# SYDNEY BOYS HIGH SCHOOL

# EXTENSION 2

# MATHEMATICS COURSE

## July 2001

## Assessment Task # 2

Time Allowed: 2 hours (plus 5 minutes Reading Time)

Examiner: Mr E Choy

**INSTRUCTIONS:**

- Attempt *all* questions.
- *All* questions are of equal value.
- All *necessary* working should be shown in every question. Full marks *may not* be awarded if work is careless or badly arranged.
- Standard integrals are provided on the back of this page. Approved calculators may be used.
- Return your answers in **4** sections: Question 1, Question 2, Question 3 and Question 4. Each booklet **MUST** show your name.
- If required, additional Answer Booklets may be obtained from the Examination Supervisor upon request.

**Question 1** **Marks**

(a) By using a suitable substitution find **2**

$$\int x\cos(\pi x^2)\,dx$$

(b) Find $\int x^3 \ln x\,dx$ **2**

(c) (i) Find constants $A$, $B$, $C$ such that **3**

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

(ii) Hence evaluate **2**

$$\int_0^1 \frac{dx}{(x+1)(x^2+1)}$$

(d) By considering $\cos^2 x = 1 - \sin^2 x$, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1+\sin x}$$

(e) Using $x = \cos 2\theta$ show that **3**

$$\int_{-1}^1 \sqrt{\frac{1-x}{1+x}}\,dx = \pi$$

**Question 1 is continued on the next page**

**Question 1 continued** **Marks**

(f) (i) Given that **5**

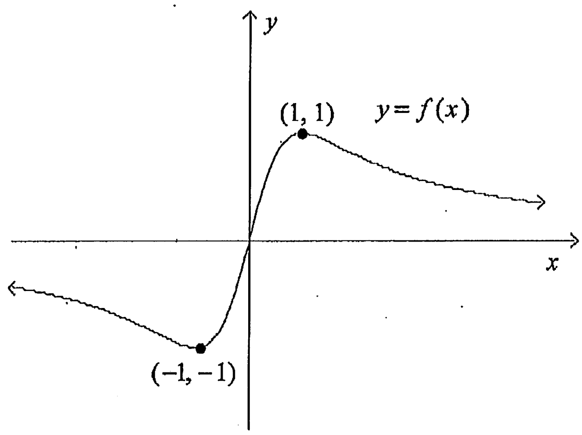
$$I_{n,a} = \int \frac{dx}{(x^2 + a^2)^{n+\frac{1}{2}}}$$

Show that $I_{n,a} = \dfrac{1}{a^{2n}} \displaystyle\int \cos^{2n-1} u \, du$

(ii) Hence, find $I_{2,1}$ leaving your answer in terms of $x$.

**Section 2 (Start a new answer booklet)** **Marks**

(a) The diagram below represents the curve $f(x) = \frac{2x}{x^2+1}$ **10**




Sketch the following on separate number planes, without using calculus.

(i) $y = f(-x)$

(ii) $y = f(|x|)$

(iii) $|y| = f(x)$

(iv) $y = f(2x)$

(v) $y \times f(x) = 1$

(vi) $y = e^{f(x)}$

(vii) $y = \max\left(f(x), \frac{1}{2}\right)$, where $\max(a, b)$ is defined below:

$$\max(a,b) = \begin{cases} a \text{ if } a \geq b \\ b \text{ if } b \geq a \end{cases}$$

**Question 2 is continued on the next page**

(b) Sketch the graph over the interval $-3 \le x \le 3$ of the function which is defined for all $x \in \mathbb{R}$ by **4**

$$\begin{cases} f(x) = x - \frac{1}{2} \text{ for } 0 \le x \le 1 \\ f(-x) = f(x) \text{ for all } x \\ f(x+2) = f(x) \text{ for all } x \end{cases}$$

(c) Two students, A and B, were asked to find $\frac{dy}{dx}$ for the curve $\frac{x^2}{y} + y = 3$. **2**

Student A worked at it directly, using the quotient rule and obtained the answer $\frac{2xy}{x^2 - y^2}$.

Student B made life easier by multiplying through by $y$ and the differentiating. B obtained the answer $\frac{2x}{3-2y}$.

Has somebody made a mistake or can the two solutions be reconciled?

(d) (i) Express in the form rcis α **4**

$$\frac{(\sqrt{3}i+1)^4}{(1-i)^3}$$

(ii) By considering $(1-i)^2$, show that

$$(1-i)^6 = 8i$$

**Question 3 (Start a new answer booklet)** **Marks**

(a) If $P(x)=2x^4-20x^3+74x^2-120x+72$ has two double roots, then factorise $P(x)$. **3**

(b) A person wishes to make up as many different parties as he can out of 20 persons, each party consisting of the same number. **2**

(i) How many should he invite?

(ii) To how many of these parties will the same person be invited?

(c) (i) Express $i$ and $(1+i)^n$ in mod-arg form **7**

(ii) Simplify $\text{cis}\theta+\text{cis}(-\theta)$

(iii) Let $M$, $N$ be positive integers.
The polynomial $x^M(1-x)^N$ when divided by $1+x^2$ has a remainder of $ax+b$.
Using the results from (i) and (ii) above, or otherwise, show that
$a=\left(\sqrt{2}\right)^N\sin\left(\dfrac{(2M-N)\pi}{4}\right)$ and $b=\left(\sqrt{2}\right)^N\cos\left(\dfrac{(2M-N)\pi}{4}\right)$

(d) A 200 g mass moving on a smooth horizontal surface encounters a resistance of $\dfrac{14v^3}{30}$ N when its speed is $v$ m/s. **4**
Show that its speed is reduced from $3\frac{1}{3}$ m/s to 1 m/s in a distance of 30 cm.

(e) An object of mass $m$ kg is thrown vertically upwards. **4**
Air resistance is given by $R=0\cdot05mv^2$ where $R$ is in newtons and $v$ $\text{ms}^{-1}$ is the speed of the object.
(Take $g=9\cdot8$ $\text{ms}^{-2}$.)
If the velocity of projection is $50$ $\text{ms}^{-1}$, find the time taken to reach the highest point.

**Question 4 (Start a new answer booklet)** **Marks**

(a) (i) Show that $2-i$ is a root of the equation $z^4-2z^3-z^2+2z+10=0$ **5**

(ii) Hence solve $z^4-2z^3-z^2+2z+10=0$

(b) Eight people are to be seated at a round table. **7**

(i) In how many ways is this possible?

(ii) In how many ways can the people be seated if two of the eight people must not sit in adjacent seats?

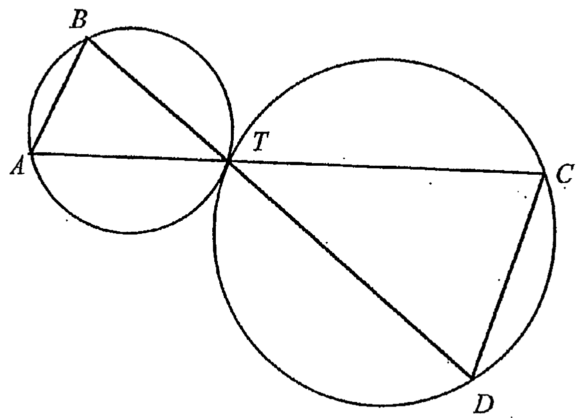
(iii) If the eight people are four men and four ladies, how many ways can they be seated, if no two men are to be in adjacent seats?

(iv) If the eight people are four married couples, how many ways can they seated if no husband and wife, as well as no two men, are to be in adjacent seats?

**Question 4 is continued on the next page**

**Question 4 continued** **Marks**

(c) Two circles touch externally at a point $T$. **8**
$A$ and $B$ are points on the <u>first circle</u> such that $AT = BT$, and $C$ and $D$ are points on the second circle such that $AC$ and $BD$ meet at $T$.




(i) Copy the diagram and include the information above.

(ii) Prove that $\angle BAC = \angle ACD$

(iii) Prove that $ABCD$ is a trapezium with two equal sides.

The line $BC$ cuts the first circle in $V$ and the second circle in $W$, and the line $AD$ cuts the first circle in $U$ and the second circle in $X$.

(iv) Prove that the points $U$, $V$, $W$ and $X$ are concyclic.

**THIS IS THE END OF THE PAPER**

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

**2001**
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

# Mathematics Extension 2

# Sample Solutions

(1) (a) $\int x\cos(\pi x^2)\,dx$ ; $u=\pi x^2$, $du = 2\pi x\,dx$

$= \frac{1}{2\pi}\int \cos(\pi x^2)\,x(2\pi x\,dx)$

$= \frac{1}{2\pi}\int \cos u\,du$

$= \frac{1}{2\pi}\sin u + C$

$= \frac{1}{2\pi}\sin(\pi x^2) + C$

(b) $\int x^3 \ln x\,dx$

$= \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^4 \times \frac{1}{x}\,dx$

$= \frac{1}{4}x^4\ln x - \int \frac{x^3}{4}\,dx$

$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

(c) (i) $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$\therefore\ 1 = A(x^2+1) + (Bx+C)(x+1)$

$x=-1$ : $1 = 2A$

$A = \frac{1}{2}$

$A+B=0 \Rightarrow B = \frac{-1}{2}$

$A+C = 1 \Rightarrow C = \frac{1}{2}$

(c)(ii)

$$\int_0^1 \frac{dx}{(x+1)(x^2+1)} = \int_0^1\left(\frac{\frac{1}{2}}{x+1} - \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\right)dx$$

$$= \frac{1}{2}\ln|x+1|\Big]_0^1 - \frac{1}{4}\int_0^1 \frac{2x\,dx}{x^2+1} + \frac{1}{2}\int_0^1 \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln\left(x^2+1\right)\Big]_0^1 + \frac{1}{2}\tan^{-1}x\Big]_0^1$$

$$= \frac{1}{4}\ln 2 + \frac{\pi}{8}$$

(d) $$\int_0^{\frac{\pi}{8}} \frac{dx}{1+\sin x} = \int_0^{\frac{\pi}{8}}\left(\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}\right)dx = \int_0^{\frac{\pi}{8}}\frac{(1-\sin x)}{1-\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{8}} \sec^2 x\,dx - \int_0^{\frac{\pi}{8}} \frac{\sin x}{\cos^2 x}dx$$

$$= \tan\frac{\pi}{4} - \int_0^{\frac{\pi}{8}} \sec x \tan x\,dx$$

$$= 1 - \left(\sqrt{2}-1\right) = 2-\sqrt{2}$$

1(e) $\int_0^{\frac{\pi}{4}} \sqrt{\frac{1-x}{1+x}}\,dx$

$$\left[\begin{array}{l} x=\cos 2\theta \Rightarrow dx=-2\sin 2\theta d\theta=-4\sin\theta\cos\theta d\theta \\ x=-1\Rightarrow \theta=\frac{\pi}{2}; x=1\Rightarrow \theta=0 \\ 1-\cos 2\theta=2\sin^2\theta; 1+\cos 2\theta=2\cos^2\theta \end{array}\right]$$

$$=\int_{\frac{\pi}{2}}^{0}|\tan\theta|\times(-4\sin\theta\cos\theta d\theta)$$

$$=2\int_0^{\frac{\pi}{2}} 2\sin^2\theta\, d\theta \qquad \left[\because |\tan\theta|=\tan\theta \text{ for } 0\le\theta\le\frac{\pi}{2}\right]$$

$$=2\int_0^{\frac{\pi}{2}}(1-\cos 2\theta)d\theta=2\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$=2\times\frac{\pi}{2}=\pi$$

1 f (i)

$$I_{n,a} = \int \frac{dx}{\left(x^2 + a^2\right)^{n+\frac{1}{2}}}$$

Let $x = a\tan\theta$

$\therefore dx = a\sec^2\theta d\theta$

$$I_{n,a} = \int \frac{a\sec^2\theta d\theta}{\left(a^2\tan^2\theta + a^2\right)^{n+\frac{1}{2}}}$$

$$= \int \frac{a\sec^2\theta d\theta}{\left(a^2\sec^2\theta\right)^{n+\frac{1}{2}}}$$

$\tan^2\theta + 1 = \sec^2\theta$

$$= \int \frac{a\sec^2\theta d\theta}{a^{2n+1}\sec^{2n+1}\theta}$$

$$= \frac{1}{a^{2n}} \int \frac{d\theta}{\sec^{2n-1}\theta}$$

$$= \frac{1}{a^{2n}} \int \cos^{2n-1}\theta d\theta$$

(ii) $n = 2,\ a = 1$

$$I_{2,1} = \int \cos^3\theta d\theta$$

$$= \int \left(\cos^2\theta \times \cos\theta\right) d\theta$$

$$= \int \left(1 - \sin^2\theta\right)\cos\theta d\theta$$

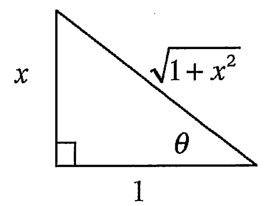
$$= \int \left(\cos\theta - \sin^2\theta\cos\theta\right) d\theta$$

$$= \sin\theta - \int \sin^2\theta\cos\theta d\theta$$

$$= \sin\theta - \int \sin^2\theta\, d(\sin\theta)$$

$$= \sin\theta - \frac{1}{3}\sin^3\theta + c$$

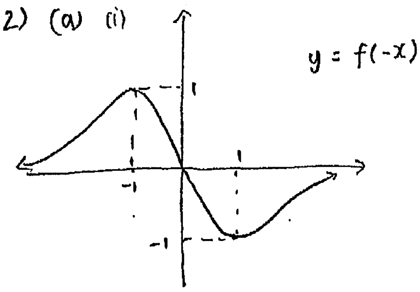
$x = \tan\theta$



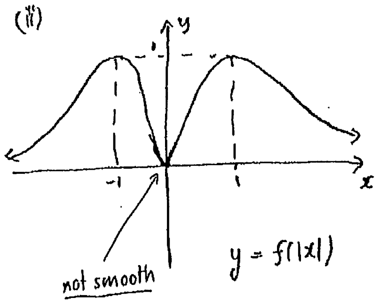

$$\therefore I_{2,1} = \frac{x}{\sqrt{1+x^2}} - \frac{x^3}{3(1+x^2)\sqrt{1+x^2}} + c$$

$$= \frac{x(3+2x^2)}{3(1+x^2)\sqrt{1+x^2}} + c$$

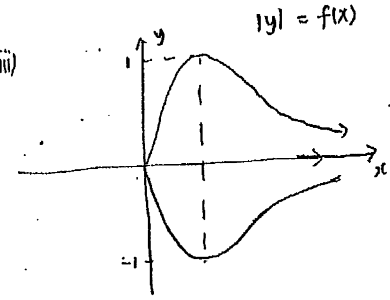
(2) (a) (i)



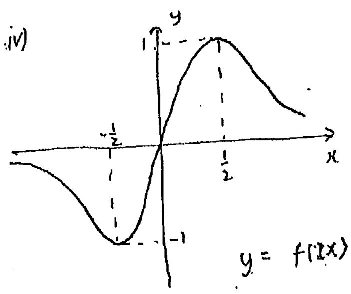

(ii)



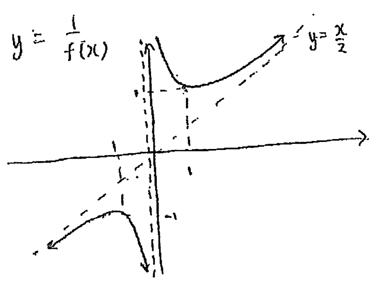

(iii)




(iv)

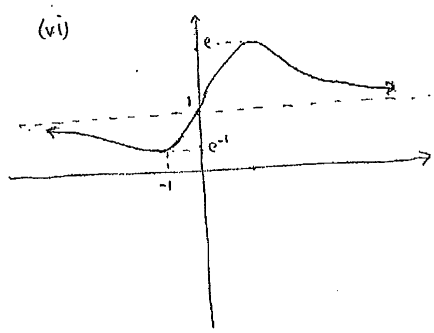



(v) $y = \frac{1}{f(x)}$

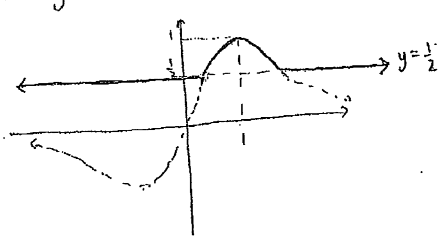



vertical asymptote at $x = 0$

(vi)



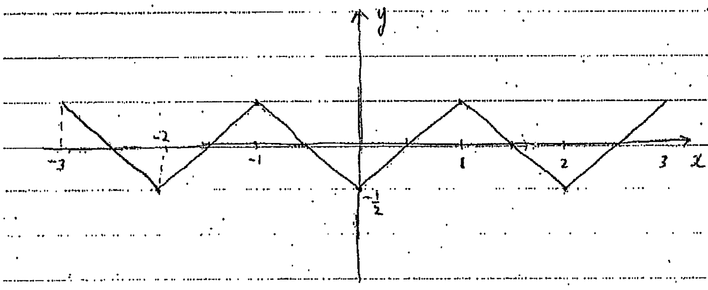

(vii) $y = \max\left(f(x), \frac{1}{2}\right)$



2(b) $f(x) = x - \frac{1}{2}$

$f(-x) = f(x)$ — even

$f(x+2) = f(x)$ — period of 2




(c) $\frac{x^2}{y} + y = 3$

A: $\therefore \frac{y \times 2x - x^2 \times \frac{dy}{dx}}{y^2} + \frac{dy}{dx} = 0$

$2xy - x^2\frac{dy}{dx} + y^2\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx}(x^2 - y^2) = 2xy$

$\therefore \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ — (1)

B: $x^2 + y^2 = 3y$ — (*)

$2x + 2y\frac{dy}{dx} = 3\frac{dy}{dx}$

$\therefore \frac{dy}{dx}(3 - 2y) = 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{3 - 2y}$ — (2)

Apply (*) to (1) $\frac{2xy}{3y - y^2 - y^2} = \frac{2xy}{3y - 2y^2} = \frac{2x}{3 - 2y}$ which is (2)

$\therefore$ reconciled

i.e. NO mistakes.

2(d)(i) $\frac{(\sqrt{3}i+1)^4}{(1-i)^3}$ $\left[\begin{array}{l}\sqrt{3}i+1=2\operatorname{cis}\frac{\pi}{3}\\ 1-i=\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\end{array}\right]$

$$=\frac{16\operatorname{cis}\frac{4\pi}{3}}{2\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)} \quad \text{[DMT]}$$

$$=4\sqrt{2}\operatorname{cis}\frac{25\pi}{12}=4\sqrt{2}\operatorname{cis}\frac{\pi}{12}$$

(ii) $(1-i)^2=-2i$

$\therefore (1-i)^6=((1-i)^2)^3=(-2i)^3=-8i^3$

$=-8\times -i$

$=8i$

(3) $P(x)=2x^4-20x^3+74x^2-120x+72$

(a) This has two double roots, call them $\alpha,\alpha,\beta \,\&\, \beta$

$2\alpha+2\beta=20/2=10 \Rightarrow \alpha+\beta=5$

$\alpha^2+\beta^2+4\alpha\beta=36$

$2\alpha^2\beta+2\alpha\beta^2=60 \Rightarrow 2\alpha\beta(\alpha+\beta)=60 \Rightarrow 5\alpha\beta=30 \Rightarrow \alpha\beta=6$

$\therefore \left.\begin{array}{l}\alpha+\beta=5\\ \alpha\beta=6\end{array}\right\} \Rightarrow \alpha,\beta=2,3$

So the double roots are $x=2,2,3,3$

**Alternatively** $P(x)=2x^4-20x^3+74x^2-120x+72=2(x^4-10x^3+37x^2-60x+36)$
So applying the Remainder Theorem by testing 36 ie $x=\pm1,\pm2,\pm3,\pm4,\pm6,\pm9,\pm12,\pm18,\pm36$
So getting to $x=2 \Rightarrow P(2)=0$. Then use the fact that $(x-2)^2$ is a factor to get $P(x)=2(x-2)^2(x-3)^2$ and hence the same result.

(b) (i) He has to maximise $\binom{20}{n}$ $\therefore$ $n=10$

(ii) How many is x invited to? If he is invited only 9 others can be.

$\therefore \binom{19}{9}$

OR $\frac{1}{2}\binom{20}{10}$ (half of the parties available)

$\binom{19}{9}=\frac{1}{2}\binom{20}{10}$ ☺

(3) (c) (i) $i = \text{cis}\,\pi/2$

$1+i = \sqrt{2}\,\text{cis}\,\pi/4 \Rightarrow (1+i)^n = (\sqrt{2})^n\,\text{cis}\left(\frac{n\pi}{4}\right)$ (D.M.T.)

(ii) $\text{cis}\,\theta + \text{cis}(-\theta) = 2\cos\theta$ $(z+\bar{z} = 2\text{Re}\,z)$

(iii) $x^M(1-x)^N = (1+x^2)Q(x) + ax + b$

sub $x = i$ : $\therefore i^M(1-i)^N = 0 + ai + b$

$\therefore \text{cis}\frac{M\pi}{2} \times (\sqrt{2})^N \text{cis}\left(-\frac{N\pi}{4}\right) = ai + b$

$\therefore (\sqrt{2})^N \text{cis}\left(\frac{M\pi}{2} - \frac{\pi}{4}\right) = (\sqrt{2})^N \text{cis}\left(\frac{(2M-N)\pi}{4}\right) = ai+b$ —(1)

sub $x = -i$ : $(-i)^M(1+i)^N = 0 + a(-i) + b$

$\therefore \text{cis}\left(-\frac{M\pi}{2}\right) \times (\sqrt{2})^N \text{cis}\left(\frac{N\pi}{4}\right) = -ai+b$

$\therefore (\sqrt{2})^N \text{cis}\left(\frac{(-2M+N)\pi}{4}\right) = -ai+b$

$\therefore (\sqrt{2})^N \text{cis}\left(-\left(\frac{2M-N}{4}\right)\pi\right) = -ai+b$ —(2)

(1)+(2) $\Rightarrow 2b = 2.(\sqrt{2})^N \cos\left(\frac{2M-N}{4}\right)\pi$

$\Rightarrow \boxed{b = (\sqrt{2})^N \cos\left(\frac{2M-N}{4}\right)\pi}$

(1)−(2) $\Rightarrow 2ai = 2i(\sqrt{2})^N \sin\left(\frac{2M-N}{4}\right)\pi$ $(z-\bar{z} = 2i\,\text{Im}\,z)$

$\therefore \boxed{a = (\sqrt{2})^N \sin\left(\frac{2M-N}{4}\right)\pi}$

(3) (d)

$$0.2\ddot{x} = -\frac{14v^3}{30} = -\frac{7v^3}{15}$$

$$\therefore \ddot{x} = -\frac{7}{3}v^3$$

$$\therefore v\frac{dv}{dx} = -\frac{7}{3}v^3 \Rightarrow \frac{dv}{dx} = -\frac{7}{3}v^2$$

$$\therefore \frac{dx}{dv} = -\frac{3}{7v^2}$$

$$x = \frac{3}{7}\times\frac{1}{v} + c$$

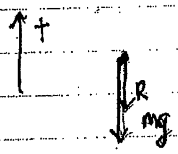
take $x=0,\ v=\frac{10}{3} \Rightarrow c = -\frac{9}{70}$

$$\therefore x = \frac{3}{7}\frac{1}{v} - \frac{9}{70}$$

when $v=1,\ x = \frac{3}{7}\times 1 - \frac{9}{70} = 0.3\text{ m} = 30\text{cm}$

$\therefore$ velocity goes from $3\frac{1}{3}$ m/s to 1 m/s in 30 cm

(e)




$t=0,\ \dot{x}=50,\ x=0$

$$\therefore m\ddot{x} = -mg - 0.05mv^2$$

$$\ddot{x} = -\left(g + \frac{1}{20}v^2\right) = -\frac{1}{20}\left(20g + v^2\right)$$

$$\therefore \frac{dv}{dt} = -\frac{1}{20}\left(20g + v^2\right) = -\frac{1}{20}\left(196 + v^2\right)$$

$$\therefore \frac{dt}{dv} = \frac{-20}{196+v^2}$$

$$\therefore t = \int \frac{-20}{196+v^2}\,dv = \frac{-20}{14}\int \frac{14}{196+v^2}\,dv = -\frac{20}{14}\tan^{-1}\left(\frac{v}{14}\right) + c$$

$t=0,\ \dot{x}=v=50 \Rightarrow 0 = -\frac{20}{14}\tan^{-1}\left(\frac{50}{14}\right) + c \Rightarrow c = \frac{20}{14}\tan^{-1}\left(\frac{50}{14}\right)$

$$\therefore t = \frac{20}{14}\left[\tan^{-1}\frac{50}{14} - \tan^{-1}\left(\frac{v}{14}\right)\right]$$

Highest point $(v=0) \Rightarrow t = \frac{20}{14}\tan^{-1}\left(\frac{50}{14}\right)$ secs.

4(a) (i) If $2-i$ is a root then

$x-(2-i)$ and $x-(2+i)$ are factors (real coeff.)

$\therefore [x-(2-i)][x-(2+i)] = x^2-4x+5$ is a factor

$$\begin{array}{r|l} & z^2+2z+2 \\ z^2-4z+5 & z^4-2z^3-z^2+2z+10 \\ & z^4-4z^3+5z^2 \\ z^2-4z+5 & 2z^3-6z^2+2z+10 \\ & 2z^3-8z^2+10z \\ z^2-4z+5 & -2z^2-8z+10 \\ & 2z^2-8z+10 \\ & 0 \end{array}$$

$\therefore z^2-4z+5$ is a factor $\Rightarrow z=2-i$ is a root

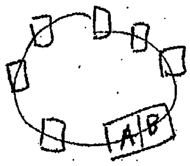
$z^2+2z+2 = (z+1)^2+1$

$\therefore$ roots are $z = 2\pm i, -1\pm i$

(b) 8 people

(i) $7! =$

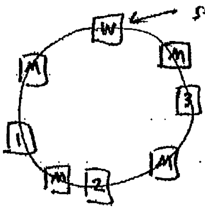
(ii) I. Put them in adjacent seats. call them A & B




$\therefore$ 7 "objects" around the circle

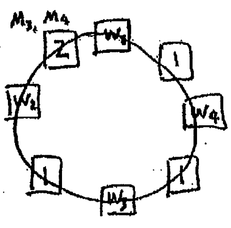
$\therefore 6! \times 2! = 1440$

(iii) seat a woman first




$\therefore 3! \times 4! =$

(iv)




. Let the women be $W_1, W_2, W_3, W_4$

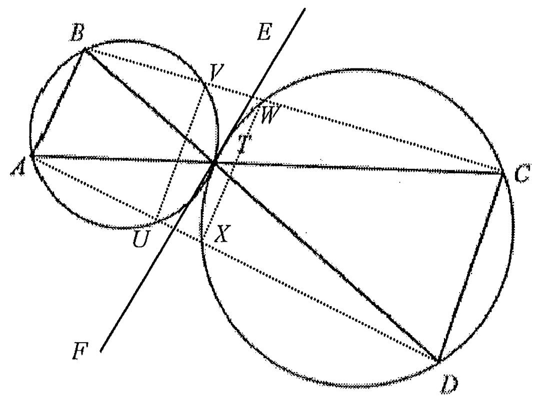
. Seat the women down first $(3!)$

Then 2 choices for men b/w $W_1$ & $W_2$

this then locks in the remaining men

$\therefore 3! \times 2 = 12$

(4) (c) (i)




(ii) $\angle BAC = \angle BTE$ (alternate segment theorem)
$\angle BTE = \angle DTF$ (vertically opposite angles)
$\angle DTF = \angle ACD$ (alternate segment theorem)

(iii) $\therefore AB \parallel CD$ (alternate angles are equal) $\Rightarrow ABCD$ is a trapezium and $\Delta DTC$ is isosceles.
$\Delta ATD \equiv \Delta BTC$ (SAS) $\Rightarrow AD = BC$ & $\angle ADC = \angle BCD$.

(iv) *ABCD* is a cyclic quad (*ABCD* is an isosceles trapezium)
*WXCD* and *ABUV* are cyclic quads too.
Let $\angle BCD = x \Rightarrow \angle UXW = x$ (exterior angle of a cyclic quad).
As well $\angle BAC = x$ (opposite angles in a cyclic quad *ABCD*)
$\therefore \angle BVU = x$ (opposite angles in a cyclic quad *ABUV*)
$\therefore \angle BVU = \angle UXW = x$
Hence *UVWX* is a cyclic quad since the exterior angle of the quad is equal to the opposite interior angle.