NORTH SYDNEY GIRLS HIGH SCHOOL

Mathematics Extension 2
2015 HSC TASK 3

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this exam.
- Show all necessary working in questions 6 & 7

Total Marks – 41

Section I Pages 2–3

5 Marks

- Attempt Questions 1–5
- Allow about 7 minutes for this section.

Section II Pages 4–7

36 marks

- Attempt Questions 6 & 7
- Allow about 48 minutes for this section

NAME: ______________________ **TEACHER** ________________

NUMBER: ____________________

	Q1-2	Q3-5	Q6 a	Q6 b – f	Q7 a	Q7 b c	Q7 d	
E4	/2		/7			/9		/18
E8		/3		/12	/3		/5	23
								/41

Section I

5 marks
Attempt Questions 1–5
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

1. Which pair of equations are the directrices of the hyperbola $16x^2 - 25y^2 = 400$?

(A) $x = \pm\frac{25}{\sqrt{41}}$

(B) $x = \pm\frac{1}{\sqrt{41}}$

(C) $x = \pm\frac{\sqrt{41}}{4}$

(D) $x = \pm\frac{\sqrt{41}}{25}$

2. What is the equation of the chord of contact from $(5,-2)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$?

(A) $\frac{x}{2} - \frac{5y}{16} = 1$

(B) $\frac{5x}{16} - \frac{2y}{9} = 1$

(C) $\frac{5x}{16} + \frac{2y}{9} = 1$

(D) $\frac{5x}{9} - \frac{y}{2} = 1$

3. Which expression is equal to $\int \cot x \, dx$?

(A) $-\operatorname{cosec}^2 x + C$.

(B) $-\ln(\operatorname{cosec} x) + C$

(C) $\frac{\cot^2 x}{2} + C$

(D) $-\ln(\operatorname{cosec} x + \cot x) + C$

4. Given that $2\cos A\cos B=\cos(A-B)+\cos(A+B)$, which of the following is equal to

$$\int \cos 3x \cos 4x \, dx?$$

(A) $\frac{1}{2}\sin x+\frac{1}{2}\sin 7x+C$

(B) $\frac{1}{2}\cos(-x)+\frac{1}{14}\cos 7x+C$

(C) $\frac{1}{2}\sin x+\frac{1}{14}\sin 7x+C$

(D) $-\frac{1}{2}\sin(-x)-\frac{1}{14}\sin 7x+C$

5. Which of the following definite integrals is greater than zero for $0<a<1$?

(A) $\int_{-a}^{a}\frac{\tan^{-1}x}{\cos x}dx$

(B) $\int_{-a}^{a}-\frac{\tan^{-1}x}{\sin^{-1}x}dx$

(C) $\int_{-a}^{a}e^{-x^2}dx$

(D) $\int_{-a}^{a}\frac{\sin^{-1}x}{1+x^2}dx$

END OF SECTION I

Section II

36 marks
Attempt Questions 6 & 7
Allow about 48 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

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Question 6 (19 marks)

(a) The point $P\,(3\sec\theta, \tan\theta)$ lies on an hyperbola with centre at the origin and major axis $y=0$.

(i) Write down the Cartesian equation of the hyperbola. **1**

(ii) Find the eccentricity of the hyperbola. **1**

(ii) Draw a neat **ONE-THIRD** page sketch of this hyperbola, showing the coordinates of the vertices and the foci, as well as the equations of the directrices and asymptotes. **3**

(iv) Derive the equation of the tangent to the hyperbola at the point P in parametric form. **2**

(b) Find $\int \cos^3 x\ \sin^5 x\,dx$ **2**

(c) Find $\int x\tan^{-1} x\,dx$. **3**

(d) (i) Write $\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}$ in the form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ **2**

(ii) Hence find $\displaystyle\int \frac{3x^2+3x+2}{(x-1)(x^2+1)}dx$ **3**

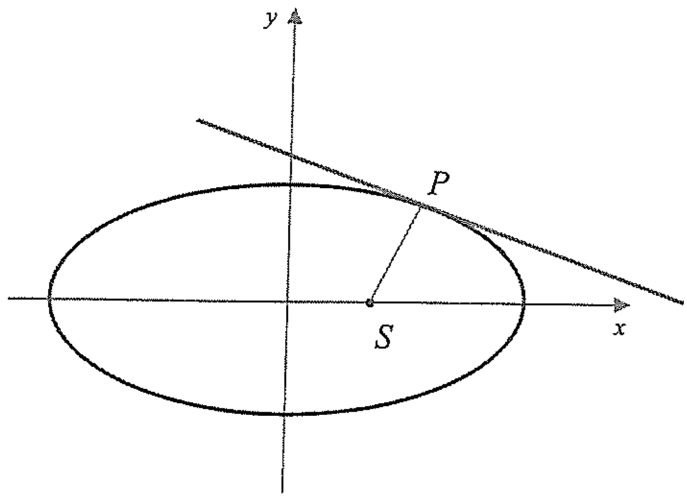
(e) Find $\displaystyle\int \frac{dx}{\sqrt{5-8x-4x^2}}$ **2**

End of Question 6

Question 7 (17 marks)

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta + 2}$ using the substitution $t = \tan\frac{\theta}{2}$ **3**

(b) The diagram below shows the variable point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the focus S on the positive x-axis.



(i) Find the gradient of the tangent at P in terms of a, b and θ. **1**

(ii) Show that the product of the gradient of SP and the gradient of the tangent at P is: **2**

$$\frac{\cos\theta\left(1-e^2\right)}{e-\cos\theta}$$

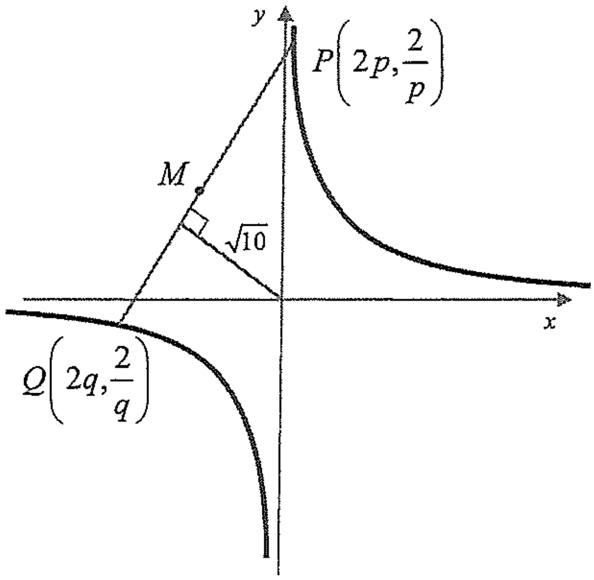
(iii) If P is at the end points of the major axis then PS is perpendicular to the tangent at P. **2**

Use part (ii) to show that there are no other points on the curve where PS is perpendicular to the tangent at P.

Question 7 continues on the next page.

Question 7 (Continued)

(c) In the diagram below $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$ are variable points on the rectangular hyperbola $xy=4$. M is the midpoint of the chord PQ and the perpendicular distance from PQ to the origin is fixed at $\sqrt{10}$ units.



Given that PQ has equation $x+pqy=2(p+q)$

(i) Show that $4(p+q)^2=10(1+p^2q^2)$ **2**

(ii) Find the Cartesian equation of the locus of M. **2**
You are not required to consider the restrictions on this locus.

(d) Let $I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n dx$ for all positive integers n.

(i) Show that $I_{n+1} = \frac{n+1}{n} I_n - \frac{1}{n \times 2^n}$ **3**

(ii) It can be shown that:

$$\frac{1}{n+1} I_{n+1} = I_1 - \sum_{r=1}^{n} \frac{1}{r(r+1)2^r}$$ (DO NOT PROVE THIS)

Hence find the limiting sum of the series: **2**

$$\frac{1}{1\times 2\times 2^1} + \frac{1}{2\times 3\times 2^2} + \frac{1}{3\times 4\times 2^3} + \frac{1}{4\times 5\times 2^4} + \ldots$$

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

1. Which pair of equations are the directrices of the hyperbola $16x^2 - 25y^2 = 400$?

(A) $x = \pm\frac{25}{\sqrt{41}}$

(B) $x = \pm\frac{1}{\sqrt{41}}$

(C) $x = \pm\frac{\sqrt{41}}{4}$

(D) $x = \pm\frac{\sqrt{41}}{25}$

$16x^2 - 25y^2 = 400$ $\frac{x^2}{25} - \frac{y^2}{16} = 1$ $a^2 = 25$ and $b^2 = 16$ $16 = 25(e^2 - 1)$ $\frac{16}{25} = e^2 - 1$ $\frac{41}{25} = e^2$ $e = \frac{\sqrt{41}}{5}$	Directrices are: $x = \pm\frac{a}{e}$ $= \pm\frac{5}{\frac{\sqrt{41}}{5}}$ $= \pm\frac{25}{\sqrt{41}}$

2. What is the equation of the chord of contact from $(5,-2)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$?

(A) $\frac{x}{2} - \frac{5y}{16} = 1$

(B) $\frac{5x}{16} - \frac{2y}{9} = 1$

(C) $\frac{5x}{16} + \frac{2y}{9} = 1$

(D) $\frac{5x}{9} - \frac{y}{2} = 1$

Chord of contact is
$\frac{x_0 x}{9} + \frac{y_0 y}{4} = 1$ $\frac{5x}{9} - \frac{2y}{4} = 1$ $\frac{5x}{9} - \frac{y}{2} = 1$

3. Which expression is equal to $\int \cot x \, dx$?

(A) $-\operatorname{cosec}^2 x + C$

(B) $-\ln(\operatorname{cosec} x) + C$

(C) $\dfrac{\cot^2 x}{2} + C$

(D) $-\ln(\operatorname{cosec} x + \cot x) + C$

$$\begin{aligned}\int \cot x \, dx &= \int \frac{\cos x}{\sin x} dx \\ &= \ln(\sin x) + C \\ &= \ln\left(\frac{1}{\sin x}\right)^{-1} + C \\ &= \ln(\operatorname{cosec} x)^{-1} + C \\ &= -\ln(\operatorname{cosec} x) + C\end{aligned}$$

4. Given that $2\cos A \cos B = \cos(A-B) + \cos(A+B)$, which of the following is equal to

$$\int \cos 3x \cos 4x \; dx ?$$

(A) $\frac{1}{2}\sin x + \frac{1}{2}\sin 7x + C$

(B) $\frac{1}{2}\cos(-x) + \frac{1}{14}\cos 7x + C$

(C) $\frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

(D) $-\frac{1}{2}\sin(-x) - \frac{1}{14}\sin 7x + C$

$$\begin{aligned}\int \cos 3x \cos 4x \; dx &= \frac{1}{2}\int \cos(4x-3x) + \cos(4x+3x) \\ &= \frac{1}{2}\int \cos(x) + \cos(7x) \\ &= \frac{1}{2}\left(\sin(x) + \frac{\sin(7x)}{7}\right) + C \\ &= \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C\end{aligned}$$

5. Which of the following definite integrals is greater than zero for $0 < a < 1$?

(A) $\displaystyle\int_{-a}^{a} \frac{\tan^{-1} x}{\cos x} dx$

(B) $\displaystyle\int_{-a}^{a} -\frac{\tan^{-1} x}{\sin^{-1} x} dx$

(C) $\displaystyle\int_{-a}^{a} e^{-x^2} dx$

(D) $\displaystyle\int_{-a}^{a} \frac{\sin^{-1} x}{1+x^2} dx$

Since $\dfrac{\tan^{-1} x}{\cos x}$, $-\dfrac{\tan^{-1} x}{\sin^{-1} x}$ and $\dfrac{\sin^{-1} x}{1+x^2}$ are all odd functions then the integrals for these will all be zero.

And since e^{-x^2} is an even function and always positive then $\displaystyle\int_{-a}^{a} e^{-x^2} dx > 0$

Question 6 (19 marks)

(a) The point $P\ (3\sec\theta, \tan\theta)$ lies on an hyperbola with centre at the origin and major axis $y = 0$.

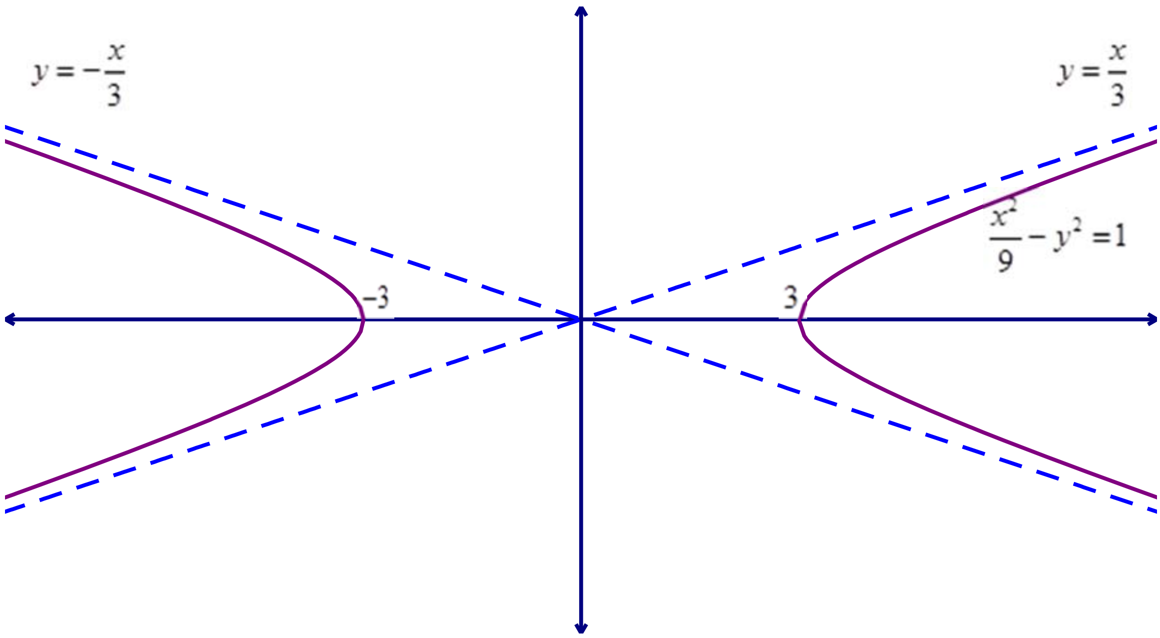
(i) Write down the Cartesian equation of the hyperbola. **1**

$$\frac{(3\sec\theta)^2}{9} - (\tan\theta)^2 = 1$$
$$\frac{x^2}{9} - y^2 = 1$$

(ii) Find the eccentricity of the hyperbola. **1**

$$\frac{x^2}{9} - y^2 = 1$$
$$a^2 = 9 \quad \text{and} \quad b^2 = 1$$
$$1 = 9(e^2 - 1)$$
$$\frac{1}{9} = e^2 - 1$$
$$\frac{10}{9} = e^2$$
$$e = \frac{\sqrt{10}}{3} \qquad \text{since } e > 0$$

(ii) Draw a neat **ONE-THIRD** page sketch of this hyperbola, showing the coordinates of the vertices and the foci, as well as the equations of the directrices and asymptotes. **3**



(iv) Derive the equation of the tangent to the hyperbola at the point P in parametric form. **2**

By differentiating parametrically $x = 3\sec\theta$ $\frac{dx}{d\theta} = 3\sec\theta\tan\theta$ $y = \tan\theta$ $\frac{dy}{d\theta} = \sec^2\theta$ $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ $= \frac{\sec^2\theta}{3\sec\theta\tan\theta}$ $= \frac{\sec\theta}{3\tan\theta}$	The equation of the tangent is : $y - \tan\theta = \frac{\sec\theta}{3\tan\theta}(x - 3\sec\theta)$ $3y\tan\theta - 3\tan^2\theta = x\sec\theta - 3\sec^2\theta$ $3\sec^2\theta - 3\tan^2\theta = x\sec\theta - 3y\tan\theta$ $3(\sec^2\theta - \tan^2\theta) = x\sec\theta - 3y\tan\theta$ $1 = \frac{x\sec\theta}{3} - y\tan\theta$

(b) Find $\int \cos^3 x \ \sin^5 x \, dx$ **2**

$$\begin{aligned}\int \cos^3 x \ \sin^5 x \, dx &= \int \cos x \cos^2 x \sin^5 x \, dx \\ &= \int \cos x \ (1 - \sin^2 x)\sin^5 x \, dx \\ &= \int \cos x \ \sin^5 x \, dx - \int \cos x \ \sin^7 x \, dx \\ &= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C\end{aligned}$$

(c) Find $\int x \tan^{-1} x \, dx$ **3**

$\int x\tan^{-1} x \, dx = \int \frac{d}{dx}\left(\frac{1}{2}x^2\right)\tan^{-1} x \, dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \int \left(\frac{1}{2}x^2\right)\left(\frac{1}{1+x^2}\right) dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} \, dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2} \, dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} \, dx$ $= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x \, dx + C$	

(d) (i) Write $\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}$ in the form $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ **2**

$\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ $=\dfrac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$ $3x^2+3x+2=A(x^2+1)+(Bx+C)(x-1)$ Let $x=1$ $3+3+2=A(1+1)+0$ $8=2A$ $A=4$ Let $x=0$ $2=A(0+1)+(0+C)(0-1)$ $2=A-C$ $2=4-C$ $C=2$ Let $x=-1$ $3-3+2=A(1+1)+(-B+C)(-2)$ $2=2A+2B-2C$ $2=8+2B-4$ $-2=2B$ $B=-1$ $\dfrac{3x^2+3x+2}{(x-1)(x^2+1)}=\dfrac{8}{x-1}+\dfrac{-x+2}{x^2+1}$	

(ii) Hence find $\displaystyle\int \frac{3x^2+3x+2}{(x-1)(x^2+1)}dx$ **3**

$\displaystyle\int \frac{3x^2+3x+2}{(x-1)(x^2+1)}dx=\int\left(\frac{8}{x-1}+\frac{-x+2}{x^2+1}\right)dx$ $\displaystyle=\int\frac{8}{x-1}dx+\int\frac{-x+2}{x^2+1}dx$ $\displaystyle=\int\frac{8}{x-1}dx-\int\frac{x}{x^2+1}dx+\int\frac{2}{x^2+1}dx$ $\displaystyle=8\int\frac{1}{x-1}dx-\frac{1}{2}\int\frac{2x}{x^2+1}dx+2\int\frac{1}{x^2+1}dx$ $\displaystyle=8\ln(x-1)-\frac{1}{2}\ln(x^2+1)+2\tan^{-1}x+C$	

(e) Find $\int \frac{dx}{\sqrt{5-8x-4x^2}}$ **2**

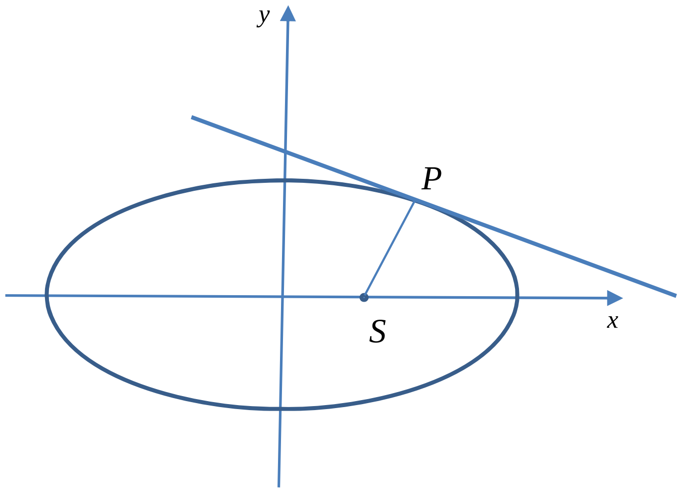
$\int \frac{dx}{\sqrt{5-8x-4x^2}} = \int \frac{dx}{\sqrt{9-4\left(x^2+2x+1\right)}}$ $= \int \frac{dx}{\sqrt{9-4\left(x+1\right)^2}}$ $= \int \frac{dx}{2\sqrt{\left(\frac{3}{2}\right)^2-\left(x+1\right)^2}}$ $= \frac{1}{2}\sin^{-1}\frac{2\left(x+1\right)}{3}+C$	

Question 7 (17 marks)

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta+2}$ using the substitution $t=\tan\frac{\theta}{2}$ **3**

$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos\theta+2}$ $\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2}+2}\times\frac{2}{1+t^2}\,dt = \int_0^1 \frac{2}{1-t^2+2\left(1+t^2\right)}dt$ $= \int_0^1 \frac{2}{3+t^2}dt$ $= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$ $= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}}-\tan^{-1}\frac{0}{\sqrt{3}}\right]$ $= \frac{2}{\sqrt{3}}\times\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$	$t=\tan\frac{\theta}{2}$ $\frac{\theta}{2}=\tan^{-1}t$ $\theta = 2\tan^{-1}t$ $\frac{d\theta}{dt}=\frac{2}{1+t^2}$ $d\theta=\frac{2dt}{1+t^2}$ When $\theta=\frac{\pi}{2}$ $\quad t=\tan\frac{\pi}{4}=1$ $\theta=0$ $\quad t=\tan 0=0$

(b) The diagram below shows the variable point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the focus S on the positive x-axis.



(i) Find the gradient of the tangent at P in terms of a, b and θ. **1**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Differentiate Implicitly $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ $\frac{y}{b^2}\frac{dy}{dx}=-\frac{x}{a^2}$ $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$	At the Point $P(a\cos\theta, b\sin\theta)$ $\frac{dy}{dx}=-\frac{b^2a\cos\theta}{a^2b\sin\theta}$ $m_{Tangent}=-\frac{b\cos\theta}{a\sin\theta}$

(ii) Show that the product of the gradient of SP and the gradient of the tangent at P is: **2**

$$\frac{\cos\theta\left(1-e^2\right)}{e-\cos\theta}$$

Gradient of PS $P(a\cos\theta, b\sin\theta)$ and $S(ae,0)$ $m_{PS}=\frac{b\sin\theta-0}{a\cos\theta-ae}$ $=\frac{b\sin\theta}{a\cos\theta-ae}$	$m_{PS}\times m_{Tangent}=\frac{b\sin\theta}{a\cos\theta-ae}\times-\frac{b\cos\theta}{a\sin\theta}$ $=-\frac{b^2\cos\theta}{a^2(\cos\theta-e)}$ $=\frac{a^2(1-e^2)\cos\theta}{a^2(\text{e}-\cos\theta)}$ $=\frac{(1-e^2)\cos\theta}{(\text{e}-\cos\theta)}$

(iii) If P is at the end points of the major axis then PS is perpendicular to the tangent at P. **2**

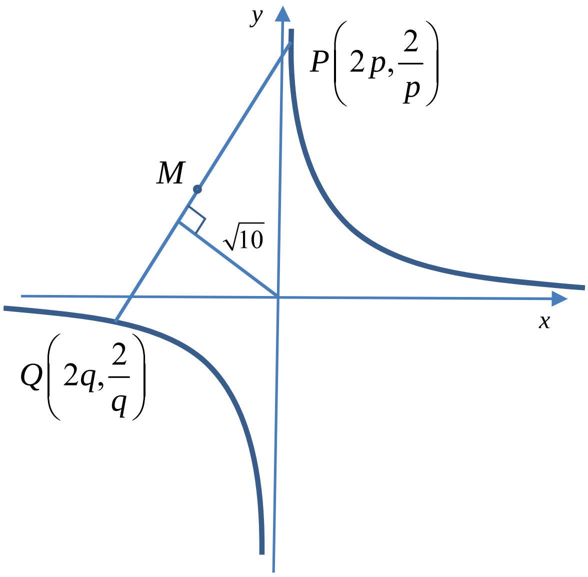
Use part (ii) to show that there are no other points on the curve where PS is perpendicular to the tangent at P.

Let $m_{PS} \times m_{Tangent} = -1$ $\frac{\cos\theta\left(1-e^2\right)}{e-\cos\theta} = -1$ $\cos\theta\left(1-e^2\right) = \cos\theta - e$ $-e^2\cos\theta = -e$ $\cos\theta = \frac{e}{e^2}$ $\cos\theta = \frac{1}{e}$ Since $O < e < 1$ For an ellipse Then $\frac{1}{e} > 1$ Therefore there are no values for θ where $\cos\theta > 1$	letting the product of the two gradients be negative 1 is only valid when neither the tangent at P nor PS is vertical. When PS is vertical then $a\cos\theta = ae$ $\cos\theta = e$ Thus the gradient of the tangent $m_{Tangent}$ is not zero, thus the tangent and PS are not perpendicular. Thus the only place where the tangent and PS are perpendicular is at the end points of the major axis

Question 7 continues on the next page.

Question 7 (Continued)

(c) In the diagram below $P\left(2p, \frac{2}{p}\right)$ and $Q\left(2q, \frac{2}{q}\right)$ are variable points on the rectangular hyperbola $xy = 4$. M is the midpoint of the chord PQ and the perpendicular distance from PQ to the origin is fixed at $\sqrt{10}$ units.



Given that PQ has equation $x + pqy = 2(p+q)$

(i) Show that $4(p+q)^2 = 10(1+p^2q^2)$ **2**

Equation of PQ is $x + pqy = 2(p+q)$ Then perpendicular distance from PQ to O $d = \left\lvert \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right\rvert$ $\sqrt{10} = \left\lvert \frac{(1)(0)+(pq)(0)-2(p+q)}{\sqrt{(1)^2+(pq)^2}} \right\rvert$ $10 = \frac{4(p+q)^2}{1+(pq)^2}$ $4(p+q)^2 = 10(1+p^2q^2)$	

(ii) Find the Cartesian equation of the locus of M. You are not required to consider the restrictions on this locus. **2**

The Midpoint M of PQ:	$4(p+q)^2 = 10(1+p^2q^2)$ From part i)
$x = \frac{2p+2q}{2}$ $= p+q$; $y = \frac{\frac{2}{p}+\frac{2}{q}}{2}$ $= \frac{p+q}{pq}$; $y = \frac{x}{pq}$; $pq = \frac{x}{y}$	$4x^2 = 10 + \frac{10x^2}{y^2}$; $4x^2y^2 = 10y^2 + 10x^2$; $4x^2y^2 - 10y^2 = 10x^2$; $y^2 = \frac{10x^2}{4x^2-10}$; $y^2 = \frac{5x^2}{2x^2-5}$

(d) Let $I_n = \int_1^2 \left(1-\frac{1}{x}\right)^n dx$ for all positive integers n.

(i) Show that $I_{n+1} = \frac{n+1}{n} I_n - \frac{1}{n \times 2^n}$ **3**

Method 1

$$I_n = \int_1^2 \left(1-\frac{1}{x}\right)^n dx = \int_1^2 \left(\frac{d}{dx}(x)\right)\left(1-\frac{1}{x}\right)^n dx$$

$$= \left[x\left(1-\frac{1}{x}\right)^n\right]_1^2 - \int_1^2 xn\left(1-\frac{1}{x}\right)^{n-1}\left(\frac{1}{x^2}\right)dx$$

$$= 2\left(1-\frac{1}{2}\right)^n - 1\left(1-\frac{1}{1}\right)^n - n\int_1^2 \frac{1}{x}\left(1-\frac{1}{x}\right)^{n-1} dx$$

$$= \left(\frac{2}{2^n}\right) - (0) - n\int_1^2 \frac{1}{x}\left(1-\frac{1}{x}\right)^{n-1} dx$$

$$= \left(\frac{1}{2^{n-1}}\right) \quad + n\int_1^2 \left(1-\frac{1}{x}-1\right)\left(1-\frac{1}{x}\right)^{n-1} dx$$

$$= \left(\frac{1}{2^{n-1}}\right) \quad + n\int_1^2 \left(1-\frac{1}{x}\right)^n - \left(1-\frac{1}{x}\right)^{n-1} dx$$

$$I_n = \left(\frac{1}{2^{n-1}}\right) \quad + nI_n - nI_{n-1}$$

$$I_n - nI_n = \frac{1}{2^{n-1}} \quad - nI_{n-1}$$

$$I_n(n-1) = nI_{n-1} - \frac{1}{2^{n-1}}$$

Replace n with $n-1$

$$I_{n+1}(n+1-1)=(n+1)I_{n+1-1}-\frac{1}{2^{n+1-1}}$$

$$nI_{n+1}=(n+1)I_n-\frac{1}{2^n}$$

$$I_{n+1}=\frac{(n+1)}{n}I_n-\frac{1}{n2^n}$$

Method 2

$$I_n=\int_1^2\left(1-\frac{1}{x}\right)^n dx=\int_1^2\left(\frac{x-1}{x}\right)^n dx$$

$$=\int_1^2\frac{(x-1)^n}{x^n}dx$$

$$=\int_1^2\frac{d}{dx}\left(\frac{(x-1)^{n+1}}{n+1}\right)\left(\frac{1}{x^n}\right)dx$$

$$=\int_1^2\frac{d}{dx}\left(\frac{(x-1)^{n+1}}{n+1}\right)\left(x^{-n}\right)dx$$

$$=\left[\frac{(x-1)^{n+1}}{n+1}\frac{1}{x^n}\right]_1^2-\int_1^2\frac{(x-1)^{n+1}}{n+1}\left(-nx^{-n-1}\right)dx$$

$$=\left[\frac{(x-1)^{n+1}}{n+1}\frac{1}{x^n}\right]_1^2+\frac{n}{n+1}\int_1^2\frac{(x-1)^{n+1}}{x^{n+1}}dx$$

$$=\left[\left(\frac{(2-1)^{n+1}}{n+1}\frac{1}{2^n}\right)-(0)\right]+\frac{n}{n+1}I_{n+1}$$

$$I_n=\frac{1}{(n+1)2^n}+\frac{n}{n+1}I_{n+1}$$

$$(n+1)I_n=\frac{1}{2^n}+nI_{n+1}$$

$$\left(\frac{n+1}{n}\right)I_n=\frac{1}{n2^n}+I_{n+1}$$

$$I_{n+1}=\left(\frac{n+1}{n}\right)I_n-\frac{1}{n2^n}$$

(ii) It can be shown that:

$$\frac{1}{n+1}I_{n+1}=I_1-\sum_{r=1}^{n}\frac{1}{r\left(r+1\right)2^r} \qquad \text{(DO NOT PROVE THIS)}$$

Hence find the limiting sum of the series: **2**

$$\frac{1}{1\times2\times2^1}+\frac{1}{2\times3\times2^2}+\frac{1}{3\times4\times2^3}+\frac{1}{4\times5\times2^4}+\ldots$$

Solution

$$\frac{1}{1\times2\times2^1}+\frac{1}{2\times3\times2^2}+\frac{1}{3\times4\times2^3}+\frac{1}{4\times5\times2^4}+\ldots=\sum_{r=1}^{\infty}\frac{1}{r\left(r+1\right)2^r}$$

$$\frac{1}{n+1}I_{n+1}=I_1-\sum_{r=1}^{n}\frac{1}{r\left(r+1\right)2^r}$$

$$\sum_{r=1}^{n}\frac{1}{r\left(r+1\right)2^r}=I_1-\frac{1}{n+1}I_{n+1}$$

As $n\to\infty$

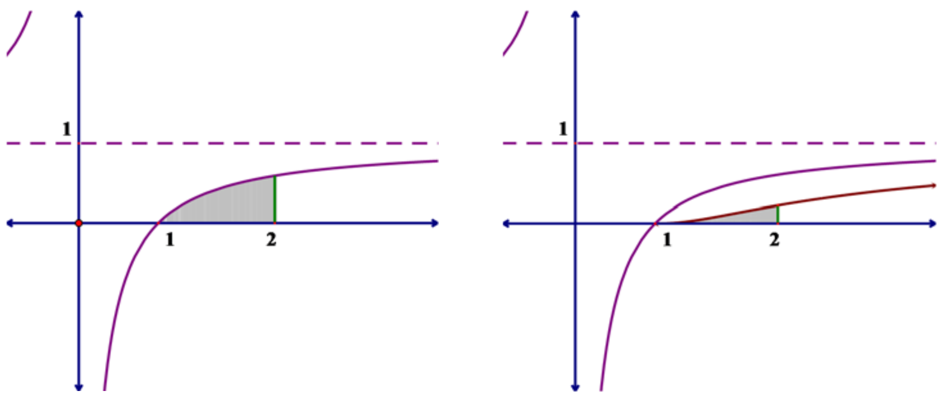
$$\frac{1}{n+1}\to0$$

From the Graph below:

For $1<x<2 \quad 0<1-\dfrac{1}{x}<1$

$$\therefore 0<\left(1-\frac{1}{x}\right)^n<1-\frac{1}{x}$$

$\therefore I_{n+1}<I_n$ for all $n>1$



As $n\to\infty$

$I_{n+1}\to0$

$$\frac{1}{1\times2\times2^1}+\frac{1}{2\times3\times2^2}+\frac{1}{3\times4\times2^3}+\frac{1}{4\times5\times2^4}+\ldots=I_1$$

$$=\int_1^2\left(1-\frac{1}{x}\right)^n dx$$

$$=\left[x-\ln x\right]_1^2$$

$$=1-\ln2$$