The Scots College

Year 12 Mathematics Extension 2

Task 3

10 June 2010

Name:_______________

General Instructions

- Working time - 45 minutes
- Write using blue or black pen
- Board approved calculators may be Used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: 32

WEIGHTING: 20 %

Question 1 (Marks 17)

a) Evaluate [2]

$$\int_{3}^{4} \frac{dx}{\sqrt{x^2 - 9}}$$

b) By completing the square, find [2]

$$\int \frac{1}{x^2 - 10x + 34}\, dx$$

c) i) Find the real numbers a , b and c such that [2]

$$\frac{3 - x}{(1 + 2x^2)(1 + 6x)} \equiv \frac{ax + b}{1 + 2x^2} + \frac{c}{1 + 6x}$$

ii) Hence find [3]

$$\int_{0}^{2} \frac{3 - x}{(1 + 2x^2)(1 + 6x)}\, dx$$

d) Find [3]

$$\int (x \log x)^2\, dx$$

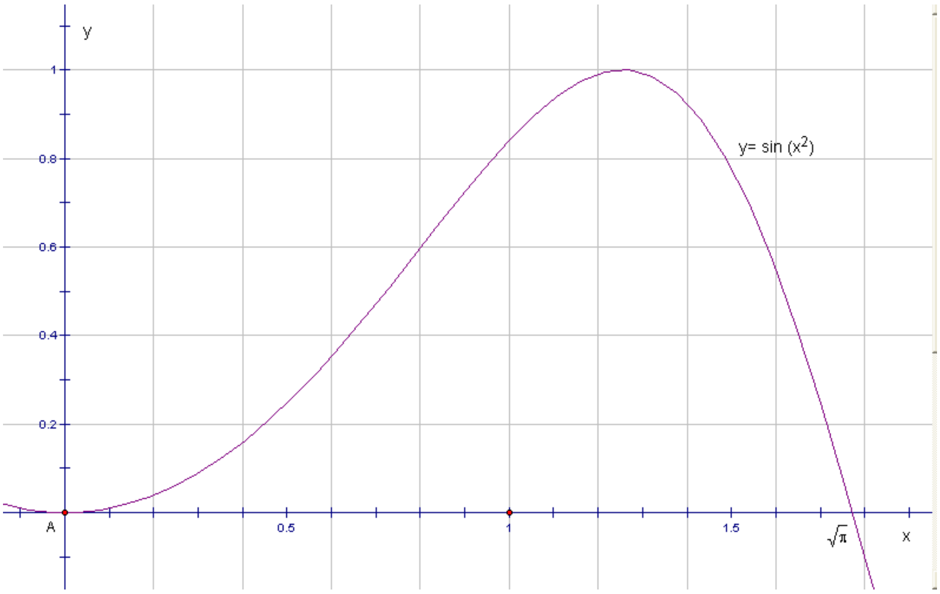
e) i) If $I_n = \int_0^{\frac{\pi}{2}} cos^n x\ dx$, show that $I_n = \frac{n-1}{n} I_{n-2}$ where $n \in J$ and $n \geq 2$. [5]

ii) Hence evaluate I_3 .

Question 2 (Marks 17)

a) A hollow glass container is to be formed by rotating the curve $y = \cos x$ about the line $y = -1$ between $x = 0$ and $= \pi$. If one unit in the Cartesian plane represents 5 cm, find the volume of liquid it can hold to the nearest millilitre. [5]

b) Copy the figure below into your worksheet and find the volume of the solid generated by rotating the shaded region about the y-axis. Use the cylindrical shell method. [5]



c) The base of a particular solid is $x^2 + y^2 = 9$. Find the volume of the solid if every cross-section perpendicular to the x-axis is a semi-ellipse with minor axis in the base of the solid and semi-major axis equal to its minor axis. [5]

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solution Ext 2 Task 3 June 2010

Q.1.

(a) $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-9}}\,dx$

$= \displaystyle\int_3^4 \frac{1}{\sqrt{x^2-3^2}}\,dx$

$= \ln\left(x+\sqrt{x^2-9}\right)\Big]_3^4$ ✓

$= \ln\left(4+\sqrt{16-9}\right) - \ln\left(3+\sqrt{0}\right)$

$= \ln\left(\frac{4+\sqrt{7}}{3}\right)$ ✓

(b) $\displaystyle\int \frac{1}{x^2-10x+34}\,dx$

$= \displaystyle\int \frac{1}{x^2-10x+25+9}\,dx$

$= \displaystyle\int \frac{1}{(x-5)^2+9}\,dx$ ✓

$= \frac{1}{3}\tan^{-1}\left(\frac{x-5}{3}\right)+c$ ✓

(c) (i) $\dfrac{3-x}{(1+2x^2)(1+6x)} \equiv \dfrac{ax+b}{1+2x^2} + \dfrac{c}{1+6x}$

$3-x \equiv (ax+b)(1+6x) + c(1+2x^2)$

$x=-\frac{1}{6} \Rightarrow \quad c\left(1+\frac{\cancel{2}^{1}}{\cancel{36}_{18}}\right) = 3+\frac{1}{6}$

$\frac{19}{18}c = \frac{19}{6}$

$c = 3$ #

coeff. of x^2 $\quad 6a+2c=0$

$3a = -c$

$3a = -3$

$a=-1$ #

$x=0 \Rightarrow \quad b+c=3$

$b+3=3$

$b=0$ #

(ii) $\displaystyle\int_0^2 \frac{3-x}{(1+2x^2)(1+6x)}\,dx$

$= \displaystyle\int_0^2 \frac{-x}{1+2x^2}\,dx + \int_0^2 \frac{3}{1+6x}\,dx$

$= \left[-\frac{1}{4}\ln(1+2x^2)\right]_0^2 + \frac{1}{2}\left[\ln(1+6x)\right]_0^2$

$= -\frac{1}{4}\ln 9 + \frac{1}{4}\ln 1 + \frac{1}{2}\ln 13 - \frac{1}{2}\ln 1$

$= \frac{1}{2}\ln 13 - \frac{1}{2}\ln 9^{1/2}$

$= \frac{1}{2}\ln(13/3)$

(d) $\displaystyle\int (x\log x)^2\,dx$

$= \displaystyle\int x^2(\log x)^2\,dx$

$= (\log x)^2\frac{x^3}{3} - \displaystyle\int 2(\log x)\cdot\frac{1}{x}\cdot\frac{x^3}{3}\,dx$ ✓

$= \frac{x^3}{3}(\log x)^2 - \frac{2}{3}\displaystyle\int x^2\log x\,dx$

$= \frac{x^3}{3}(\log x)^2 - \frac{2}{3}\left[(\log x)\frac{x^3}{3} - \displaystyle\int \frac{1}{x}\cdot\frac{x^3}{3}\,dx\right]$ ✓

$= \frac{x^3}{3}(\log x)^2 - \frac{2}{9}x^3\log x + \frac{2}{3}\displaystyle\int x^2\,dx$

$= \frac{x^3}{3}(\log x)^2 - \frac{2}{9}x^3\log x + \frac{2}{9}x^3 + c$ ✓

(f) $I_n = \int_0^{\pi/2} \cos^n x \, dx$

$= \int_0^{\pi/2} \cos^{n-1} x . \cos x \, dx$

$u = \cos^{n-1} x \qquad dv = \cos x \, dx$

$\frac{du}{dx} = -(n-1) \cos^{n-2} x \sin x \qquad v = \sin x$

$= \left[\cos^{n-1} x . \sin x \right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} \sin x . \sin x \, dx .$

$= 0 + (n-1) \int_0^{\pi/2} \cos^{n-2} x . \sin^2 x \, dx$

$= (n-1) \int_0^{\pi/2} \cos^{n-2} x \, (1 - \cos^2 x) \, dx$

$= (n-1) \left[\int_0^{\pi/2} \cos^{n-2} x \, dx - \int_0^{\pi/2} \cos^n x \, dx \right]$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{(n-1)}{n} I_{n-2}$

$I_3 = \frac{2}{3} I_1$

$= \frac{2}{3} \int_0^{\pi/2} \cos x \, dx$

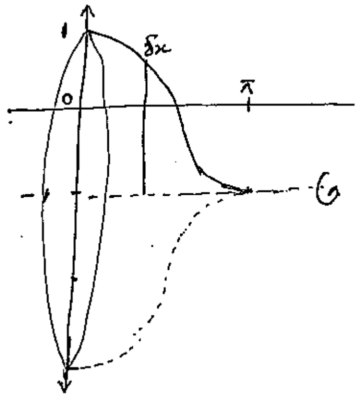
$= \frac{2}{3} \left[\sin x \right]_0^{\pi/2}$

$= \frac{2}{3} \left[1 - 0 \right]$

$= \frac{2}{3}$

Q2 (a)

$$\delta V = \pi (y+1)^2 \, \delta x$$
$$\delta V = \pi (\cos x + 1)^2 \delta x$$

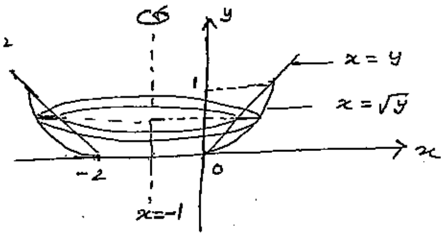


$$V = \pi \int_0^{\pi} (\cos x + 1)^2 \, dx$$
$$= \pi \int_0^{\pi} (\cos^2 x + 2\cos x + 1) \, dx$$
$$= \pi \int_0^{\pi} \left(\tfrac{1}{2}(1 + \cos 2x) + 2\cos x + 1 \right) dx$$
$$= \pi \left[\frac{3x}{2} + \sin 2x + 2 \sin x \right]_0^{\pi}$$
$$= \pi \left\{ \left[\frac{3\pi}{2} + 0 + 0 \right] - [0] \right\}$$
$$= \frac{3\pi^2}{2} \quad \text{units}^3.$$

Now 1 cubic unit $= 5 \times 5 \times 5 = 125 \text{ cm}^3$

$$\therefore \quad V = \frac{3\pi^2}{2} \times 125$$
$$\doteqdot 589 \text{ cm}^3$$
$$\doteqdot 589 \text{ mL}.$$

(b)

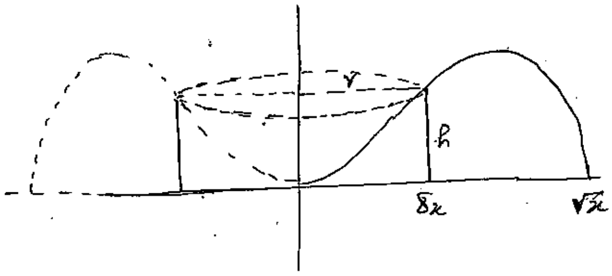


$$A = \pi (\text{outer radius})^2 - \pi (\text{inner radius})^2$$
$$= \pi (1 + \sqrt{y})^2 - \pi (1 + y)^2$$
$$= \pi \left(2\sqrt{y} - y - y^2 \right)$$

$$V = \pi \int_0^1 (2\sqrt{y} - y - y^2) \, dy$$
$$= \pi \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$
$$= \frac{\pi}{2}$$

(c)

$\delta V = 2\pi r h \, \delta x$



$V = 2\pi \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$

$u = x^2$

$du = \cancel{dx}\ 2x\, dx.$

$\frac{du}{2x} = dx$

$\frac{dv}{dx} = \sin(x^2)$

$v = -\frac{\cos(x^2)}{2x}$

$V = \cancel{2}\pi \int_0^{\sqrt{\pi}} \cancel{x} \sin u \frac{du}{\cancel{2x}}$

$x = 0 \quad u = 0$

$x = \sqrt{\pi} \quad u = \pi$

$= \pi \int_0^{\pi} \sin u \, du$

$= \pi \left[-\cos u\right]_0^{\pi}$

$= \pi \left[-\cos \pi + \cos 0\right]_0^{\pi}$

$= \pi$

(d)

Volume of cross-section :

$\delta V = \frac{8y^2}{3} \delta x$

$\delta V = \frac{8}{3}(9 - x^2)^2 \delta x$

$V = \frac{8}{3} \int_{-3}^{3} (9 - x^2)\, dx$ ✓

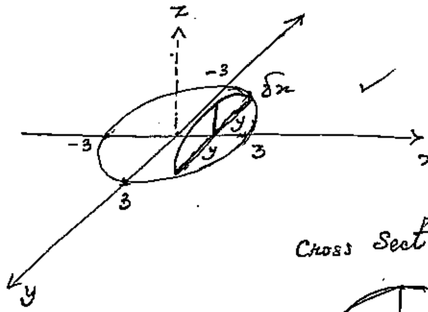
$= \frac{8}{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$

$= \frac{8}{3}\left[(27 - 9) - (-27 + 9)\right]$

$= \frac{8}{3}\left[18 + 18\right]$

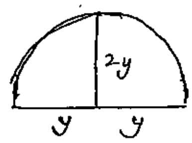
$= 8 \times 12$

$= 96$ cubic units. ✓



✓

Cross Section



✓

simpsons Rule.

$A = \frac{y}{3}\left\{0 + 4(2y) + 0\right\}$

$= \frac{8y^3}{3}$ ✓