THE SCOTS COLLEGE

2014

HSC Task 3 – In Class Test
10th June 2014
20%

Mathematics Extension 2

General Instructions
• Working time – 75 minutes
• Reading time – 5 minutes
• Write using black or blue pen
Black pen is preferred
• Board-approved calculators may be used
• Show all necessary working

Total marks – 43

Learning Intentions:

Polynomials

Integration

Volumes

QUESTION 1 - $\int (4-x^2)^{-\frac{1}{2}}\ dx$

A) $\dfrac{(4-x^2)^{\frac{1}{2}}}{-4x}+C$

C) $ln\left|x+\sqrt{(4-x^2)}\right|+C$

B) $-x\left(4-x^2\right)^{\frac{1}{2}}+C$

D) $sin^{-1}\dfrac{x}{2}+C$

QUESTION 2 - Which integral below gives the volume of the solid of revolution obtained by rotating the bounded region between $y=\sqrt{x},\ x=4$ and the line $y=0$ around the $y-axis$?

A) $\int_0^4 \pi x\, dx$

C) $\int_0^2 \pi(16-y)\, dy$

B) $2\int_0^4 \pi x\sqrt{x}\, dx$

D) $2\int_0^2 \pi y\, dy$

QUESTION 3 - $\int xsec^2x\ dx$

A) $xtanx+ln|cosx|+C$

C) $\dfrac{x^2}{2}tanx+C$

B) $\dfrac{x^2}{2}(sec^2x-tan^2x)+C$

D) $\dfrac{sec^3x}{3x}+C$

QUESTION FOUR (12 MARKS) BEGIN A NEW SHEET OF PAPER

a) Evaluate

$$\int_0^{\frac{\pi}{6}} \cos\theta\, sin^3\theta \; d\theta$$ *(2)*

b) Find

$$\int \frac{\sqrt{x}}{1+x} \; dx$$ *(3)*

c) *i)* Express $\dfrac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+1}$ where A,B and C are constants. *(2)*

ii) Hence find $\displaystyle\int \frac{x^2+x+2}{(x^2+1)(x+1)} dx$ *(2)*

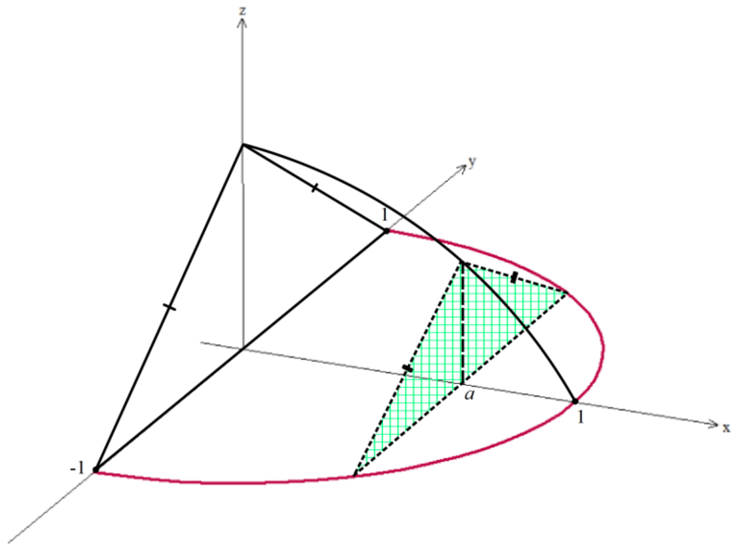
d) Using the substitution $t = \tan\frac{\theta}{2}$, evaluate, *(3)*

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta+\cos\theta} d\theta$$

QUESTION FIVE (8 MARKS) BEGIN A NEW SHEET OF PAPER

a) Given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a zero of multiplicity 3, find all the zeros of $P(x)$. *(3)*

b) The base of a solid is the semi-circular region of radius 1 unit in the *x*-*y* plane as illustrated in the diagram below.



Each cross-section perpendicular to the x-axis is an isosceles triangle. Each of the two equal sides are three quarters the length of the third side.

i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}(1-a^2)$ *(3)*

ii) Hence find the volume of the solid. *(2)*

QUESTION SIX (10 MARKS) **BEGIN A NEW SHEET OF PAPER**

a) *i)* If $I_n = \int_0^{\frac{\pi}{2}} x^n sinx\, dx$ for $n \geq 2$ prove that *(3)*

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

ii) Hence evaluate *(2)*

$$\int_0^{\frac{\pi}{2}} x^2 sinx\, dx$$

b) *i)* Prove that

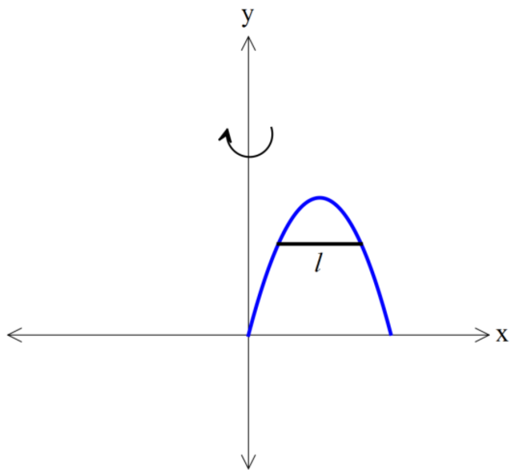
$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$ *(2)*

ii) Hence or otherwise, evaluate

$$\int_0^{\pi} xsin^3x\, dx$$ *(3)*

QUESTION SEVEN (10 MARKS) BEGIN A NEW SHEET OF PAPER

a) The region bounded by the curve $y = x - x^2$ and the *x-axis is* rotated around the *y-axis* to form a solid. When the region is rotated, the horizontal line segment *l* at height *y* sweeps out an annulus.

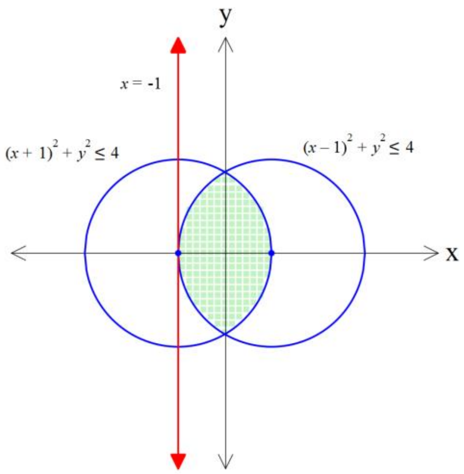


i) Show that the area of the annulus at height *y* is given by $2\pi\sqrt{\frac{1}{4} - y}$ *(3)*

ii) Find the volume of the solid *(2)*

b) The region enclosed by the circles $(x+1)^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 4$ is rotated about the line $x = -1$.

Using the method of cylindrical shells,



i) Show that the volume is given by *(3)*

$$8\pi \int_0^1 \sqrt{4 - (x+1)^2}\, dx$$

ii) Hence calculate the volume *(2)*

END OF EXAM

Ext 2 - task 3 Solutions - 2014

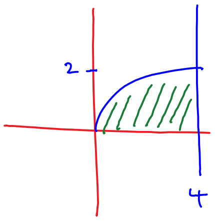
Thursday, 29 May 2014
2:23 PM

Multiple Choice

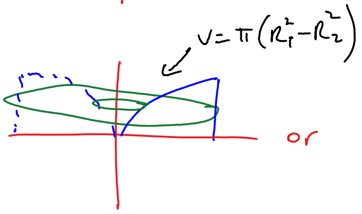
(1) $$\int (4-x^2)^{-\frac{1}{2}}\,dx = \int \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C \qquad \boxed{D}$$

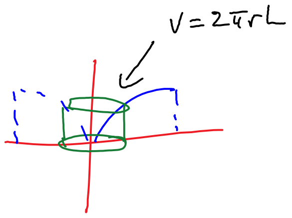
(2)



$y = \sqrt{x}$



or



$$\pi\int_0^2 (16 - x^2)\,dy \qquad 2\int_0^4 \pi x y\,dx$$

$$\pi\int_0^2 (16 - y^4)\,dy \qquad 2\int_0^4 \pi x\sqrt{x}\,dx \quad ✓$$

$$\boxed{B}$$

(3) $$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$

$$= x\tan x - \int \frac{\sin x}{\cos x}\,dx$$

reverse chain

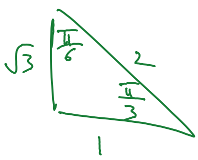
$= x\tan x - \ln|\cos x| + c$ $\boxed{A}$

Long Response

<u>Q4.</u>

a) $\displaystyle\int_0^{\pi/6} \cos\theta \sin^3\theta \, d\theta$

$\left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/6}$



$\left[\frac{1}{4}\left(\sin\frac{\pi}{6}\right)^4 - 0\right]$

$\left[\frac{1}{4}\left(\frac{1}{2}\right)^4\right]$

$\boxed{\frac{1}{64}}$

b) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

let $u = \sqrt{x}$

$u^2 = x$

$\frac{dx}{du} = 2u$

$dx = 2u\,du$

$\displaystyle\int \frac{u}{1+u^2}\,2u\,du$

$$2\int \frac{u^2}{1+u^2}\,du$$

$$2\int \frac{u^2+1-1}{u^2+1}\,du$$

$$2\int 1-\frac{1}{u^2+1}\,du$$

$$2\left[u-\tan^{-1}u\right]+c$$

$$\boxed{2\sqrt{x}-2\tan^{-1}(\sqrt{x})+c}$$

c) i) $$\frac{x^2+x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1}+\frac{C}{x+1}$$

$$x^2+x+2 = (Ax+B)(x+1)+C(x^2+1)$$

$\therefore$

$A+C = 1$

$A = 1-C \quad \ldots (1)$

$B+C = 2 \quad \ldots (2)$

let $x=-1$

$\therefore$

$2 = C+C$

$C = 1$

$A = 0$

$B = 1$

$\therefore$

$$\frac{x^2+x+2}{(x^2+1)(x+1)} = \frac{1}{x^2+1}+\frac{1}{x+1}$$

ii) $$\int \frac{1}{x^2+1}+\frac{1}{x+1}\,dx$$

$$\boxed{\tan^{-1}(x) + \ln|x+1| + c}$$

d) $$\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\, d\theta$$

$\tan\frac{\theta}{2} = t$ $\qquad$ $\sin\theta = \frac{2t}{1+t^2}$

$\theta = 2\tan^{-1} t$ $\qquad$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$\frac{d\theta}{dt} = \frac{2}{t^2+1}$

$$\boxed{\begin{array}{l} \theta = \frac{\pi}{2} \\ t = \tan\frac{\pi}{4} = 1 \\ \theta = 0 \\ t = 0 \end{array}}$$

$$\int_0^1 \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{t^2+1}\, dt$$

$$\int_0^1 \frac{2}{1+t^2+2t+1-t^2}\, dt$$

$$\int_0^1 \frac{2}{2t+2}\, dt = \int_0^1 \frac{1}{t+1}\, dt$$

$$= \Big[\ln|t+1|\Big]_0^1$$

$$= \ln 2 - \ln 1$$

$$\boxed{= \ln 2}$$

* Question 5:

a) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$

$= 6(2x^2 + x - 1)$

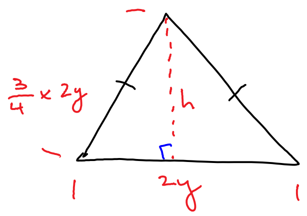
$= 6(2x-1)(x+1)$

$x = \frac{1}{2}$ or -1 $\qquad$ $P(x)$ is monic $\therefore$ $(x+1)$ is triple root

$$P(x) = (x+1)^3(x-2)$$

b)

i)



$x^2 + y^2 = 1$

$y = \pm\sqrt{1-x^2}$

$h^2 = \left(\frac{3}{2}y\right)^2 - y^2$

$= \frac{9}{4}y^2 - \frac{4}{4}y^2 = \frac{5}{4}y^2$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 2y \times \sqrt{\frac{5}{4}y^2}$

$A = y \times \frac{\sqrt{5}}{2} \times y$

$= \frac{\sqrt{5}}{2}(1-x^2)$ $\qquad$ at $x=a$

$A = \frac{\sqrt{5}}{2}(1-a^2)$

ii)

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} A \, \delta x$$

$$= \int_0^1 \frac{\sqrt{5}}{2}(1-x^2)\,dx$$

$$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{\sqrt{5}}{2}\left[\left(1-\frac{1}{3}\right)-0\right]$$

$$= \frac{\sqrt{5}}{3} \text{ units}^3$$

*Question 6

a) $$I_n = \int_0^{\frac{\pi}{2}} \underbrace{x^n}_{\text{2nd}} \underbrace{\sin x}_{\text{1st}} \, dx$$

$$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} -x^{n-1} \cos x \, dx$$

$$I_n = 0 + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx$$

$$I_n = +n\left(\left[+x^{n-1} \sin x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} -(n-1)x^{n-2} \sin x \, dx\right)$$

$$I_n = +n\left(\left[+\left(\frac{\pi}{2}\right)^{n-1}\right] - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx\right)$$

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, I_{n-2}$$

ii) $$\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx = 2\left(\frac{\pi}{2}\right)^1 - 2(2-1)\int_0^{\frac{\pi}{2}} x^0 \sin x \, dx$$

$\boxed{n=2}$

$$= \pi - 2\int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \pi - 2\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \pi - 2\left[0 - -1\right]$$

$$= \pi - 2$$

b)

i) $\int_0^a f(x)\,dx$

let $x = a-u \iff u = a-x$

$dx = -du$

if $x=a$
$u=0$

or $x=0$
$u=a$

substitute

$\int_a^0 f(a-u)\,-du$

$-\int_a^0 f(a-u)\,du$

$\int_0^a f(a-u)\,du$

substitute $u=x$

$\int_0^a f(a-x)\,dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

ii) $\int_0^\pi x\sin^3 x\,dx$ using part (i)

$\int_0^\pi x\sin^3 x\,dx = \int_0^\pi (\pi - x)\sin^3(\pi - x)\,dx$

$= \int_0^\pi \pi\sin^3(\pi - x)\,dx - \int_0^\pi x\sin^3(\pi - x)\,dx$

$\sin(\pi - x) = \sin x$

$= \pi\int_0^\pi \sin^3(x)\,dx - \int_0^\pi x\sin^3(x)\,dx$

$2\int_0^\pi x\sin^3 x\,dx = \pi\int_0^\pi \sin^3(x)\,dx$

$\int_0^\pi x\sin^3 x\,dx = \frac{\pi}{2}\int_0^\pi \sin^3 x\,dx$

let $u = \cos x$

$= \frac{\pi}{2}\int_0^\pi \sin^2 x . \sin x\,dx$

$du = -\sin x\,dx$

-1

$$= \frac{\pi}{2} \int_0 \sin^2 x \cdot \sin x \, dx \qquad du = -\sin x \, dx$$

$$= \frac{\pi}{2} \int_1^{-1} (1-u^2) \cdot -du$$

$$= \frac{\pi}{2} \int_{-1}^{1} 1-u^2 \, du$$

$$= \frac{\pi}{2} \left[u - \frac{u^3}{3} \right]_{-1}^{1} \qquad = \frac{\pi}{2} \left[\frac{2}{3} + \frac{2}{3} \right]$$

$$= \frac{2\pi}{3}$$

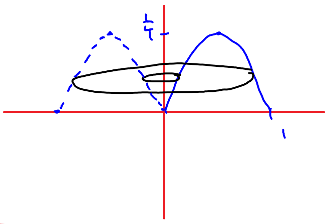
* Question 7

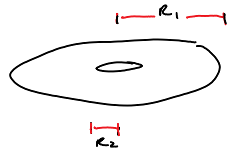
$$y = \frac{1}{2} - \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

a) $y = x - x^2$

$= x(1-x)$





$R_1 = x_1 \quad = \frac{1}{2} + \sqrt{\frac{1}{4} - y}$

$R_2 = x_2 \quad = \frac{1}{2} - \sqrt{\frac{1}{4} - y}$

$$V = \int_0^{1/4} \pi \left(R_1^2 - R_2^2\right) dy$$

$$\boxed{\begin{aligned} -y &= x^2 - x \\ -y &= \left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} \\ \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - y} &= x \end{aligned}}$$

$$= \int_0^{\frac{1}{4}} \pi \left(x_1^2 - x_2^2\right) dy$$

$$= \pi \int_0^{\frac{1}{4}} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - y}\right)^2 - \left(\frac{1}{2} - \sqrt{\frac{1}{4} - y}\right)^2 dy$$

$$= \pi \int_0^{\frac{1}{4}} \left(\frac{1}{4} + \sqrt{\frac{1}{4} - y} + \frac{1}{4} - y\right) - \left(\frac{1}{4} - \sqrt{\frac{1}{4} - y} + \frac{1}{4} - y\right) dy$$

$$= \pi \int_0^{\frac{1}{4}} 2\sqrt{\frac{1}{4} - y} \; dy$$

$$= 2\pi \int_0^{\frac{1}{4}} \left(\frac{1}{4} - y\right)^{\frac{1}{2}} dy$$

$$= 2\pi \left[-\frac{2}{3} \left(\frac{1}{4} - y\right)^{\frac{3}{2}} \right]^{\frac{1}{4}}$$

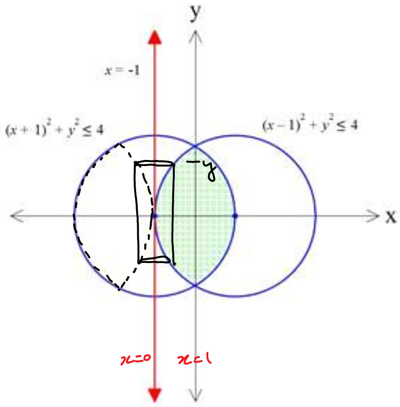
$$= 2\pi \left[-\frac{2}{3}\left(\frac{1}{4}-y\right)^{3/2} \right]_0^{1/4}$$

$$= 2\pi \left[0 - \left(-\frac{2}{3}\left(\frac{1}{4}\right)^{3/2} \right) \right]$$

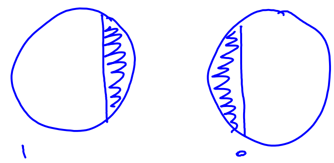
$$= 2\pi \left[\frac{2}{3}\left(\frac{1}{8}\right) \right]$$

$$= \frac{\pi}{6} \text{ units}^3$$

b)



$(x+1)^2 + y^2 = 4$ $\quad$ $(x-1) + y^2 = 4$



$\int_0^1 A_1\,dx$ $\quad$ $\int_{-1}^0 A_2\,dx$

Remember – Radius & Height MUST be positive values.

$\delta V = 2\pi r h \delta x$

Radius will always be bigger than 1.

$R = 1 + x$

Radius will always be smaller than 1 BUT x-value is negative

$\therefore$

$R = 1 + x$

2 parts

$$V = \int_0^1 A_1\,dx + \int_{-1}^0 A_2\,dx$$

Height for both volumes will be

$H = 2y$

$$V = \int_0^1 2\pi(1+x)2\sqrt{4-(x+1)^2}\,dx + \int_{-1}^0 2\pi(1+x)\,2\sqrt{4-(x-1)^2}\,dx$$

let $u = -x$

$$V = 4\pi\int_0^1 (1+x)\sqrt{4-(x+1)^2}\,dx + 4\pi\int_1^0 (-u+1)\sqrt{4-(-u-1)^2}\,-du$$

$(-u-1)^2 = (u+1)^2$

$$V = 4\pi\int_0^1 (1+x)\sqrt{4-(x+1)^2}\,dx + 4\pi\int_1^0 (u-1)\sqrt{4-(u+1)^2}\,du$$

$\int_a^b f(x) = -\int_b^a f(x)$

$$V = 4\pi\int_0^1 (1+x)\sqrt{4-(x+1)^2}\,dx + 4\pi\int_0^1 (1-u) \ldots$$

$$\int_a^b f(x) = -\int_b^a f(x)$$

$$V = 4\pi \int_0^1 (1+x)\sqrt{4-(x+1)^2}\,dx + 4\pi \int_0^1 (1-u)\sqrt{4-(u+1)^2}\,du$$

$x = u$

$$V = 4\pi \int_0^1 (1+x)\sqrt{4-(x+1)^2}\,dx + 4\pi \int_0^1 (1-x)\sqrt{4-(x+1)^2}\,dx$$

expand & simplify

$$V = 4\pi \int_0^1 2\sqrt{4-(x+1)^2}\,dx$$

$$V = 8\pi \int_0^1 \sqrt{4-(x+1)^2}\,dx \quad \text{as required}$$

ii)

$$V = 8\pi \int_0^1 \sqrt{4-(x+1)^2}\,dx$$

let $x+1 = 2\sin\theta$ when $x=1$ $\theta = \frac{\pi}{2}$

$dx = 2\cos\theta\,d\theta$ $x=0$ $\theta = \frac{\pi}{6}$

$\therefore$

$$V = 8\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2} \cdot 2\cos\theta\,d\theta$$

$$= 8\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos^2\theta\,d\theta$$

$$= 32\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 16\pi \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 16\pi\left(\frac{\pi}{2}+0\right) - 16\pi\left(\frac{\pi}{6}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= 16\pi\left[\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= 16\pi\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

$= 16\pi^2$ [illegible]

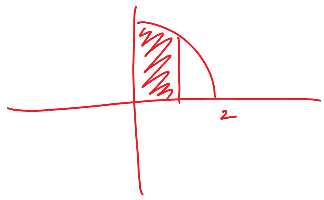
$$= \frac{16\pi^2}{3} - 4\pi\sqrt{3} \quad \text{units}^3$$

#

ii) $$V = 8\pi \int_0^1 \sqrt{4-(x-1)^2}\, dx$$

let $x - 1 = u$

$$V = 8\pi \int_{-1}^{0} \sqrt{4-u^2}\, du$$



let $u = 2\cos\theta$ $\quad u = 0 \quad \theta = \frac{\pi}{2}$

$du = -2\sin\theta \, d\theta$ $\quad u = -1 \quad \theta = \frac{2\pi}{3}$

$$= 8\pi \int_{\frac{2\pi}{3}}^{\frac{\pi}{2}} \sqrt{4 - 4\cos^2\theta} \cdot -2\sin\theta \, d\theta$$

$$= 8\pi \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 2\sin\theta \cdot -2\sin\theta \, d\theta$$

$$= +32\pi \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin^2\theta \, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - \sin^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= +16\pi \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 1 - \cos 2\theta \, d\theta$$

$$= +16\pi \left[\theta - \frac{1}{2}\sin 2\theta \right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$$

$$= +16\pi\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_{\frac{\pi}{2}}$$

$$= 16\pi\left[\left(\frac{2\pi}{3} - \frac{1}{2}\left(\sin\frac{4\pi}{3}\right)\right) - \left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right)\right]$$

$$= 16\pi\left[\left(\frac{2\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{\pi}{2} - \frac{1}{2}(0)\right)\right]$$

$$= 16\pi\left[\frac{2\pi}{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{6}\right]$$

$$= 16\pi\left[\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{8\pi^2}{3} + 4\sqrt{3}\,\pi$$