx + 9)$

Substitute $x = 1$: $\quad 1 - 1 + 1 - 24 + 36 = 1(10 + D)$

$\therefore D = 3$

$\therefore P(x) = x^4 - x^3 + x^2 - 24x + 36 = (x - 2)^2 (x^2 + 3x + 9)$

Now $x^2+3x+9=x^2+3x+2\tfrac{1}{4}+8\tfrac{3}{4}=\left(x+\frac{3}{2}\right)^2+\frac{27}{4}=\left(x+\frac{3}{2}\right)^2-\frac{27}{4}i^2$

$$\therefore x^2+3x+9=\left(x+\frac{3}{2}+i\frac{3\sqrt{3}}{2}\right)\left(x+\frac{3}{2}-i\frac{3\sqrt{3}}{2}\right)$$

$$P(x)=(x-2)(x-2)\left(x+\frac{3}{2}+i\frac{3\sqrt{3}}{2}\right)\left(x+\frac{3}{2}-i\frac{3\sqrt{3}}{2}\right)$$

$$=\frac{1}{4}(x-2)(x-2)\left(2x+3+i3\sqrt{3}\right)\left(2x+3-i3\sqrt{3}\right)$$

(c) (i) $z^5=-1=\text{cis}(\pi+2k\pi)=\text{cis}(2k+1)\pi,\ k\in\mathbb{Z}$

$\therefore z=\text{cis}\left(\frac{2k+1}{5}\right)\pi$ [de Moivre's Theorem]

$$z=\text{cis}\left(\frac{\pi}{5}+\frac{2\pi k}{5}\right)$$

$\therefore |z|=1$ and $\arg z$ are $\frac{2\pi}{5}$ radians apart. So the solutions are equally spaced around a unit circle.

$$\therefore z=\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right),\text{cis}\left(\frac{5\pi}{5}\right),\text{cis}\left(\frac{7\pi}{5}\right),\text{cis}\left(\frac{9\pi}{5}\right)$$

$$\therefore z=-1,\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(-\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right),\text{cis}\left(-\frac{3\pi}{5}\right)$$

(ii) $z^5+1=(z+1)\left[z-\text{cis}\left(\frac{\pi}{5}\right)\right]\left[z-\text{cis}\left(-\frac{\pi}{5}\right)\right]\left[z-\text{cis}\left(\frac{3\pi}{5}\right)\right]\left[z-\text{cis}\left(-\frac{3\pi}{5}\right)\right]$

$$=(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$$

$$\left[\because (z-\alpha)(z-\bar{\alpha})=z^2-2z\,\text{Re}\,\alpha+|\alpha|^2\right]$$

(iii) Now $-1+\text{cis}\left(\frac{\pi}{5}\right)+\text{cis}\left(-\frac{\pi}{5}\right)+\text{cis}\left(\frac{3\pi}{5}\right)+\text{cis}\left(-\frac{3\pi}{5}\right)=0$ (sum of roots of $z^5+1=0$)

$\therefore 2\cos\frac{\pi}{5}+2\cos\frac{3\pi}{5}=-1 \qquad (\alpha+\bar{\alpha}=2\,\text{Re}\,\alpha)$

$$\therefore \cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=-\frac{1}{2}$$

Substitute $z = i$ into $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$

$\therefore i^5 + 1 = (i+1)\left(-2i\cos\frac{\pi}{5}\right)\left(-2i\cos\frac{3\pi}{5}\right)$

$\therefore 1 = -4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$

$\therefore \cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$

Alternative 1

Using $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$

Equating the coefficients of z of both sides obtains:

$$0 = 1 - 2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5}$$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2} \quad -(1)$

Equating the coefficients of z^2 of both sides obtains:

$$0 = 1 + 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} - 2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5}$$

$\therefore 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = 2\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) - 2 = -1 \qquad [\text{from } (1)]$

$\therefore \cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}$

Alternative 2

$z^5 + 1 = (z+1)(1 - z + z^2 - z^3 + z^4)$

$\quad = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$

$\therefore 1 - z + z^2 - z^3 + z^4 = \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$

Equating the coefficients of z of both sides obtains:

$$-2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5} = -1$$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Equating the coefficients of z^2 of both sides obtains:

$$1 + 1 + 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = 1$$

$$\therefore 4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-1$$

$$\therefore \cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$$

Q 3

a) Ellipse when both

$6-k>0 \Rightarrow k<6$

and $4-k>0 \Rightarrow k<4$

ie $k<4 \Rightarrow$ Ellipse

Hyperbola when both

$6-k>0 \Rightarrow k<6$ }

AND $4-k<0 \Rightarrow k>4$ }

ie $4<k<6$

(ii) $k=5$

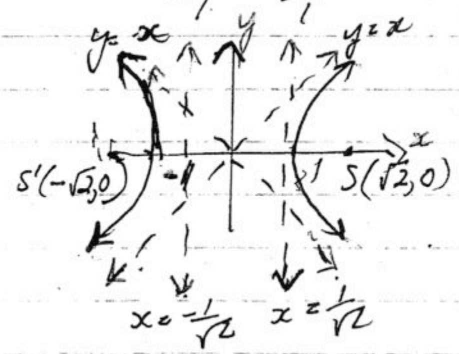
$$\frac{x^2}{1}-\frac{y^2}{1}=1$$

$y=\pm 1x$ asymptotes

$b^2=a^2(e^2-1)$

$1=e^2-1$

$e=\sqrt{2}$

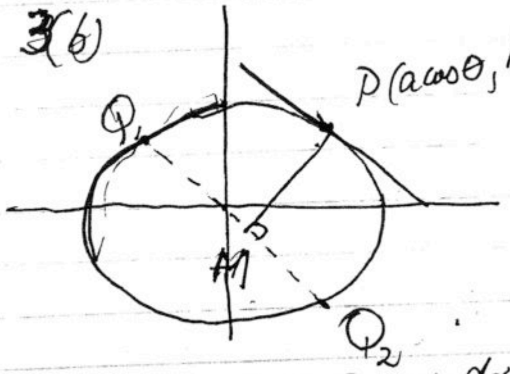


(iii) As '6' is approached by 'k', a^2 becomes small.

eg: $\frac{x^2}{\frac{1}{1000}}+\frac{y^2}{-2}=1$ ie $\frac{x^2}{(\frac{1}{1000})}-\frac{y^2}{2}=1$ This shows the length of the transverse axis will approach zero

a) The length of the transverse axis approaches zero. Hence the branches become flatter, approaching a limiting position which is the y axis

3(b)



(i) $x = a\cos\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta \Rightarrow \frac{dy}{d\theta} = b\cos\theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= -\frac{b\cos\theta}{a\sin\theta}$

Since OQ is of the form $y = mx$

$y = -\frac{b\cos\theta}{a\sin\theta}x$

Solve

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ①

$y = -\frac{b\cos\theta}{a\sin\theta}x$ ②

simultaneously

$\frac{x^2}{a^2} + \frac{\left(-\frac{b\cos\theta}{a\sin\theta}x\right)^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{b^2\cos^2\theta}{a^2 b^2\sin^2\theta}x^2 = 1$

(×) $a^2\sin^2\theta$

$x^2\sin^2\theta + x^2\cos^2\theta = a^2\sin^2\theta$

$x^2 = a^2\sin^2\theta$

$x = \pm a\sin\theta$

By substitution into ①

② becomes $xb\cos\theta + ya\sin\theta = 0$

$Q_2(a\sin\theta, -b\cos\theta)$

$Q_1(-a\sin\theta, b\cos\theta)$

(iii) Area $= \frac{1}{2} PM . OQ_2$

$PM = \frac{|b\cos\theta . a\cos\theta + a\sin\theta b\sin\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$= \frac{ab(\cos^2\theta + \sin^2\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

$OQ_2 = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$A = \frac{1}{2}\frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} * \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$A = \frac{1}{2}ab$ and a, b are independent of θ