NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 3 Term 2 2014

Name: ________________________ **Mathematics Class:**__________

Student Number: ______________

Time Allowed: 50 minutes + 2 minutes reading time

Available Marks: 41

Instructions:

- Questions are ***not*** of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Section	I 1-2	I 3	I 4-5	Q6	Q7 a)	Q7 b)	Q7 c)	Total
E3	/2		/2		/7			/11
E4						/6		/6
E8		/1		/16			/7	/24
								/41

Section I: Multiple Choices

5 marks
Attempt all questions.
Answer on the multiple choice answer sheet provided for Section I.

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1. What type of curve does the equation $\left|z-3\right|-\left|z+3\right|=4$ corresponds to in the Argand diagram?

(A) Circle

(B) Ellipse

(C) Parabola

(D) Hyperbola

2. The point $T(a\cos\theta, a\sin\theta)$ lies on the circle $x^2+y^2=a^2$. Which of the following is the equation of the tangent at T?

(A) $x\cos\theta + y\sin\theta = a$

(B) $x\cos\theta + y\sin\theta = a^2$

(C) $x\cos\theta - y\sin\theta = a$

(D) $x\cos\theta - y\sin\theta = a^2$

3. Given that $\sin\left(A-B\right)x+\sin\left(A+B\right)x = 2\sin Ax\cos Bx$, which of the following is equal to

$$\int \sin 3x \cos 2x \; dx \, ?$$

(A) $\frac{1}{2}\sin 5x + \frac{1}{2}\sin x + C$

(B) $\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$

(C) $-\frac{1}{2}\sin 5x - \frac{1}{2}\sin x + C$

(D) $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$

4. Which of the following are the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$?

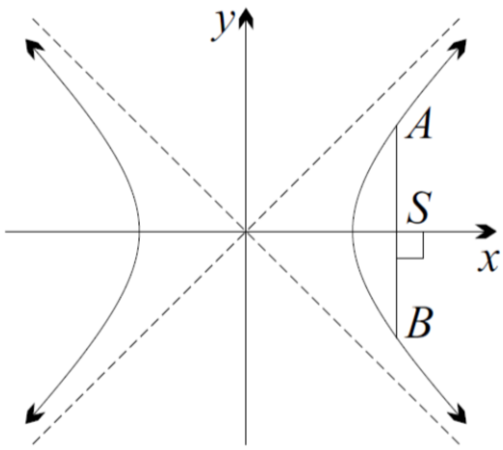
(A) $(2,0)$ and $(-2,0)$

(B) $(3\sqrt{5},0)$ and $(-3\sqrt{5},0)$

(D) $\left(\frac{3\sqrt{5}}{2},0\right)$ and $\left(-\frac{3\sqrt{5}}{2},0\right)$

(C) $(\sqrt{5},0)$ and $(-\sqrt{5},0)$

5. The graph of the hyperbola $\frac{x^2}{9}-\frac{y^2}{5}=1$ is shown below. *S* is one of the foci and *AB* is a chord of the hyperbola that is perpendicular to the *x*-axis at *S*.



What is the length of *AB?*

(A) $\frac{10}{3}$

(B) $\frac{18}{\sqrt{5}}$

(C) $\frac{10}{9}$

(D) $\frac{18}{5}$

END OF SECTION I

Section II

Attempt all questions.
Answer each question in a separate writing booklet.
Your responses should include relevant mathematical reasoning and/or calculations.

<u>Question 6</u> (16 marks)

(a) Evaluate $\int_0^7 \frac{2}{\sqrt{x+9}}\,dx$. **2**

(b) Find $\int \sin^3 x\,dx$ **3**

(c) (i) Write $\frac{2x+3}{(x-2)(x^2+2)}$ in the form $\frac{A}{x-2}+\frac{Bx+C}{x^2+2}$ **2**

(ii) Hence find $\int \frac{2x+3}{(x-2)(x^2+2)}\,dx$ **3**

(d) Find $\int \frac{dx}{\sqrt{13+4x-x^2}}$ **3**

(e) Find $\int x\cos 2x\,dx$ **3**

End of Question 6

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Question 7 (20 marks)

(a) The point $P\left(4\sec\theta, 3\tan\theta\right)$ lies on a hyperbola that has both vertices on the *x*-axis.

(i) Find the equation of the hyperbola in Cartesian form. **1**

(ii) Find the eccentricity of the hyperbola. **1**

(iii) Draw a neat **HALF** page sketch of this hyperbola, showing the coordinates of the vertices and the foci, and the equations of the directrices and asymptotes. **3**

(iii) Derive the equation of the tangent to the hyperbola at the point *P* in parametric form. **2**

(b) The points $P\left(2p, \dfrac{2}{p}\right)$ and $Q\left(2q, \dfrac{2}{q}\right)$ lie on the rectangular hyperbola $xy = 4$.

M is the midpoint of *PQ*.

P and *Q* move on the hyperbola so that the chord *PQ* will always pass through the point *R*(4,2) and $pq > 0$

(i) Show that the chord *PQ* has the equation $x + pqy = 2\left(p+q\right)$. **2**

(ii) Show that $pq = p + q - 2$. **1**

(iii) Hence find the equation of the locus of *M* as *P* moves on the hyperbola. **3**

Question 7 (continued)

(c) (i) Using the substitution $x = \tan \frac{1}{2} x$ or otherwise, show that $\int_0^{\frac{\pi}{2}} \frac{1}{1-\frac{1}{2}\sin x}\, dx = \frac{4\pi}{3\sqrt{3}}$ **3**

(ii) Show that $\int_0^{2a} f(x)\, dx = \int_0^{a} \left[f(x) + f\left(2a - x\right)\right] dx$ **2**

(iii) Hence evaluate $\int_0^{\pi} \frac{x}{1-\frac{1}{2}\sin x}\, dx$ **2**

END OF EXAMINATION

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SECTION I
Q1 D
Q2 A
Q3 D
Q4 C
Q5 A

SECTION II
Question 16
(a)

$$\int_0^7 \frac{2}{\sqrt{x+9}}\,dx = \int_0^7 2(x+9)^{-\frac{1}{2}}\,dx$$
$$= \left[4(x+9)^{\frac{1}{2}}\right]_0^7 = \left[4\sqrt{x+9}\right]_0^7$$
$$= \left[\left(4\sqrt{7+9}\right)-\left(4\sqrt{0+9}\right)\right]$$
$$= 4$$

(b)

$$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx$$
$$= \int \sin x\left(1-\cos^2 x\right)dx$$
$$= \int \sin x - \sin x\cos^2 x\,dx$$
$$= -\cos x + \frac{\cos^3 x}{3} + C$$

(c) (i)

$$\frac{2x+3}{(x-2)(x^2+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2}$$
$$2x+3 = A(x^2+2)+(Bx+C)(x-2)$$

Sub $x=2$

$$7 = A(4+2)$$
$$A = \frac{7}{6}$$

Sub $x=0$

$$3 = A(0+2)+C(-2)$$
$$3 = \frac{7}{6}(2)+C(-2)$$
$$C = -\frac{1}{3}$$

Sub $x=1$

$$5 = A(1+2)+(B+C)(-1)$$
$$5 = 3\left(\frac{7}{6}\right) - B - \frac{1}{3}(-1)$$
$$B = \frac{7}{3} - 5 + \frac{1}{3}$$
$$B = -\frac{7}{6}$$

$$\frac{2x+3}{(x-2)(x^2+2)} = \frac{\frac{7}{6}}{x-2} + \frac{-\frac{7}{6}x-\frac{1}{3}}{x^2+2}$$

(ii) **ho**

$$\int \frac{2x+3}{(x-2)(x^2+2)}\,dx$$
$$= \int \frac{\frac{7}{6}}{x-2}\,dx + \int \frac{-\frac{7}{6}x-\frac{1}{3}}{x^2+2}dx$$
$$= \frac{7}{6}\int \frac{1}{x-2}\,dx + \int \frac{-\frac{7}{6}x}{x^2+2} + \int \frac{-\frac{1}{3}}{x^2+2}$$
$$= \frac{7}{6}\int \frac{1}{x-2}dx - \frac{7}{12}\int \frac{2x}{x^2+2}dx - \frac{1}{3}\int \frac{1}{x^2+2}dx$$
$$= \frac{7}{6}\ln(x-2) - \frac{7}{12}\ln(x^2+2) + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

(d)

$$\int \frac{dx}{\sqrt{13+4x-x^2}} = \int \frac{dx}{\sqrt{17-(x^2-4x+4)}}$$
$$= \int \frac{dx}{\sqrt{17-(x-2)^2}}$$
$$= \sin^{-1}\frac{x-2}{\sqrt{17}} + C$$

(e)

$$\int x\cos 2x dx = \frac{x\sin 2x}{2} - \int \frac{\sin 2x}{2}dx$$
$$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

<u>Question 7</u>

a)

(i) Find the equation of the hyperbola in Cartesian form.

$(4\sec\theta, 3\tan\theta)$

$a=4$ and $b=3$

Equation is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

(ii) Find the eccentricity of the hyperbola.

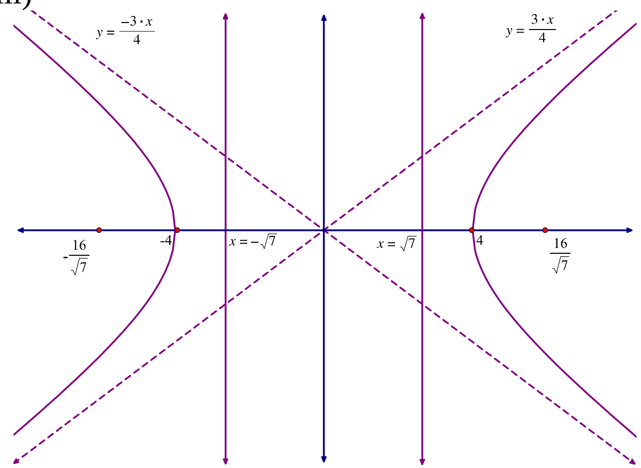
$$b^2 = a^2\left(1-e^2\right)$$

$$\frac{9}{16} = 1-e^2$$

$$\frac{7}{16} = e^2$$

$$\frac{\sqrt{7}}{4} = e$$

(iii)



(iv) Derive the equation of the tangent to the hyperbola at the point P in parametric form.

At $P\left(4\sec\theta, 3\tan\theta\right)$

By Implicit Differentiation

$$\frac{2x}{16} - \frac{2y}{9}\times\frac{dy}{dx} = 0$$

$$\frac{2y}{9}\times\frac{dy}{dx} = \frac{2x}{16}$$

$$\frac{dy}{dx} = \frac{9x}{16y}$$

At $P\left(4\sec\theta, 3\tan\theta\right)$

$$\frac{dy}{dx} = \frac{3\sec\theta}{4\tan\theta}$$

Equation of the tangent at P

At $P\left(4\sec\theta, 3\tan\theta\right)$

$$y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}\left(x - 4\sec\theta\right)$$

$$4y\tan\theta - 12\tan^2\theta = 3x\sec\theta - 12\sec^2\theta$$

$$4y\tan\theta = 3x\sec\theta - 12\sec^2\theta + 12\tan^2\theta$$

$$4y\tan\theta = 3x\sec\theta + 12$$

b)

(i) Show that the chord PQ has the equation $x + pqy = 2\left(p+q\right)$

Gradient of PQ

$$m_{PQ} = \frac{\frac{2}{p} - \frac{2}{q}}{2p - 2q}$$

$$= \frac{\frac{2\left(q-p\right)}{pq}}{2\left(p-q\right)}$$

$$= -\frac{1}{pq}$$

Equation of PQ

$$y - \frac{2}{p} = -\frac{1}{pq}\left(x - 2p\right)$$

$$pqy - 2q = -x + 2p$$

$$x + pqy = 2\left(p+q\right)$$

(ii) Show that $pq = p + q - 2$.

At $R(4,2)$ Let $x = 4$ and $y = 2$

$$x + pqy = 2\left(p+q\right)$$

$$\left(4\right) + pq\left(2\right) = 2\left(p+q\right)$$

$$2 + pq = \left(p+q\right)$$

$$pq = \left(p+q\right) - 2$$

(iii) Hence find the equation of the locus of M as P moves on the hyperbola.

Midpoint of PQ

$$x = \frac{2p + 2q}{2}$$

$$x = p + q$$

$$y = \frac{\frac{2}{p} + \frac{2}{q}}{2}$$

$$y = \frac{p+q}{pq}$$

Sub $x = p+q$ and $pq = (p+q)-2$ into

$$y = \frac{p+q}{pq}$$

$$y = \frac{x}{(p+q)-2}$$

$$= \frac{x}{x-2}$$

But $pq > 0$ so P and Q must on the same branch and as PQ must pass through R then both p and q must be positive.

Thus the equation of the locus of M is

$$y = \frac{x}{x-2} \qquad x > 2$$

Question 7 (continued)

(c) (i) Using the substitution $x = \tan\frac{x}{2}$ or otherwise, show that $\int_0^{\frac{\pi}{2}} \frac{1}{1-\frac{1}{2}\sin x}\,dx = \frac{4\pi}{3\sqrt{3}}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1-\frac{1}{2}\sin x}\,dx = \int_0^1 \frac{1}{1-\frac{1}{2}\times\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}\,dx$$

$$= \int_0^1 \frac{2dt}{1+t^2-t}$$

$$= \int_0^1 \frac{2dt}{1-\frac{1}{4}+t^2-t+\frac{1}{4}}$$

$$= 2\int_0^1 \frac{dt}{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}$$

$$= 2\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(t-\frac{1}{2}\right)}{\sqrt{3}}\right]_0^1$$

$$= \frac{4}{\sqrt{3}}\left[\tan^{-1}\frac{2\left(\frac{1}{2}\right)}{\sqrt{3}} - \tan^{-1}\frac{2\left(-\frac{1}{2}\right)}{\sqrt{3}}\right]$$

$$= \frac{4}{\sqrt{3}}\left[\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{4\pi}{3\sqrt{3}}$$

3

(ii) Show that

$$\int_0^{2a} f(x)\,dx = \int_0^a \left[f(x)+f(2a-x)\right]dx$$

$$LHS = \int_0^{2a} f(x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

and

$\int_a^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx$ by reflecting in the line $x = a$

OR

Let $u = 2a - x$

$du = -dx$

$When \quad x = a \quad u = a$

$When \quad x = 2a \quad u = 0$

$$\int_a^{2a} f(u)\,du = \int_0^a f(2a-x)\,dx$$

$$LHS = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$$= RHS$$

$$\int_0^{\pi} \frac{x}{1-\frac{1}{2}\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{x}{1-\frac{1}{2}\sin x} + \frac{\pi - x}{1-\frac{1}{2}\sin\left(\pi - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x}{1-\frac{1}{2}\sin x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\pi}{1-\frac{1}{2}\sin\left(\pi - x\right)}\,dx - \int_0^{\frac{\pi}{2}} \frac{x}{1-\frac{1}{2}\sin\left(\pi - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x}{1-\frac{1}{2}\sin x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\pi}{1-\frac{1}{2}\sin x} - \int_0^{\frac{\pi}{2}} \frac{x}{1-\frac{1}{2}\sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\pi}{1-\frac{1}{2}\sin x}\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{1-\frac{1}{2}\sin x}\,dx = \frac{4\pi^2}{3\sqrt{3}}$$