The Scots College

Year 12 Mathematics Extension 2

Task 3

15 June 2011

Name:________________

General Instructions

- Working time - 45 minutes
- Write using blue or black pen
- Board approved calculators may be Used (Non Graphic)
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: 35

WEIGHTING: 20 %

Marks : **Q 1 (Integration / 15)**
Q 2 (Polynomials /8)
Q 3 (Volumes / 7)
Q 4 (Volumes / 5)

Total : **/35** **Percentage:**

Question 1 (Marks 15)

a) Find [2]

$$\int \frac{dx}{\sqrt{x^2 - 4\mathrm{x} + 5}}$$

b) Evaluate [4]

$$\int_0^{\frac{\pi}{2}} \frac{2}{2 + \sin\theta} \quad d\theta$$

c) Find [4]

$$\int \tan^2 x \ \sec x \ dx$$

(You may assume $\int \sec x \ \ dx \ = \ln(\sec x \ + \ \tan x)$)

d) i) If $I_n = \int_0^1 x^n e^x \ dx$, where n is a non-negative integer [5]

show that $I_n \ = \ e \ - \ n \ I_{n-1}$

ii) Hence evaluate I_3 .

Question 2 (Marks 8)

The quartic polynomial $f(x) \ = \ x^4 + px^3 + qx^2 + rx + s$ has four zeroes $\alpha, \ \beta, \ \gamma \ and \ \delta$ such that the sum of $\alpha \ and \ \beta$ equals the sun of $\gamma \ and \ \delta$. [8]
Let $A = \ \alpha\beta$, $B \ = \ \gamma\delta \ \ and \ \ C = \ \alpha + \beta = \gamma + \delta.$

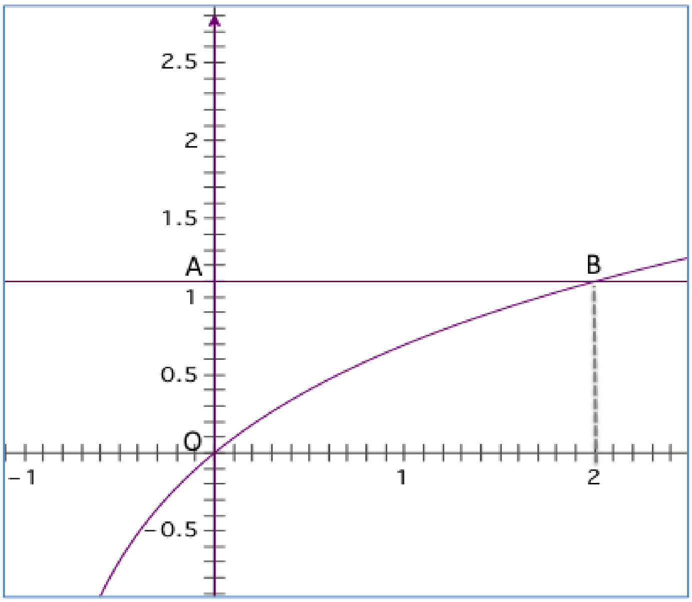
i) Find $p, \ q, \ r \ \ and \ \ s$ in terms of $A, \ B \ \ and \ \ C.$

ii) Show that the coefficients of $f(x)$ satisfy the condition
$p^3 + 8r = 4pq.$

iii) The polynomial $g(x) \ = \ x^4 - \ 18x^3 + \ 79x^2 + \ 18x - 440$ has the property that the sum of two of the zeroes equals the sum of the other two zeroes. Using the identities of part (i) or otherwise, find all the zeroes of $g(x)$.

Question 3 (Marks 7)

The diagram below is the graph of $y = \ln(x+1)$ [7]



i) Find the exact coordinates of B.

ii) Using the method of cylindrical shells, calculate the volume of solid of revolution when the shaded area is rotated about the y-axis.

Question 4 (Marks 5)

Any section perpendicular to the x-axis is a rectangle whose base lies on the semi-circle with radius 2. The other edge is bounded by the semi-ellipse whose major axis is along the x-axis and it's lengths of semi major and semi minor axes are 2 and 1 respectively. [5]

By slicing the solid perpendicular to the x-axis calculate the volume of the solid formed.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1.

(a) $\int \frac{dx}{\sqrt{x^2-4x+5}}$

$\int \frac{dx}{\sqrt{(x-2)^2+1}}$

$= \ln\left\{(x-2)+\sqrt{(x-2)^2+1}\right\}$

$= \ln\left\{x-2+\sqrt{x^2-4x+5}\right\}.$

(b) $t = \tan(\frac{\theta}{2})$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2(\theta/2)$

$= \frac{1}{2}(1+t^2)$

$d\theta = \frac{2dt}{1+t^2}$

$\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{2} \quad t = 1$

$\int_0^{\pi/2} \frac{2}{2+\sin\theta}\, d\theta$

$= \int_0^1 \frac{\cancel{2}}{2+\left(\frac{2t}{1+t^2}\right)} \frac{2\,dt}{(1+t^2)}$

$= 2\int_0^1 \frac{dt}{t^2+t+1}$

$= 2\int_0^1 \frac{dt}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$= \left[2 \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{(t+\frac{1}{2})2}{\sqrt{3}}\right)\right]_0^1$

$= 2 \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right].$

$= \frac{4}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right].$

$= \frac{4}{\sqrt{3}}\left[\frac{\pi}{6}\right].$

$= \frac{2}{3\sqrt{3}}\pi$ ✓

(c) $\int \tan^2 x \sec x\, dx$

$I = \int \sec^3 x - \sec x\, dx$

$I = \int \sec^3 x\, dx - \int \sec x\, dx.$

$u = \sec x \quad v' = \sec^2 x$

$u' = \sec x \tan x \quad v = \tan x$

$= \sec x \tan x - \int \tan x \sec x \tan x\, dx - \ln[\sec x + \tan x]$

$I = \sec x \tan x - \int \tan^2 x \sec x\, dx - \ln[\sec x + \tan x]$

$2I = \sec x \tan x - \ln[\sec x + \tan x]$

$I = \frac{1}{2}\left[\sec x \tan x - \ln[\sec x + \tan x]\right].$

(d) $I_n = \int_0^1 x^n e^x\, dx$

$u = x^n. \quad dv = e^x dx$

$\frac{du}{dx} = nx^{n-1} \quad v = e^x$

$I_n = x^n e^x\Big]_0^1 - \int_0^1 n\, x^{n-1} e^x\, dx$

$= e - n\int_0^1 x^{n-1} e^x\, dx$

$= e - n I_{n-1}$ ✓

Now $I_3 = e - 3I_2$

$= e - 3\int_0^1 x^2 e^x\, dx$

$= e - 3[e - 2I_1]$

$= e - 3[e - 2(e - I_0)]$

$I_0 = \int_0^1 e^x dx$

$e^x\Big]_0^1$

$e - 1$

$= e - 3[e - 2(e - (e-1))].$

$= e - 3[e - 2]$

$= e - 3e + 6$

$= 6 - 2e$ ✓

Q2

(i) $p = -(\alpha+\beta+\gamma+\delta)$

$= -2C$

$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\gamma + \gamma\delta$

$= A + B + \alpha(\gamma+\delta) + \beta(\gamma+\delta)$

$= A + B + C^2$

$r = -(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)$

$= -(A\gamma + B\beta + B\alpha + A\delta)$

$= -(A+B)C$

$s = \alpha\beta\gamma\delta$

$= AB$

2 ✓✓

(ii) $p^3 + 8r = 4pq$

LHS. $= -8C^3 - 8(A+B)C$

$= -8C(A+B+C^2)$

$= 4pq.$ ✓✓

(iii) $-2C = -18 \Rightarrow C = 9$ ✓

$A+B+C^2 = 79 \Rightarrow A+B = -2$

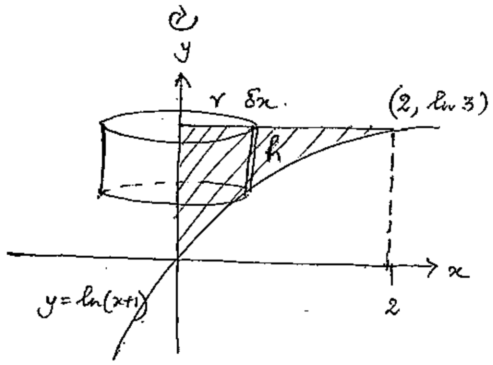
$-(A+B)C = 18$

$AB = -440$

$A = -22$ ✓, $B = 20$ ✓

$\therefore \alpha = 11., \beta = -2, \gamma = 4 \quad \delta = 5$ ✓

Q.3



$\delta v = 2\pi r h \, \delta x.$

$V = 2\pi \int_0^2 x \left[\ln 3 - \ln(x+1) \right] dx.$

$= 2\pi \left[\int_0^2 x \ln 3 \, dx - \int_0^2 x \ln(x+1) \, dx \right]$

$u = \ln(x+1) \qquad dv = x\,dx$

$\frac{du}{dx} = \frac{1}{x+1} \qquad v = \frac{x^2}{2}$

$= 2\pi \left\{ \left. \ln 3 \times \frac{x^2}{2} \right]_0^2 - \left[\frac{x^2}{2} \ln(x+1) \Big|_0^2 - \int_0^2 \frac{x^2}{2} \frac{1}{x+1} \, dx \right] \right\}$

$= 2\pi \left\{ 2 \cancel{\ln 3} - 0 - 2 \cancel{\ln 3} + 0 + \frac{1}{2} \int_0^2 \frac{x^2}{x+1} \, dx \right\}$

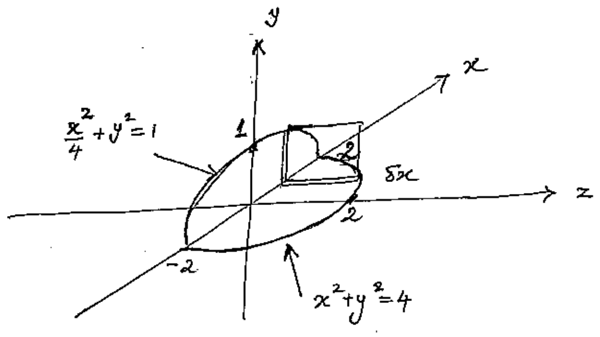
$= \cancel{2}\pi \times \frac{1}{\cancel{2}} \int_0^2 \left(x - 1 + \frac{1}{x+1} \right) dx$

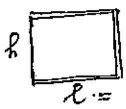
$= \pi \left[\frac{x^2}{2} - x + \ln(x+1) \right]_0^2$

$= \pi \left[\cancel{2} - \cancel{2} + \ln 3 - 0 + 0 - \ln 1 \right]$

$= \pi \ln 3$ cubic units.

94





$h = \sqrt{1-\frac{x^2}{4}} = \frac{1}{2}\sqrt{4-x^2}$

$l = \sqrt{4-x^2}$

$\delta V = h \times l \times \delta x$

$= \frac{1}{2}\sqrt{4-x^2} \times \sqrt{4-x^2}\ \delta x$

$$V = \frac{1}{2}\int_{-2}^{2} 4 - x^2\, dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{1}{2}\left\{4\times 2 - \frac{2^3}{3} - 4(-2) + \frac{(-2)^3}{3}\right\}$$

$$= \frac{1}{2}\left\{8 - \frac{8}{3} + 8 - \frac{8}{3}\right\}.$$

$$= \frac{1}{2}\left\{16 - \frac{16}{3}\right\}$$

$$= \frac{1}{2} \times \frac{32}{3}$$

$$= \frac{16}{3}$$

$= 5\frac{1}{3}$ cubic units.