NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 2: 2009

Name:_______________________________ Mathematics Class:12MZ____

Time Allowed: 60 minutes + 2 minutes reading time

Available Marks: 46

Instructions:

- Start each question on a new page.
- Put your name on the top of each page.
- Attempt all four questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Each question will be collected separately.
- If you do not attempt a question, submit a blank page with your name and the question number clearly displayed.

Question	1	2	3	4	Total	
E3		/12	b/6		/18	
E4	/11		a/5		/16	
E9				/12	/12	
	/11	/12	/11	/12	/46	%

Question 1. (11 marks) **Marks**

(a) It is given that $3+i$ is a zero of $P(x)=x^3+rx+60$ where r is real.

(i) State why $3-i$ is also a zero of $P(x)$. **1**

(ii) Factorise $P(x)$ over the real numbers. **2**

(iii) Find the value of r. **1**

(b) Decompose $\dfrac{3x^2-7x+4}{(x-3)(x^2+1)}$ into partial fractions over the real field. **3**

(c) The equation $x^3-3x-1=0$ has roots α, β, γ.

(i) Show that none of α, β, γ are integers. **2**

(ii) Form the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. **2**

Question 2. (12 marks)

(a) $P(x, y)$, $Q(8,0)$ and $R(-8,0)$ are three points in the number plane such that $PQ + PR = 24$ units.

(i) Explain why P lies on an ellipse. **1**

(ii) Find the equation of the locus of P in standard form. **3**

(b) (i) The hyperbola $x^2-y^2=8$ is rotated 45° anticlockwise until it coincides with $xy=c^2$. Write down the value of c. (No proof is required.) **1**

(ii) Draw a neat sketch showing this information using the above value of c, showing clearly the coordinates of the foci and the equations of the directrices of $xy=c^2$. **3**

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on one end of the latus rectum drawn to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(The latus rectum is the focal chord perpendicular to the major axis)

(i) Show that $\cos\theta = e$ where e is the eccentricity. **1**

(ii) Show that the length of the latus rectum is given by $\dfrac{2b^2}{a}$. **3**

Question 3. (11 marks) **Marks**

(a) (i) $P(x)=0$ is a polynomial with a root of multiplicity m. Show that $P'(x)=0$ has a root of multiplicity $(m-1)$. **2**

(ii) The polynomial $\alpha x^{n+1}+\beta x^{n}+1$ is divisible by $(x-1)^{2}$.
Show that $\alpha=n$ and $\beta=-(1+n)$. **3**

(b) $P(2p,\frac{2}{p})$ and $Q(2q,\frac{2}{q})$ are points on the hyperbola $xy = 4$. M is the midpoint of the chord PQ. P and Q move on the hyperbola so that the chord PQ always passes through the point $R(4,2)$.

(i) Show that the equation of the chord PQ is given by $x+pqy=2(p+q)$ **2**

(ii) Show that $pq=p+q-2$ **1**

(iii) Show that the locus of M is given by $y=1+\dfrac{2}{x-2}$ **3**

Question 4. (12 marks)

(a) $P(x_1,y_1)$ is a variable point on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

(i) Show that the equation of the normal at P is given by $\dfrac{a^2x}{x_1}+\dfrac{b^2y}{y_1}=a^2+b^2$ **3**

(ii) The tangent and normal at P meet the y-axis at G and H respectively and S is a focus of the hyperbola. Show that $\angle GSH$ is a right angle. **3**
(You may assume the equation of the tangent at P is given by $\dfrac{x_1x}{a^2}-\dfrac{y_1y}{b^2}=1$)

(b) (i) Show the solutions to the equation $z^9-1=0$ on the Argand plane. **1**

(ii) Show that the solutions to $z^6+z^3+1=0$ are contained in the solutions to $z^9-1=0$. **1**

(iii) Hence factorise $z^6+z^3+1=0$ into real quadratic factors. **2**

(iv) Show that $\left(1-\cos\frac{2\pi}{9}\right)\left(1-\cos\frac{4\pi}{9}\right)\left(1+\cos\frac{\pi}{9}\right)=\frac{3}{8}$. **2**

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End of paper

EXTENSION 2: 2009 TASK 2 Solutions

Question 1

(a) (i) coefficients real

$\therefore$ roots in conjugate pairs

$\therefore (3-i)$ a zero

(ii) $P(x) = x^3 + rx + 60$

$= [x^2 - (3-i+3+i)x + (3-i)(3+i)] \times (x-b)$

$= (x^2 - 6x + 10)(x-b)$

$\therefore -10b = 60$

$\boxed{\therefore b = -6}$

$\therefore P(x) = (x^2 - 6x + 10)(x+6)$

(iii) $x = -6$ a zero

$\therefore P(-6) = -6^3 - 6r + 60 = 0$

$6r = -6^3 + 60$

$r = -6^2 + 10$

$\boxed{r = -26}$

(b) $\dfrac{3x^2 - 7x + 4}{(x-3)(x^2+1)} \equiv \dfrac{a}{x-3} + \dfrac{bx+c}{x^2+1}$

$3x^2 - 7x + 4 = a(x^2+1) + (bx+c)(x-3)$

$\underline{x=3}$ $27 - 21 + 4 = 10a$

$\boxed{a = 1}$

coeff x^2: $3 = a + b$

$\therefore \boxed{b = 2}$

$x = 0$ $4 = a - 3c$

$3c = 1 - 4$

$\boxed{c = -1}$

$\therefore \dfrac{3x^2 - 7x + 4}{(x-3)(x^2+1)} = \dfrac{1}{x-3} + \dfrac{2x-1}{x^2+1}$

(c) $x^3 - 3x - 1 = 0$

(i) monic, integer coefficients

$\therefore$ any integer zeros divide constant term (-1)

ie only possible integer zeros $x = \pm 1$

$P(1) = 1 - 3 - 1 = -3 \neq 0$

$P(-1) = -1 + 3 - 1 = -5 \neq 0$

$\therefore$ No integer zeros.

(ii) $\alpha^2, \beta^2, \gamma^2$ satisfy $P(\sqrt{x}) = 0$

$\sqrt{x}^3 - 3\sqrt{x} - 1 = 0$

$\sqrt{x}(x-3) = 1$

$x(x-3)^2 = 1$

$x(x^2 - 6x + 9) = 1$

$x^3 - 6x^2 + 9x - 1 = 0$

Question 2

(a) $PQ + PR = 24$

(i) ellipse since sum of distances to 2 fixed points is a constant

(ii) $2a = 24$

$a = 12$

$ae = 8$ since $S(\pm 8, 0)$

$e = \frac{2}{3}$

$\therefore b^2 = a^2(1 - e^2)$

$b^2 = 144(1 - \frac{4}{9})$

$b^2 = 144 \times \frac{5}{9}$

$= 80$

$\therefore$ Equation $\dfrac{x^2}{144} + \dfrac{y^2}{80} = 1$

Ext. 2 : 2009 Task 2 Solution

Question 2 (cont) (b) (i) $x^2-y^2=8$

$xy=c^2$

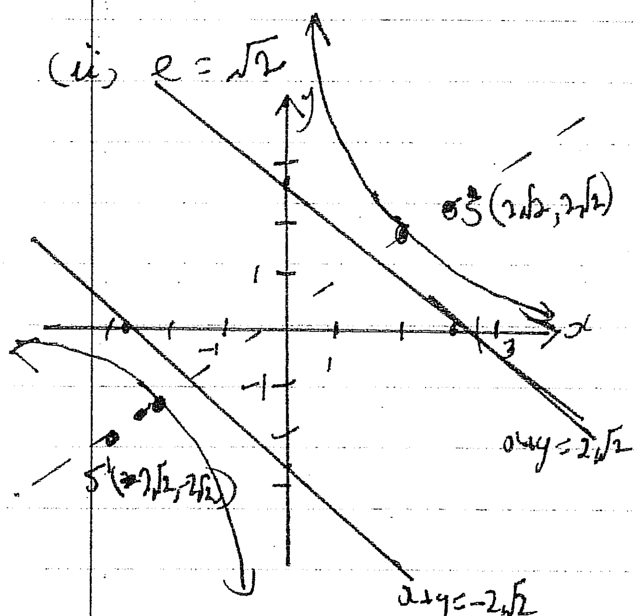
$\therefore c^2+c^2=\sqrt{8}^2$

$\therefore 2c^2=8$

$c^2=4$

$\therefore xy=4$

(ii) $e=\sqrt{2}$



(c) (i) $a\cos\theta=ae$

$\therefore \cos\theta=e$

(ii) length $LR=2b\sin\theta$

$=2b\sqrt{1-\cos^2\theta}$

$=2b\sqrt{1-e^2}$

but $b^2=a^2(1-e^2)$

$\therefore 1-e^2=\frac{b^2}{a^2}$

$\therefore$ length $=2b\cdot\frac{b}{a}$

$=\frac{2b^2}{a}$

Question 3

(a) (i) let $P(x)=(x-\alpha)^m Q(x)$

$P'(x)=m(x-\alpha)^{m-1}Q(x)+Q'(x)(x-\alpha)^m$

$=(x-\alpha)^{m-1}\left(mQ(x)+Q'(x)(x-\alpha)\right)$

$=(x-\alpha)^{m-1}\times$ polynomial.

$\therefore P'(x)$ has root multiplicity $(m-1)$

(ii) $P(x)=\alpha x^{n+1}+\beta x^n+1$

$(x-1)^2$ a factor $\therefore P(1)=P'(1)=0$

$P(1)=\alpha+\beta+1=0$ — ①

$P'(x)=\alpha(n+1)x^n+\beta nx^{n-1}$

$P'(1)=(n+1)\alpha+\beta n=0$ — ②

①$\times n$: $\alpha n+\beta n+n=0$ — ③

③ − ② : $\alpha n-\alpha n-\alpha+n=0$

$\therefore \alpha=n$

sub into ① $n+\beta+1=0$

$\beta=-(1+n)$

(b) (i) $m_{PQ}=\left(\frac{2}{q}-\frac{2}{p}\right)\div(2q-2p)$

$=\frac{p-q}{pq}\times\frac{1}{q-p}$

$=-\frac{1}{pq}$

Equation

$y-\frac{2}{p}=\frac{-1}{pq}(x-2p)$

$pqy-2q=-x+2p$

$x+pqy=2p+2q$

$x+pqy=2(p+q)$

Ext 2 : 2009 : Task 2 solution

Question 3 CTD (b) through $(4,2)$

$$\therefore 4+2pq = 2(p+q)$$
$$2+pq = p+q$$
$$pq = p+q-2$$

(iii)
$$M\left[\frac{2(p+q)}{2}, \frac{2\left(\frac{1}{p}+\frac{1}{q}\right)}{2}\right]$$
$$x = p+q$$
$$y = \frac{1}{p}+\frac{1}{q}$$
$$y = \frac{q+p}{pq}$$
$$= \frac{q+p}{p+q-2}$$
but $x = p+q$
$$y = \frac{x}{x-2}$$
$$= \frac{x-2+2}{x-2}$$
$$= 1+\frac{2}{x-2}$$

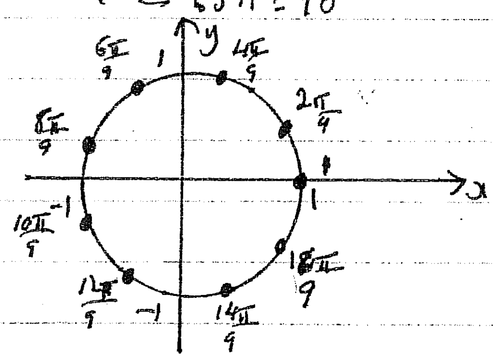
Question 4 (a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(i) $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$
$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$
$$m_N = \frac{-a^2y_1}{b^2x_1}$$
equation: $y-y_1 = \frac{-a^2y_1}{b^2x_1}(x-x_1)$
$$b^2x_1y - b^2x_1y_1 = -a^2xy_1 + a^2x_1y_1$$
$$a^2xy_1 + b^2yx_1 = x_1y_1(a^2+b^2)$$
$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2+b^2$$

(ii) at H: $x=0$
$$\therefore y = \frac{y_1(a^2+b^2)}{b^2}$$

at G ~~y~~ $x=0$ $\frac{xx_1}{a^2}-\frac{y_1y}{b^2}=1$
$$y = -\frac{b^2}{y_1}$$
$S(ae,0)$ $G\left(0,\frac{-b^2}{y_1}\right)$ $H\left(0,\frac{y_1(a^2+b^2)}{b^2}\right)$
$$m_{SG} = \frac{\frac{b^2}{y_1}}{ae} = \frac{b^2}{aey_1}$$
$$m_{SH} = \frac{\frac{y_1(a^2+b^2)}{b^2}}{-ae} = \frac{y_1(a^2+b^2)}{-b^2ae}$$
but $a^2+b^2 = a^2e^2$
$$m_{SH} = \frac{y_1(a^2e^2)}{-b^2ae} = -\frac{aey_1}{b^2}$$
$\therefore m_{SG}\times m_{SH} = -1$ $\therefore SG \perp SH$
$\therefore \angle GSH = 90^\circ$

(b)



(ii) $z^9-1 = (z^3-1)(z^6+z^3+1)$
$\therefore z^6+z^3+1$ contained in solutions to $z^9=1$ which do not satisfy $z^3=1$ ie $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

(iii) $z^6+z^3+1=0$
roots with args. $\pm\frac{2\pi}{9}, \pm\frac{4\pi}{9}, \pm\frac{8\pi}{9}$
$$\therefore z^6+z^3+1 = \left[z^2-2\cos\tfrac{2\pi}{9}x+1\right]\left[z^2-2\cos\tfrac{4\pi}{9}z+1\right]\left[z^2-2\cos\tfrac{8\pi}{9}z+1\right]$$

(iv) let $z=1$
$$1+1+1 = \left(2-2\cos\tfrac{2\pi}{9}\right)\left(2-2\cos\tfrac{4\pi}{9}\right)\left(2-2\cos\tfrac{8\pi}{9}\right)$$
$$\therefore \left(1-\cos\tfrac{2\pi}{9}\right)\left(1-\cos\tfrac{4\pi}{9}\right)\left(1-\cos\tfrac{8\pi}{9}\right) = \frac{3}{8}$$
but $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$
$$\therefore \left(1-\cos\tfrac{2\pi}{9}\right)\left(1-\cos\tfrac{4\pi}{9}\right)\left(1+\cos\tfrac{8\pi}{9}\right) = \frac{3}{8}$$