NAME: ________________________

TEACHER: ________________________

The Scots College

Year 12 HSC Mathematics Extension 2

Assessment 3

June 2013

General Instructions

- Working time - 45 minutes
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is provided at the end of the paper

TOTAL MARKS: 36

WEIGHTING: 20 %

Question	Topic	Max Marks	Marks Obtained
1	Integration	24	
2	Volumes	12	
Total		**36**	

Question 1 (24 Marks)

a) Find the following integrals

(i) $\int \frac{x-1}{\sqrt{1-x^2}} dx$ [2]

(ii) $\int \frac{e^{4x}-1}{e^x+1} dx$ [3]

(iii) $\int \frac{dx}{3\sin x+4\cos x}$ [3]

(iv) $\int x \tan^{-1} x \, dx$ [3]

b) (i) Express $\frac{x^3}{(x-1)(x-2)}$ in the form $ax + b + \frac{c}{x-1} + \frac{d}{x-2}$ where *a*, *b*, *c* and *d* are real numbers.

(ii) Hence evaluate $\int_3^4 \frac{x^3}{(x-1)(x-2)} dx$ [4]

c) (i) Let $I_n = \int_0^1 x^n \sqrt{1-x} \, dx$ where *n* is a non-negative integer.

Show that $I_n = \frac{2n}{2n+3} I_{n-1}$, for $n \geq 1$.

(ii) Hence evaluate $\int_0^1 x^3 \sqrt{1-x} \, dx$. [5]

d) (i) Show that $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$.

(ii) Hence evaluate $\int_{-\pi}^{\pi} \frac{cos^2 x}{1+e^x} dx$ [4]

…………Question 2 on next page

Question 2 (12 Marks)

a) A solid sphere is formed by rotating the circle $x^2 + y^2 = 36$ about the y - axis. A cylindrical hole of diameter 6 cm is bored through the centre of the hole in the direction *Oy*.

(i) By considering a slice perpendicular to the y – axis, use the method of cylindrical shell to determine the volume of the solid remaining.

(ii) Also determine the volume of the section cut out from the sphere.

[6]

b) The region bound by the curve $y = x\,(4 - x)$ and the x – axis is rotated about the y – axis.

(i) By considering the solid formed by taking a slice perpendicular to the y – axis, show that the volume δV of this solid is given as $8\pi\sqrt{4 - y}$ cubic units.

(ii) Hence determine the volume of the solid of revolution.

[6]

... END OF EXAMINATION …

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

9.07 am

Yr 12 HSC Mathematics Extension 2

Assessment 3 - Solutions - June 2013

1 (a) (i) $\int \frac{x-1}{\sqrt{1-x^2}}\,dx$

$= \int \left(\frac{x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right) dx$

$= -\frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} - \sin^{-1} x + C$

$= -\sqrt{1-x^2} - \sin^{-1} x + C$

(iii) $\int \frac{dx}{3\sin x + 4\cos x}$, let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx = \frac{2dt}{1+t^2}$

$$= \int \frac{2dt/1+t^2}{3\left(\frac{2t}{1+t^2}\right) + \frac{4(1-t^2)}{1+t^2}}$$

$$= \int \frac{2dt/(1+t^2)}{(6t+4-4t^2)/(1+t^2)}$$

$$= \int \frac{dt}{2+3t-2t^2}$$

$$= \int \frac{dt}{(2-t)(1+2t)}$$

let $\frac{1}{(2-t)(1+2t)} = \frac{A}{2-t} + \frac{B}{1+2t}$

$1 = A(1+2t) + B(2-t)$

$t = 2$: $1 = 5A$, $A = \frac{1}{5}$

$t = -\frac{1}{2}$: $1 = \frac{5B}{2}$, $B = \frac{2}{5}$

$$= \int \left(\frac{1}{5(2-t)} + \frac{2}{5(1+2t)}\right) dt$$

$$= \frac{1}{5}\left[-\frac{1}{1}\log_e(2-t) + \frac{2}{2}\log_e 1+2t\right] + C$$

$$= \frac{1}{5}\log_e \frac{1+2t}{2-t} + C$$

$$= \frac{1}{5}\log_e \frac{1+2\tan\frac{x}{2}}{2-\tan\frac{x}{2}} + C$$

(iii) $\int x \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad v' = x$

$u' = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{x^2}{2(1+x^2)} \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(\frac{x^2+1-1}{1+x^2} \right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \left[x - \tan^{-1} x \right] + C$

$= \frac{x^2}{2} \tan^{-1} x + \frac{1}{2} \tan^{-1} x - \frac{1}{2} x + C$

9.14.

(b) (i) $I_n = \int_0^1 x^n \sqrt{1-x}\, dx$.

$u = x^n \quad v' = (1-x)^{1/2}$

$u' = nx^{n-1} \quad v = -\frac{2}{3}(1-x)^{3/2}$

$$I_n = \left[-\frac{2}{3} x^n (1-x)^{3/2}\right]_0^1 + \frac{2n}{3}\int (1-x)^{3/2} x^{n-1}\, dx$$

$$= [0-0] + \frac{2n}{3}\int (1-x)\sqrt{1-x}\, x^{n-1}\, dx$$

$$= \frac{2n}{3}\int \left[x^{n-1}\sqrt{1-x} - x^n\sqrt{1-x}\right] dx$$

$$= \frac{2n}{3} I_{n-1} - \frac{2n}{3} I_n$$

$$I_n\left(1+\frac{2n}{3}\right) = \frac{2n}{3} I_{n-1}$$

$$I_n\left(\frac{3+2n}{3}\right) = \frac{2n}{3} I_{n-1}$$

$$\therefore I_n = \frac{2n}{2n+3} I_{n-1}$$

(ii) $\int_0^1 x^3\sqrt{1-x}\, dx = I_3 = \frac{6}{9} I_2$

$$= \frac{6}{9}\left[\frac{4}{7}\left\{\frac{2}{5} I_0\right\}\right]$$

$$I_0 = \int_0^1 \sqrt{1-x}\, dx = \left[-\frac{2}{3}(1-x)^{3/2}\right]_0^1$$

$$= 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$$

$$\therefore I_3 = \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}\times\frac{2}{3} = \underline{\underline{\frac{96}{945}}}.$$

9.26

(d) (i) $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

let $I_1 = \int_{-a}^{0} f(x)\,dx$ let $y = -x$ $dy = -dx$

or $-dy = dx$.

when $x = 0,\ y = 0$

$x = -a,\ y = a$

$$I_1 = \int_{a}^{0} f(-y)(-dy)$$

$$= \int_{0}^{a} f(-y)\,dy$$

$$= \int_{0}^{a} f(-x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [f(x) + f(-x)]\,dx$$

(ii) $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,dx = \int_{0}^{\pi} \left[\frac{\cos^2 x}{1+e^x} + \frac{[\cos(-x)]^2}{1+e^{-x}}\right] dx$

$$= \int_{0}^{\pi} \left[\frac{\cos^2 x}{1+e^x} + \frac{\cos^2 x}{1+\frac{1}{e^x}}\right] dx$$

$$= \int_{0}^{\pi} \left(\frac{\cos^2 x}{1+e^x} + \frac{e^x \cos^2 x}{e^x+1}\right) dx$$

$$= \int_{0}^{\pi} \frac{\cos^2 x\,(1+e^x)}{1+e^x}\,dx = \int_{0}^{\pi} \cos^2 x\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi} [1+\cos 2x]\,dx = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{2}[\pi + 0 - 0 - 0]$$

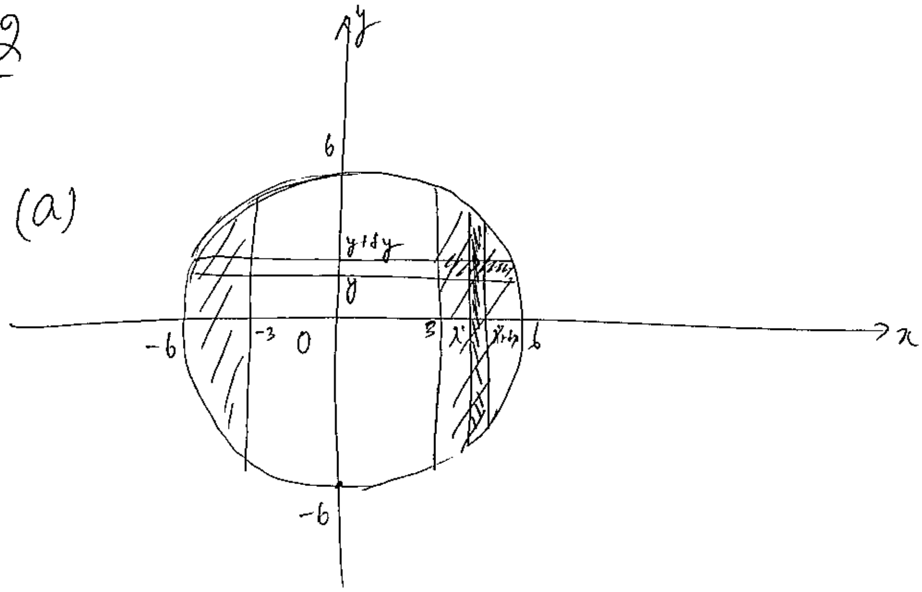
$$= \pi/2$$

9.30

9.59.

Q2

(a)



$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]\cdot 2y$$

$$= \pi\left[x^2 + 2x\delta x + (\delta x)^2 - x^2\right]\cdot 2y$$

$$= \pi\left[2x\delta x\right]2y$$

$$V = 4\pi\int_3^6 xy\,dy$$

$$= 4\pi\int_3^6 x\sqrt{36-x^2}\,dx$$

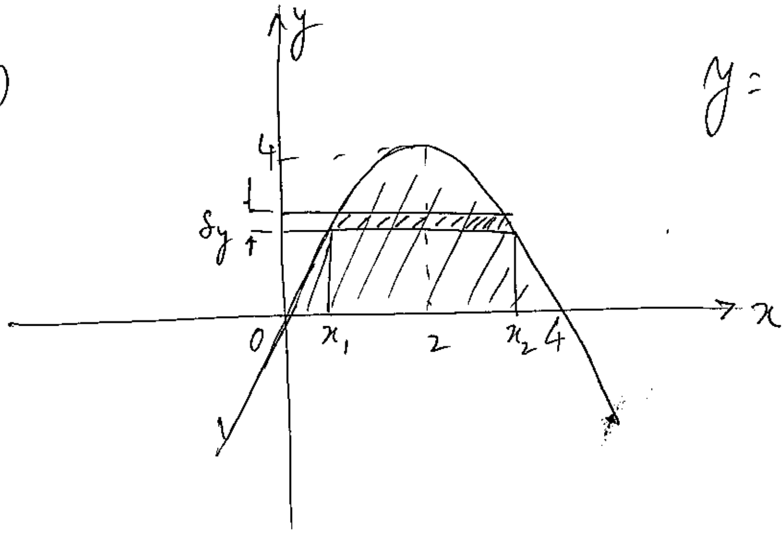
$$= 4\pi\left[-\frac{1}{2}\frac{(36-x^2)^{3/2}}{3/2}\right]_3^6 dx$$

$$= 4\pi\left[0 + \frac{1}{3}(36-9)^{3/2}\right]$$

$$= 4\pi\left[0 + \frac{1}{3}25^{3/2}\right]$$

$$= \frac{500\pi}{3}\ u^3.$$

(b)



$y = x(4-x)$

$y = 4x - x^2$

$x^2 - 4x + y = 0$

$x = \frac{4 \pm \sqrt{16-4y}}{2}$

$= \frac{4 \pm 2\sqrt{4-y}}{2}$

$= 2 \pm \sqrt{4-y}.$

$x_1 = 2 - \sqrt{4-y}$

$x_2 = 2 + \sqrt{4-y}$

$\delta V = \pi \left[x_2^2 - x_1^2\right] \delta y.$

$= \pi \left[x_2 - x_1\right]\left[x_2 + x_1\right] \delta y$

$= \pi \left[4 \left(2\sqrt{4-y}\right)\right] \delta y$

$= 8\pi \sqrt{4-y}\ \delta y$

(ii) $V = 8\pi \int_0^4 (4-y)^{1/2}\, dy$

$= 8\pi \left[\frac{-2(4-y)^{3/2}}{3}\right]_0^4$

$= 8\pi \left[-\frac{2}{3}(0) + \frac{2}{3} 4^{3/2}\right]$

$= \frac{8\pi}{3}\left(\frac{16}{3}\right) = \frac{128\pi}{9}\ u^3.$