STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \; x > 0$

SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

S·H·S

VERITATE ET VIRTUTE

Year 12 HSC Task # 2
June 1998

MATHEMATICS

4 UNIT ADDITIONAL

Time allowed — 2 Hours

Examiner: P.S. Parker

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Use a new booklet for each question.
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

ion 1. (Start a new booklet) — **Marks**

Evaluate the following

(i) $\int x^2 e^{x^3+1}\,dx$ — 2

(ii) $\int_1^3 \frac{dx}{\sqrt{1-9x^2}}$ — 3

(iii) $\int \frac{dx}{(x-3)(x+1)}$ — 2

(iv) $\int_0^1 \tan^{-1} x\,dx$ — 3

Explain, with the aid of a sketch, why $\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\sin^2 x}\,dx = \int_{\pi}^{\frac{3\pi}{2}} -\cos x\,dx$ — 4

Hence evaluate $\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\sin^2 x}\,dx$

Question 2. (Start a new booklet) — **Marks**

(a) (i) Sketch the graph of $y = x(x^2 - 1)$, showing the intercepts on the axes. — 1

(ii) Without using calculus, use the graph of $y = x(x^2 - 1)$ to sketch on separate axes the graphs of:

(α) $y = \left| x(x^2 - 1) \right|$ — 2

(β) $y = \frac{1}{x(x^2 - 1)}$ — 2

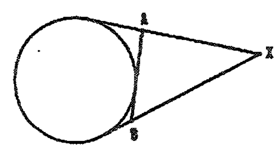
(γ) $y^2 = x(x^2 - 1)$ — 2

(b) (i) Sketch the curve $y = \frac{x^2}{x^3 + 1}$, showing stationary points, asymptotes and any other important features. [You do not need to locate points of inflexions.] — 3

(ii) Hence, or otherwise, determine the set of values of k, for which the following equation has three roots. — 2

$$kx^3 - x^2 + k = 0$$

(c) From the point X, tangents are drawn to a circle. The line segment AB is also tangent to the circle. — 2



Show that the perimeter of $\Delta\ ABX$ has constant length for all positions of A and B.

ion 3. (Start a new booklet) | **Marks**

$z_1 = 1 - i$ and $z_2 = -1 + \sqrt{3}\, i$

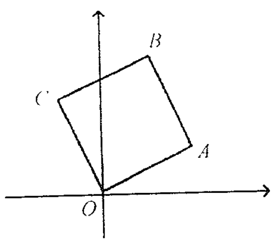
(i) Find the moduli and principal arguments of z_1 and z_2. **2**

(ii) Hence express $z_1 z_2$ in mod-arg form. **1**

(iii) Express $z_1 z_2$ in the form $a + ib$, where a and b are real numbers. **1**

(iv) Use the results in (ii) and (iii) to evaluate $\cos\frac{5\pi}{12}$ as a surd. **2**

A represents the complex number z. **2**



If $OABC$ is a square, what complex number does

(i) C represent?

(ii) B represent?

Sketch the locus of z represented by: **6**

(i) $\{z : |z - 2| = \text{Re}(z)\}$

(ii) $\{z : |z + 2i| < 2\} \cap \{z : 0 \le \arg(z + 2i) \le \frac{\pi}{4}\}$

(iii) $\{z : |z - 2i| < |z - 3 + i|\}$

Question 4. (Start a new booklet) | **Marks**

(a) Find the values which the real numbers a and b must take for $z = 1$ to be a root of the complex equation $iz^2 + (ia - 1)z + (i - b) = 0$ **2**

(b) $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + qx^2 + rx + s = 0$.

(i) Form equations whose roots are:

(α) $\alpha + 1, \beta + 1, \gamma + 1, \delta + 1$ **2**

(β) $\alpha^2, \beta^2, \gamma^2, \delta^2$ **2**

(ii) Hence, or otherwise, show that **3**

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = (1 - q + s)^2 + r^2$$

(c) (i) Use $t = \tan\frac{x}{2}$ to prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \sin x} = \frac{\pi}{3\sqrt{3}}$ **2**

(ii) Show that $\int_0^{2a} f(x)\, dx = \int_0^{a} [f(x) + f(2a - x)]\, dx$ **3**

Hence, evaluate $\int_0^{\pi} \frac{x\, dx}{2 + \sin x}$

Marks

ion 5. (Start a new booklet)

A particle of unit mass, moving in a straight line, is subject to a variable resistive force which is proportional to the cube of the velocity of the particle at any instant. The initial velocity is 2 m/s and the magnitude of the initial retardation is 1 m/s^2.

(i) Find the speed of the particle after 6 seconds. 3

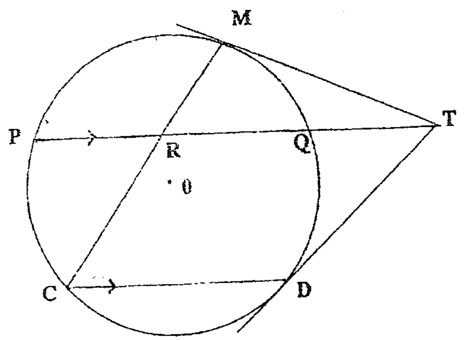
(ii) Find the distance travelled by the particle in the first 15 seconds. 3

) It is required to seat n people at a round table.

(i) How many possible arrangements are possible if 3 specified people are to sit together? 4

(ii) Use the result of (i) to find the probability (in simplest form) of 3 particular people sitting together if n people are seated at random at a round table.

(c) The figure below shows a circle at O. 4
Chords PQ and CD are parallel.
The tangent at D meets PQ produced at T.
TM is the other tangent from T.



Draw the figure in your answer booklet and prove that MC bisects PQ.

Marks

Question 6. (Start a new booklet)

(a) If α is a zero of the polynomial $P(z) = az^2 + bz + c$, which has real coefficients a, b and c.

(i) Show that $\overline{\alpha}$ is a zero of $P(z)$. 3

(ii) Find the monic polynomial of least degree, with integral coefficients which have no common factor, having $1 - i$ as a root of multiplicity 2. 3

(b) There are six people from whom a game of tennis is to be made up, two on each side. How many different matches could be arranged; a change in either pair giving a different match? 3

(c) A particle of mass m moves through a medium which offers resistance $av + bv^3$, where a and b are constants. Given that this is the only force acting and that the particle is given an initial speed u, show that the speed becomes $\frac{1}{2}u$ after a time $\frac{m}{2a}\log\frac{4a + bu^2}{a + bu^2}$ 5

THIS IS THE END OF THE PAPER.