2018 | **HSC Assessment Task 3**

Mathematics Extension 2

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 5 – 7, show relevant mathematical reasoning and/or calculations

Total marks – 37

Section I – 4 marks (pages 2 – 3)

- Attempt Questions 1 – 4
- Allow about 6 minutes for this section

Section II – 33 marks (pages 4 – 7)

- Attempt Questions 5 – 7
- Allow about 49 minutes for this section

NAME:____________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

Question	1 – 4	5	6	7	Total
Integration	/2	/8	/8		**/18**
Conics	/2	/5	/3	/9	**/19**
	/4	**/13**	**/11**	**/9**	**/37**

Section I

4 marks
Attempt Questions 1-4
Allow about 6 minutes for this section

Use the multiple choice answer sheet for Questions 1-4.

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1. Which function is a primitive of $\int \frac{2x}{2x-1}$?

(A) $x+\ln|2x-1|$

(B) $\ln|2x-1|$

(C) $x+\frac{1}{2}\ln|2x-1|$

(D) $\frac{1}{2}\ln|2x-1|$

2. Which integral is obtained when the substitution $t=\tan\frac{x}{2}$ is applied to $\int \frac{dx}{5+4\cos x}$?

(A) $\int \frac{2}{9-4t^2}\,dt$

(B) $\int \frac{2}{9+t^2}\,dt$

(C) $\int \frac{1+t^2}{9+t^2}\,dt$

(D) $\int \frac{2\left(1-t^2\right)}{\left(1+t^2\right)\left(9-t^2\right)}\,dt$

3. Every point on a certain conic is twice as far from the line $x = 4$ as it is from the point $(1,0)$. What is a possible equation of the conic?

(A) $\dfrac{x^2}{3} - \dfrac{y^2}{4} = 1$

(B) $\dfrac{x^2}{4} - \dfrac{y^2}{3} = 1$

(C) $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

(D) $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

4. The hyperbolae $\mathcal{H}_1$ and $\mathcal{H}_2$ have equations $x^2 - y^2 = a^2$ and $xy = a^2$ respectively, where a is a positive constant.

Which of the following is a point of intersection of one of the directrices of $\mathcal{H}_2$ and one of the asymptotes of $\mathcal{H}_1$?

(A) $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$

(B) $(2a, 2a)$

(C) $\left(\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$

(D) $\left(a\sqrt{2}, a\sqrt{2}\right)$

Section II

33 marks
Attempt Questions 5-7
Allow about 49 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 5-7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (13 marks) Use a SEPARATE writing booklet

(a) Find $\int x^2\sqrt{3+x^3}\,dx$. **1**

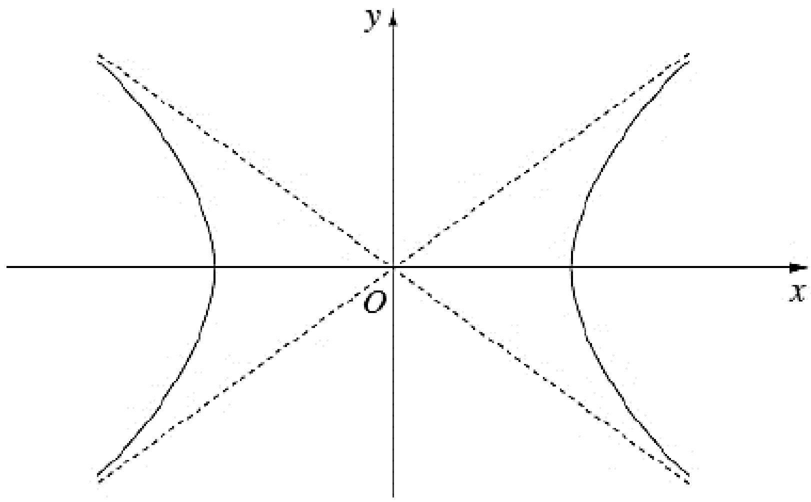
(b) By completing the square in the denominator, or otherwise, evaluate $\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}$. **3**

(c) (i) Find real numbers A and B such that $\frac{1}{\cos x} \equiv \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$. Show working. **2**

(ii) Hence evaluate $\int_{0}^{\frac{\pi}{6}}\sec x\,dx$. **2**

Question 5 continues on page 5

(d) The diagram shows the graph of the hyperbola $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$.



(i) Find the coordinates of the points where the hyperbola intersects the x-axis. **1**

(ii) Find the coordinates of the foci of the hyperbola. **2**

(iii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

Question 6 starts on page 6

Question 6 (11 marks) Use a SEPARATE writing booklet

(a) Let $I_n = \int_1^e (1+\ln x)^n \, dx$, where n is a non-negative integer.

(i) Use integration by parts to show that $I_n = (2^n)e - 1 - nI_{n-1}$ for $n \geq 1$. **2**

(ii) Hence find the exact value of $\int_1^e (2+\ln x)(1+\ln x)\, dx$. **2**

(b) $P\left(p, \frac{1}{p}\right)$ and $Q\left(q, \frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$.
You are given that the equation of the secant PQ is $x + pqy - (p+q) = 0$.
(DO NOT PROVE THIS)

(i) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the secant PQ from the origin $O(0,0)$ is always $\sqrt{2}$, show that $(p+q)^2 = 2(1+p^2q^2)$. **1**

(ii) M is the midpoint of the chord PQ. Find the equation of the locus of M, leaving your answer in implicit form. (There is no need to consider restrictions on the locus) **2**

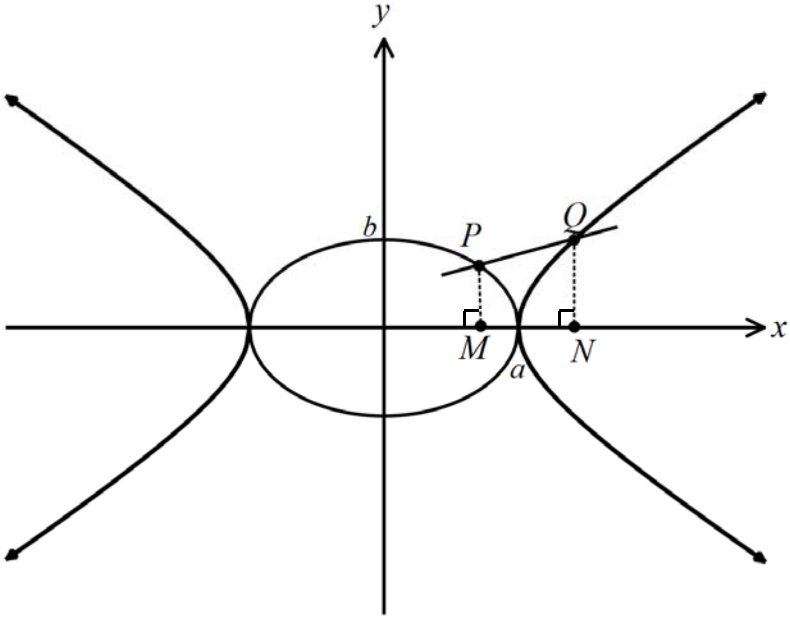
(c) (i) Show that $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **2**

(ii) Hence evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}}\, dx$. **2**

Question 7 starts on page 7

Question 7 (9 marks) Use a SEPARATE writing booklet

For a given value of θ, distinct points $P(a\cos\theta, b\sin\theta)$ and $Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ respectively, as shown, where θ varies such that $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. The points M and N are the feet of the perpendiculars from P and Q respectively to the x-axis.



(i) The line PQ meets the x-axis at T. Without finding the coordinate of T, show that $\frac{TN}{TM}=\sec\theta$. **1**

(ii) Hence find the coordinates of T. **2**

(iii) Show that the tangent to the hyperbola at Q has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. Deduce that it passes through M. **3**

(iv) The tangent to the ellipse at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ and passes through N. (DO NOT PROVE THIS). Let G be the point of intersection of PN and QM.

(α) Show that G always lies on the same vertical line and state its equation. **1**

(β) Where on this line can G lie? Justify your answer. **2**

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2018 | **HSC Assessment Task 3**

Mathematics Extension 2: Solutions

1. Which function is a primitive of $\int \frac{2x}{2x-1}$?

Answer: C

$$\frac{2x}{2x-1}=\frac{2x-1}{2x-1}+\frac{1}{2x-1}$$

$$\therefore \int 1+\frac{1}{2x-1}\,dx = x+\frac{1}{2}\ln|2x-1|$$

2. Which integral is obtained when the substitution $t=\tan\frac{x}{2}$ is applied to $\int \frac{dx}{5+4\cos x}$?

Answer: B

$$\int \frac{1}{5+4\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}\,dt$$

$$=\int \frac{2}{5+5t^2+4-4t^2}\,dt$$

$$=\int \frac{2}{9+t^2}\,dt$$

Let $t=\tan\frac{x}{2} \quad\Rightarrow\quad x=2\tan^{-1}t$

$$dx=\frac{2}{1+t^2}\,dt$$

3. Every point on a certain conic is twice as far from the line $x=4$ as it is from the point $(1,0)$. What is a possible equation of the conic?

Answer: D

$PS=ePM$

$PS=\frac{1}{2}PM$

Since $0<e<1$, conic is an ellipse with focus at $(ae=1,0)$ and directrix at $x=\frac{a}{e}=4$.

$\therefore ae\times\frac{a}{e}=1\times 4 \ \Rightarrow a=2$

Conic is an ellipse with possible equation $\frac{x^2}{4}+\frac{y^2}{3}=1$.

4. The hyperbolae $\mathcal{H}_1$ and $\mathcal{H}_2$ have equations $x^2-y^2=a^2$ and $xy=a^2$ respectively, where a is a positive constant.

Which of the following is a point of intersection of one of the directrices of $\mathcal{H}_2$ and one of the asymptotes of $\mathcal{H}_1$?

Answer: C

The asymptotes of $\mathcal{H}_1$ are $y=\pm x$ and the directrices of $\mathcal{H}_2$ are $x+y=\pm a\sqrt{2}$. Since all options are in first quadrant solve $y=x$ simultaneously with $x+y=a\sqrt{2}$ to get $\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)$.

Question 5

(a)	Find $\displaystyle\int x^2\sqrt{3+x^3}\,dx.$	**1**

$$\int x^2\sqrt{3+x^3}\ dx$$
$$=\int 3x^2\sqrt{3+x^3}\,dx$$
$$=\frac{1}{3}\frac{2\left(3+x^3\right)^{\frac{3}{2}}}{3}+C$$
$$=\frac{2}{9}\sqrt{\left(3+x^3\right)^3}+C$$

Generally very well done. Quite a few students used a substitution instead of just using the reverse chain rule process, which would have been quite time consuming.

(b)	By completing the square in the denominator, or otherwise, evaluate $\displaystyle\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}.$	**3**

$$\int_{-1}^{5}\frac{dx}{\sqrt{32+4x-x^2}}=\int_{-1}^{5}\frac{dx}{\sqrt{32-4-(x^2-4x+(-2)^2}}$$
$$=\int_{-1}^{5}\frac{dx}{\sqrt{32+4-(x^2-4x+(-2)^2)}}$$
$$=\int_{-1}^{5}\frac{dx}{\sqrt{36-(x-2)^2}}$$
$$=\left[\sin^{-1}\frac{x-2}{6}\right]_{-1}^{5}$$
$$=\sin^{-1}\frac{1}{2}-\sin^{-1}\left(-\frac{1}{2}\right)$$
$$=\frac{\pi}{3}$$

Exceptionally well done. Completing the square was done competently and the only errors were
- Dropping the square root and then using inverse tan
- Evaluating the substitution into the integral incorrectly

(c)	(i)	Find real numbers A and B such that $\dfrac{1}{\cos x}\equiv\dfrac{A\cos x}{1-\sin x}+\dfrac{B\cos x}{1+\sin x}$. Show working.	**2**
	(ii)	Hence evaluate $\displaystyle\int_0^{\frac{\pi}{6}}\sec x\,dx.$	**2**

(i)

$$\frac{1}{\cos x}\equiv\frac{A\cos x}{1-\sin x}+\frac{B\cos x}{1+\sin x}$$
$$=\frac{A\cos x\left(1+\sin x\right)+B\cos x\left(1-\sin x\right)}{1-\sin^2 x}$$
$$=\frac{\left(A+B\right)+\left(A-B\right)\sin x}{\cos x}$$

Equating coefficients $A+B=1$

$$A-B=0 \qquad \Rightarrow \qquad A=\frac{1}{2},\ B=\frac{1}{2}$$

Many students had some trouble with part i). Most common errors

- Not noticing the denominators were not the same on both sides and so the numerators were not necessarily equivalent.
- Subsisting values for x into LHS and RHS was an acceptable method but many students chose unsuitable values for x that produced expressions that were difficult to then evaluate A and B.
- Many students were able to arrive at $A+B=1$ but just wrote down $A-B=0$ without any evidence to show where that expression had come from.

(ii)

$$\int_0^{\frac{\pi}{6}} \sec x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(\frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}\right)dx$$

$$=\frac{1}{2}\Big[-\ln|1-\sin x|+\ln|1+\sin x|\Big]_0^{\frac{\pi}{6}}$$

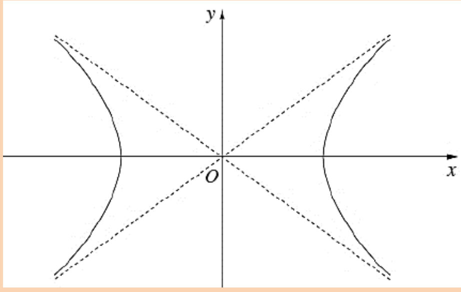
$$=\frac{1}{2}\left[\ln\frac{|1+\sin x|}{|1+\sin x|}\right]_0^{\frac{\pi}{6}}$$

$$=\frac{1}{2}\left(\ln\frac{3/2}{1/2}-\ln 1\right)$$

$$=\frac{1}{2}\ln 3$$

- Most students who were able to correctly obtain A and B were able to go on and evaluate the integral properly.
- It is recommended that If you are unable to evaluate A and B then choose non-trival values so that you are able to complete the second section.
- Since this was a “hence” question, any other method for evaluating the integral was not paid any marks.
- The only common error was leaving off the negative in front of $-\ln|1-\sin x|$ in the second line of working.

(d) The diagram shows the graph of the hyperbola $\frac{x^2}{144}-\frac{y^2}{25}=1$.



(i) Find the coordinates of the points where the hyperbola intersects the x-axis. **1**

(ii) Find the coordinates of the foci of the hyperbola. **2**

(iii) Find the equations of the directrices and the asymptotes of the hyperbola. **2**

$$\frac{x^2}{144}-\frac{y^2}{25}=1.$$

(i) $y=0$: $\quad \frac{x^2}{144}=1$

$$x=\pm 12.$$

$\therefore$ The hyperbola intersects the x-axis at $(12,0)$ and $(-12,0)$.

(ii) The foci have coordinates $(\pm ae, 0)$.

$$\begin{aligned} a &= 12 \\ b^2 &= a^2\left(e^2-1\right) \\ 25 &= 144\left(e^2-1\right) \\ e^2-1 &= \frac{25}{144} \\ e^2 &= \frac{169}{144} \\ e &= \frac{13}{12}, \quad \text{as } e>0. \end{aligned}$$

$\therefore$ The foci have coordinates $(13,0)$ and $(-13,0)$.

(iii) Directrices: $x=\pm\frac{a}{e}$

$$x=\frac{144}{13} \text{ and } x=-\frac{144}{13}.$$

Asymptotes: $y=\pm\frac{b}{a}x$

$$y=\frac{5}{12}x \text{ and } y=-\frac{5}{12}x.$$

Question 6

(a) Let $I_n = \int_1^e (1+\ln x)^n \, dx$, where n is a non-negative integer.

(i) Use integration by parts to show that $I_n = (2^n)e - 1 - nI_{n-1}$ for $n \geq 1$. **2**

(ii) Hence find the exact value of $\int_1^e (2+\ln x)(1+\ln x)\, dx$. **2**

(i)

$$\text{Let } I_n = \int_1^e (1+\ln x)^n \, dx,$$

$$u = (1+\ln x)^n, \qquad v' = 1$$

$$u' = n(1+\ln x)^{n-1}\frac{1}{x}, \quad v = x$$

$$I_n = \left[x(1+\ln x)^n\right]_1^e - n\int_1^e x.\frac{1}{x}(1+\ln x)^{n-1}\, dx \qquad \text{(by parts)}$$

$$= e(1+\ln e)^n - 1.(1+\ln 1)^n - n\int_1^e (1+\ln x)^{n-1}\, dx$$

$$= e.2^n - 1 - nI_{n-1}$$

a)i This is a 2 mark question and is also a ‘show that’ style question. Students should show enough working to earn 2 marks and <u>not</u> skip steps. The first mark was paid for correctly applying integration by parts. The second mark was for showing correct substitution of the limits into the formula and then arriving at the result. Many students barely did enough to earn this second mark. It was marked generously this time but may not be in the future.

When you use integration by parts, state it beside your working in brackets.

(ii)

$$\int_1^e (2+\ln x)(1+\ln x)\, dx = \int_1^e \left(1+(1+\ln x)\right)(1+\ln x)\, dx$$

$$= \int_1^e (1+\ln x) + (1+\ln x)^2\, dx$$

$$= I_1 + I_2$$

$$I_0 = \int_1^e dx = e - 1$$

$$I_1 = 2e - 1 - I_0 = 2e - 1 - e + 1 = e$$

$$I_2 = 2^2 e - 1 - 2I_1 = 4e - 1 - 2e = 2e - 1$$

$$\int_1^e (2+\ln x)(1+\ln x)\, dx = 2e - 1 + e = 3e - 1$$

a)ii Mostly well done. Some students attempted to apply integration by parts again, this time setting $v' = (1+\ln x)$. They then incorrectly tried to find v by using the reduction formula I_n. This is incorrect because I_n is the formula for a particular definite integral from $x = 1$ to e, not an expression for the primitive.

(b)	$P\left(p,\dfrac{1}{p}\right)$ and $Q\left(q,\dfrac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$. You are given that the equation of the secant PQ is $x+pqy-\left(p+q\right)=0$. (DO NOT PROVE THIS)		
	(i)	If P and Q move on the rectangular hyperbola such that the perpendicular distance of the secant PQ from the origin $O\left(0,0\right)$ is always $\sqrt{2}$, show that $\left(p+q\right)^2=2\left(1+p^2q^2\right)$.	**1**
	(ii)	M is the midpoint of the chord PQ. Find the equation of the locus of M, leaving your answer in implicit form. (There is no need to consider restrictions on the locus)	**2**

(i)

$$\left|\frac{0+pq(0)-(p+q)}{\sqrt{1^2+p^2q^2}}\right|=\sqrt{2}$$

$$\frac{(p+q)^2}{1+p^2q^2}=2$$

$$\therefore \quad (p+q)^2=2(1+p^2q^2)$$

b)i	Many students incorrectly stated that $\left|-(p+q)\right|=p+q$. This is not always the case (think about if $(p+q)$ is negative). This was penalised ½ a mark. Best to leave it inside absolute value signs and then square it.

(ii)

$$x_M=\frac{p+q}{2} \qquad (1)$$

$$y_M=\frac{\frac{1}{p}+\frac{1}{q}}{2}=\frac{p+q}{2pq} \qquad (2)$$

$$(p+q)^2=2(1+p^2q^2) \qquad (3)$$

Rearrange (1) $\quad 2x=p+q$

Sub. Into (2) $\quad y=\dfrac{x}{pq}\Rightarrow \qquad pq=\dfrac{x}{y}$

Sub. Into (3) $\quad \left(2x\right)^2=2\left(1+\dfrac{x^2}{y^2}\right)$

$$2x^2=1+\frac{x^2}{y^2}$$

$$2x^2y^2=x^2+y^2$$

b)ii	When finding the locus of a point, you are looking for a Cartesian equation in x and y. You need to eliminate the parameters so you cannot have p or q in your final answer.

(c)	(i)	Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.	**2**
	(ii)	Hence evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx$.	**2**

(i)

$$\int_0^a f(a-x)\,dx \qquad \text{Let } u = a-x$$

$$= -\int_a^0 f(u)\,du \qquad du = -dx$$

$$= \int_0^a f(u)\,du \qquad \text{when } \quad x=0,\ \ u=a$$

$$= \int_0^a f(x)\,dx \quad \text{(dummy variable)} \qquad x=a,\ \ u=0$$

c)i	Overall, well done. This is a 'show that' question. You need to <u>show</u> enough working to justify earning 2 marks.

(ii)

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{e^{\sin\left[\left(\frac{\pi}{2}\right)-x\right]}}{e^{\sin\left[\left(\frac{\pi}{2}\right)-x\right]}+e^{\cos\left[\left(\frac{\pi}{2}\right)-x\right]}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{e^{\sin\left[\left(\frac{\pi}{2}\right)-x\right]}}{e^{\sin\left[\left(\frac{\pi}{2}\right)-x\right]}+e^{\cos\left[\left(\frac{\pi}{2}\right)-x\right]}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}\,dx$$

$$\therefore \quad 2I = \int_0^{\frac{\pi}{2}} \frac{e^{\sin x}+e^{\cos x}}{e^{\sin x}+e^{\cos x}}\,dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

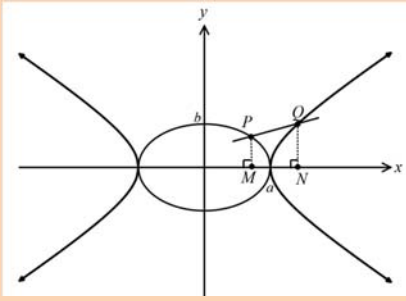
$$= \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

c)ii	Generally well done. Even if students could not complete this part, nearly all could earn the first mark by correctly applying the result in part (i).

Question 7

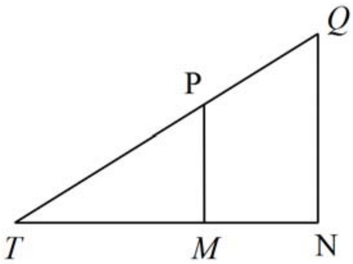
For a given value of θ, distinct points $P(a\cos\theta, b\sin\theta)$ and $Q(a\sec\theta, b\tan\theta)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ respectively, as shown, where θ varies such that $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$. The points M and N are the feet of the perpendiculars from P and Q respectively to the x-axis.



(i) The line PQ meets the x-axis at T. Without finding the coordinate of T, show that $\dfrac{TN}{TM}=\sec\theta$. **1**

(ii) Hence find the coordinates of T. **2**

(i)



ΔTPM is similar to ΔTQN (equiangular)

$$\frac{TN}{TM}=\frac{QN}{PM} \quad \text{(matching sides in similar triangles in ratio)}$$
$$=\frac{b\tan\theta}{b\sin\theta}$$
$$=\frac{b\sin\theta}{\cos\theta}\times\frac{1}{b\sin\theta}$$
$$=\frac{1}{\cos\theta}$$
$$=\sec\theta$$

(i) A large number of students quoted 'parallel lines preserve ratio', but that result doesn't explain the ratios of the sides being used. A similarity proof was not required, but a reference to similar triangles was necessary.

(ii) Let the coordinates of T be $(x,0)$.

$$\frac{|x-a\sec\theta|}{|x-a\cos\theta|}=\sec\theta \qquad \text{from (i)}$$

$$|x-a\sec\theta|=\sec\theta|x-a\cos\theta|$$

$$|x-a\sec\theta|=|x\sec\theta-a\cos\theta\sec\theta| \qquad \left(\text{since } \sec\theta>0 \text{ when } -\frac{\pi}{2}<\theta<\frac{\pi}{2}\right)$$

$$|x-a\sec\theta|=|x\sec\theta-a|$$

Squaring b.s.

$$x^2 - 2xa\sec^2\theta + a^2\sec^2\theta = x^2\sec^2\theta - 2xa\sec^2\theta + a^2$$

$$x^2\left(\sec^2\theta - 1\right) = a^2\left(\sec^2\theta - 1\right)$$

$$x^2 = a^2$$

$x = -a$ $\qquad$ $\left(x \neq a, \text{ since } P \text{ and } Q \text{ are distinct}\right)$

$\therefore T\left(-a, 0\right)$

(ii)	Many students subtracted the wrong way around to find a distance. Many students added coordinates instead of subtracting. Others indicated they knew they were performing a distance calculation, but forgot to convert to a coordinate at the end.

(iii) Show that the tangent to the hyperbola at Q has equation $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$. **3**

Deduce that it passes through M.

(iv) The tangent to the ellipse at P has equation $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ and passes through N. (DO NOT PROVE THIS). Let G be the point of intersection of PN and QM.

(α) Show that G always lies on the same vertical line and state its equation. **1**

(β) Where on this line can G lie? Justify your answer. **2**

(iii)

$$x = a\sec\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = a\sec\theta\tan\theta \qquad\qquad y = b\tan\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = b\sec^2\theta.\frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

Equation of tangent at Q:

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}\left(x - a\sec\theta\right)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$

$$= ab$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

Substitute $M\left(a\cos\theta, 0\right)$ into equation of tangent $\Rightarrow$ LHS $= \dfrac{a\cos\theta\sec\theta}{a} - 0$

= RHS

$\therefore M$ lies on tangent.

(iii)	Done well. But some students need to practise these routine questions.

(iv) (α) Solve simultaneously:

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \qquad (1)$$

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad (2)$$

Rearrange (1)	$xb\sec\theta-ay\tan\theta=ab$	(3)
Rearrange (2)	$bx\cos\theta+ay\sin\theta=ab$	(4)
$(3)\times\cos\theta$	$xb-ay\sin\theta=ab\cos\theta$	(5)
(4)+(5)	$bx(\cos\theta+1)=ab(\cos\theta+1)$	

$x=a$ which is independent of θ.

$\therefore G$ always lies on the same vertical line $x=a$.

(β) Sub. $x=a$, into (1)

$$\frac{ab}{\cos\theta}-ay\tan\theta=ab$$

$$\frac{b}{\cos\theta}-y\tan\theta=b \qquad (a\neq 0)$$

$$y\frac{\sin\theta}{\cos\theta}=\frac{b}{\cos\theta}-b$$

$$y=\frac{b(1-\cos\theta)}{\sin\theta}$$ This is the y-coordinate of G.

As $\theta\to\frac{\pi}{2}$, $y\to b$

As $\theta\to-\frac{\pi}{2}$, $y\to -b$

$\therefore G$ lies on the line $x=a$ where $-b<y<b$.

Also when $\theta=0$, P would have coordinates $(a,0)$ and Q would also be at $(a,0)$. These are not distinct and G would also have coordinates $(a,0)$..

$\therefore G$ lies on the line $x=a$ where $-b<y<b$ and $y\neq 0$.

(iv) (alpha) Solving simultaneously involves elimination. Simply subtracting the two equations does not eliminate a pronumeral.

(beta)