SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
JUNE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time — 5 minutes
- Working time — 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks — 100
- Attempt questions 1—6
- All questions are not of equal value, the mark value is shown beside each part.

Examiner: E.Choy

Note: This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 100
Attempt Questions 1–6
All questions are not of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (20 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{1}{x\ln x}\,dx$. **2**

(b) Find $\displaystyle\int_0^{\pi/3} \sin^3 x \cos x\, dx$. **3**

(c) By completing the square, find $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}}$. **3**

(d) Use integration by parts to evaluate $\displaystyle\int_0^{½} \cos^{-1} x\, dx$. **3**

(e) (i) Use partial fractions to show that: **3**

$$\int_0^1 \frac{dx}{(x+2)(2x+1)} = \frac{1}{3}\ln 2.$$

(ii) Hence evaluate $\displaystyle\int_0^{\pi/2} \frac{3}{4+5\sin x}\,dx$. **3**

(f) Show that $f(x) = x^8 \sin x$ is an odd function. **3**

Hence evaluate $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^8 \sin x\, dx$.

Marks

Question 2 (20 marks) Use a SEPARATE writing booklet.

(a) (i) Simplify i^{2002}. **2**

(ii) Solve $2z^2+(3+i)z+2 = 0$. **2**

(b) On separate Argand diagrams, shade the regions:

(i) $-2 < \mathbb{I}\text{m}(z) \leq 5$ **2**

(ii) $|z| < 6$ **2**

(iii) $2 < z+\bar{z} < 10$ **2**

(iv) $arg(z^2) = \dfrac{2\pi}{3}$. **2**

(c) In how many ways can 10 women be directed into two groups of 3 and 7 respectively? **2**

(d) If α, β, γ are the roots of the equation $x^3-2x^2+2x-2 = 0$, find the value of $\alpha^2 + \beta^2 + \gamma^2$. Explain why only one root of the equation is real. **3**

(e) A certain polynomial, $P(x)$, is an odd polynomial of degree 5. It is given that $P(1) = P(2) = 0$ and $P(3) = 240$. Find $P(x)$. **3**

Marks

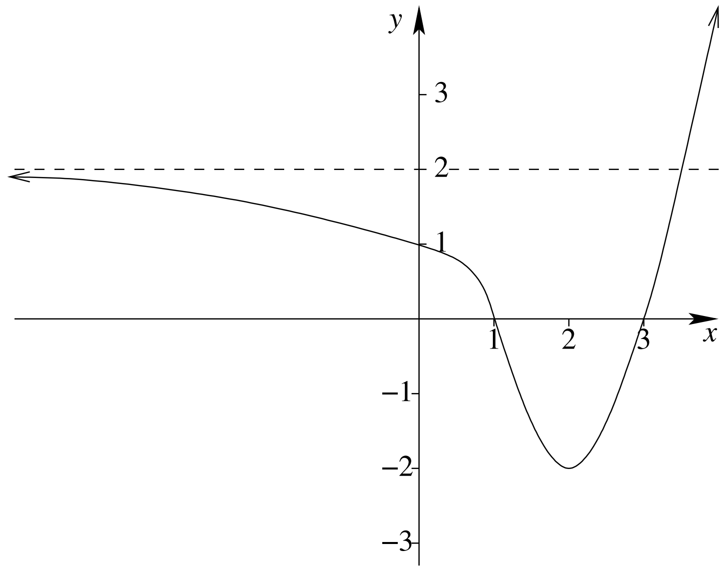
Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Solve, graphically or otherwise, **4**

$$|x^2 - 2x - 3| < 3x - 3.$$

(b) On the same set of axes, sketch and label the graphs with equations $y = x(x-3)^2$ and $y^2 = x(x-3)^2$. Clearly indicate turning points and any other critical points. **4**

(c) The diagram below shows the graph of a function, $y = f(x)$. There is an horizontal asymptote $y = 2$ as shown. For your convenience this graph is reproduced on a separate sheet. Using the given graphs as a guide, sketch the required graphs on the separate sheet.
Insert the sheet into your examination booklet for Question 3.



(i) $y = f(x+2)$,

(ii) $y = |f(x)|$,

(iii) $|y| = f(x)$,

(iv) $y = \dfrac{1}{f(x)}$,

(v) $y = \ln f(x)$.

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $x = \sin^2\theta$ to evaluate $\displaystyle\int_0^{1/2} \frac{\sqrt{x}}{(1-x)^{3/2}}\,dx.$ **3**

(b) (i) If $f(x) = f(a-x)$, prove that **4**

$$\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx.$$

(ii) Hence or otherwise, prove that $\displaystyle\int_0^{\pi} g(x)\,dx = \frac{\pi^2}{4}$, **3**

if $g(x) = \dfrac{x\sin x}{1+\cos^2 x}$.

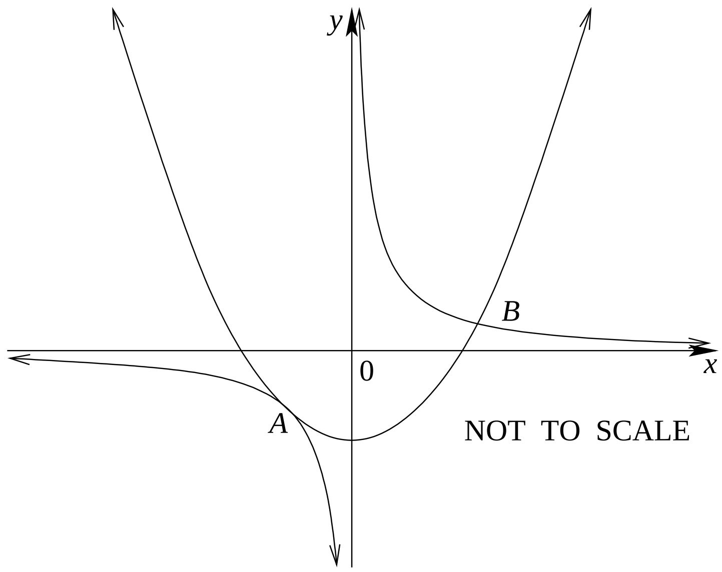
(c) (i) Given $\mathrm{I}_n = \displaystyle\int_0^1 x^n e^{2x}\,dx$, where n is a positive integer, use integration by parts to show that $\mathrm{I}_n = \frac{1}{2}\left(e^2 - n\mathrm{I}_{n-1}\right)$. **3**

(ii) Hence evaluate $\displaystyle\int_0^1 x^4 e^{2x}\,dx.$ **2**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a)



The sketch above shows the graphs of $y = x^2 - b$ and $y = \dfrac{k}{x}$, where $b > 0$ and $k > 0$. The hyperbola touches the parabola at the point A and cuts it at the point B.

(i) Show that the x-coordinates of the points of intersection of the hyperbola and the parabola are the roots of the equation $x^3 - bx - k = 0$. **2**

(ii) Explain why this equation has a double root. **2**

(iii) Show that $4b^3 = 27k^2$. **3**

(iv) If $b = 12$, find the coordinates of A and B. **3**

(b) In a hat are 10 red, 10 blue, and 10 yellow cards. Each colour group is numbered from 1 to 10. Four cards are chosen at random from the hat.

(i) How many groups of four cards can be chosen which contain at least one red card? **2**

(ii) Find the probability that a group of four cards chosen at random contains at least one of each colour, given that the group contains at least one red card. **3**

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

A particle of unit mass, initially at rest at the origin, is moving along a straight line. The particle is attracted by two objects that are to the right of the origin, one at position A and the other at B. The magnitude of the force due to the object at A is equal to the distance of the particle from A while the magnitude of the force due to B is equal to the square of the distance of the particle from B. Position A is 3 metres from the origin and B is 6 metres from the origin.

(i) Show that the acceleration of the particle for $0 \le x \le 3$ and also for $3 \le x \le 6$ is given by: **3**

$$\ddot{x} = x^2 - 13x + 39.$$

(ii) Find an expression for v^2, the square of the velocity at position x where $0 \le x \le 6$. **2**

(iii) Explain why the particle never comes to rest between the origin and B. **2**

(iv) Show that the speed of the particle when it first arrives at B is 12 m/s. **2**

(v) Find an expression for the acceleration of the particle when it is beyond B. **2**

(vi) Find an expression for the speed of the particle when it is beyond B and explain why the particle comes to rest somewhere between 11 and 12 metres from the origin. (You do not have to find the exact position.) **4**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a1>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

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Name ____

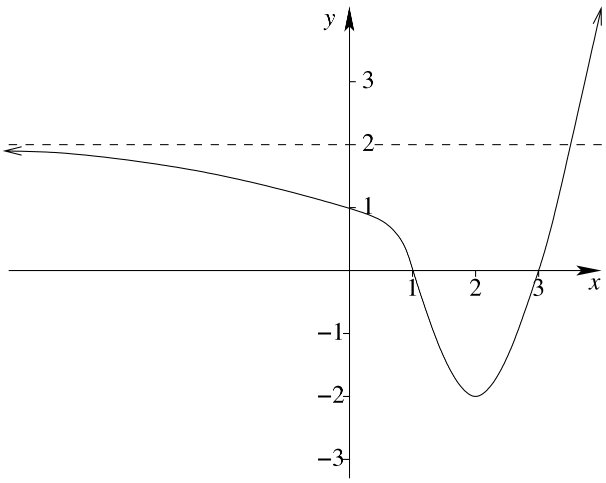
Class ______________

GRAPHS FOR QUESTION 3

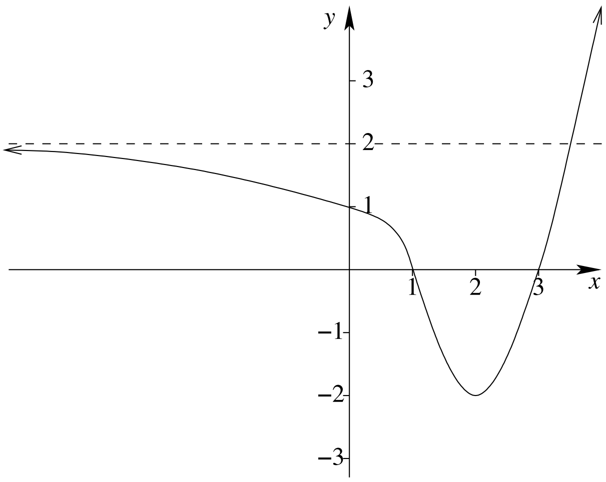
The diagrams on this sheet each show a graph of the function $y = f(x)$, as shown on page 4 of your question booklet. Using the given graphs as a guide, sketch the required graphs.

Insert this sheet into your answer booklet for Question 3. **Marks**

(vii) Sketch $y = f(x+2)$, **2**

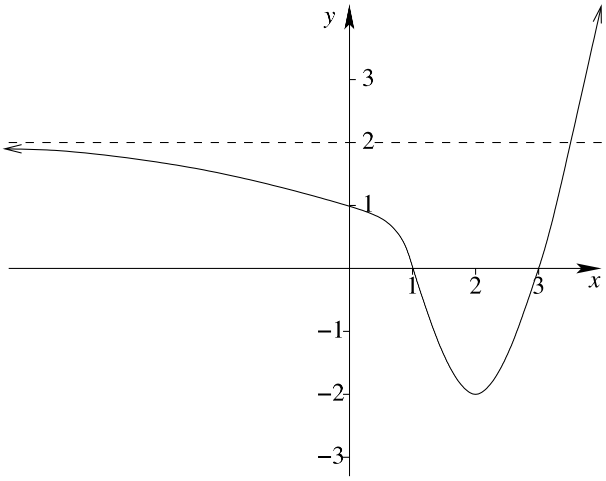


(viii) Sketch $y = |f(x)|$. **2**

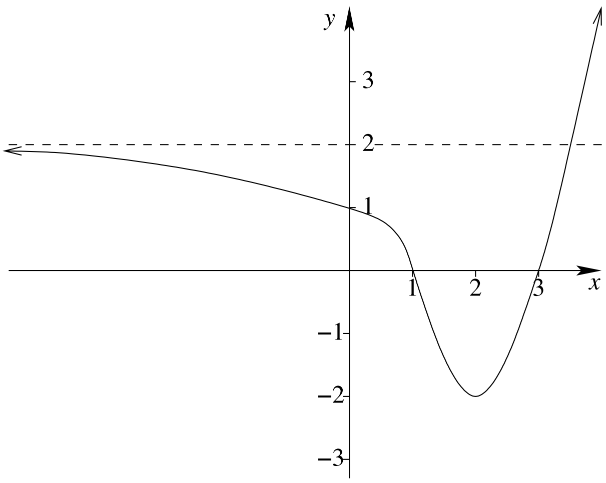


Marks

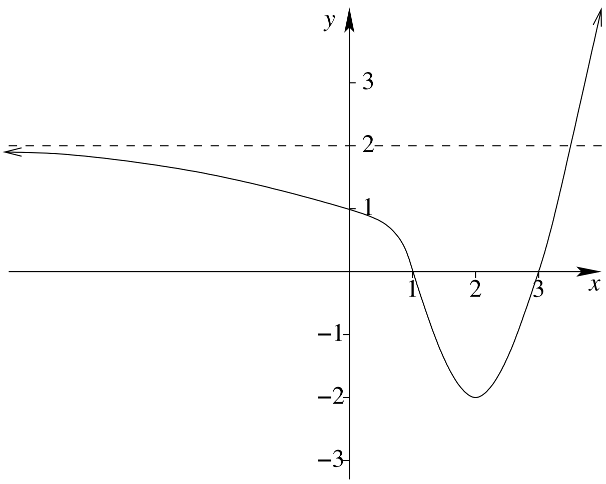
(ix) Sketch $|y| = f(x)$. **2**



(x) Sketch $y = \dfrac{1}{f(x)}$. **2**



(xi) Sketch $y = \ln f(x)$. **2**



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2002
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ASSESSMENT TASK # 2

Mathematics Extension 2

Sample Solutions

2002 Ext 2 Task # 2

(1) (a) $\int \frac{1}{x\ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx$ (let $u = \ln x$)

$= \ln|\ln x| + C$

(b) $\int_0^{\pi/3} \sin^3 x \cos x\,dx$

$\left[\begin{array}{l} \text{let } u = \sin x \Rightarrow du = \cos x\,dx \\ x = 0 \Rightarrow u = 0 \\ x = \pi/3 \Rightarrow u = \frac{\sqrt{3}}{2} \end{array}\right.$

$= \int_0^{\sqrt{3}/2} u^3\,du$

$= \frac{1}{4}u^4\Big]_0^{\sqrt{3}/2} = \frac{1}{4}\left[\left(\frac{\sqrt{3}}{2}\right)^4 - 0\right]$

$= \frac{1}{4} \times \frac{9}{16} = \frac{9}{64}$

(c) $\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{(x+2)^2+4}}$

Table of Integrals

$\left[\quad \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) \right.$

$= \ln\left| x+2 + \sqrt{x^2+4x+8} \right| + C$

(d) $\int_0^{\frac{1}{2}} \cos^{-1}x\,dx = \int_0^{\frac{1}{2}} 1 \times \cos^{-1}x\,dx$

$= x\cos^{-1}x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \times \frac{-1}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2}\cos^{-1}\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} \frac{-x}{\sqrt{1-x^2}}\,dx = \frac{1}{2} \times \frac{\pi}{3} - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x}{\sqrt{1-x^2}}\,dx$

[OR let $u = 1-x^2$]

$= \pi/6 - \frac{1}{2} \times 2\sqrt{1-x^2}\Big]_0^{\frac{1}{2}}$

$= \pi/6 - \left(\sqrt{1-\frac{1}{4}} - \sqrt{1}\right)$

$= \pi/6 + 1 - \frac{\sqrt{3}}{2}$

(e) (i) $\dfrac{1}{(x+2)(2x+1)} = \dfrac{A}{(x+2)} + \dfrac{B}{(2x+1)}$

$\therefore 1 \equiv A(2x+1) + B(x+2)$

$x = -\frac{1}{2} \Rightarrow 1 = 0 + B \times \frac{3}{2}$

$\therefore B = \frac{2}{3}$

$\because A + 2B = 1 \Rightarrow A + \frac{4}{3} = 1$

$\therefore A = -\frac{1}{3}$

$$\int_0^1 \frac{dx}{(x+2)(2x+1)} = \int_0^1 \left(\frac{-\frac{1}{3}}{x+2} + \frac{\frac{2}{3}}{2x+1} \right) dx$$

$$= -\frac{1}{3} \int_0^1 \left(\frac{1}{x+2} - \frac{2}{2x+1} \right) dx = -\frac{1}{3} \ln\left(\frac{x+2}{2x+1} \right) \Big]_0^1$$

$$= -\frac{1}{3} \left[\ln\left(\frac{3}{3}\right) - \ln 2 \right]$$

$$= \frac{1}{3} \ln 2$$

(ii) $\displaystyle\int_0^{\pi/2} \frac{3}{4+5\sin x}\, dx$

let $u = \tan x/2$

$\therefore du = \frac{1}{2}\sec^2 x/2 \, dx$

$\therefore dx = 2\cos^2 x/2 \, du$

$= \dfrac{2}{1+u^2}\, du$

$\sin x = \dfrac{2u}{1+u^2}$

$x = 0 \Rightarrow u = 0$

$x = \pi/2 \Rightarrow u = 1$

$$= \int_0^1 \frac{3}{4 + 5\left(\frac{2u}{1+u^2}\right)} \times \frac{2du}{1+u^2}$$

$$= \int_0^1 \frac{3(1+u^2)}{(4(1+u^2) + 10u)} \times \frac{2du}{(1+u^2)}$$

$$= \int_0^1 \frac{3du}{(2u^2+5u+2)}$$

$$= 3\int_0^1 \frac{du}{2u^2+5u+2} = 3\int_0^1 \frac{du}{(u+2)(2u+1)}$$

$= 3 \times \frac{1}{3} \ln 2 \quad$ [from (i)]

$= \ln 2$

(f) $f(x) = x^8 \sin x$

$\therefore f(-x) = (-x)^8 \sin(-x)$

$= x^8 \times -\sin x$

$= -x^8 \sin x$

$= -f(x) \quad \therefore$ ODD

$$\therefore \int_{-\pi/2}^{\pi/2} x^8 \sin x \, dx = 0$$

(2) (a) (i) $i^{2002} = (i^2)^{1001}$

$= (-1)^{1001}$

$= -1$

(ii) $2z^2 + (3+i)z + 2 = 0$

$$\therefore z = \frac{-(3+i) \pm \sqrt{(3+i)^2 - 4\times 2\times 2}}{4}$$

$$= \frac{-(3+i) \pm \sqrt{8+6i-16}}{4}$$

$$= \frac{-3-i \pm \sqrt{6i-8}}{4}$$

$$= \frac{-3-i \pm (1+3i)}{4}$$

$$= \frac{-2+2i}{4}, \frac{-4-4i}{4}$$

$$= \frac{-1+i}{2}, -(1+i)$$

$(x+iy)^2 = -8+6i$

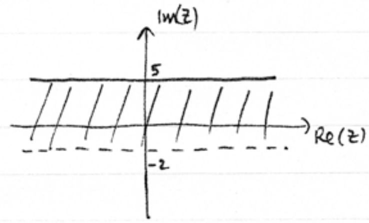
$\therefore x^2 - y^2 = -8$

$2xy = 6$

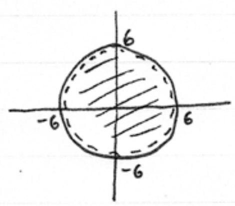
$1 - 9 = -8$

$\therefore \boxed{x=1,\ y=3}$

(b) (i) $-2 < \operatorname{Im}(z) \le 5$



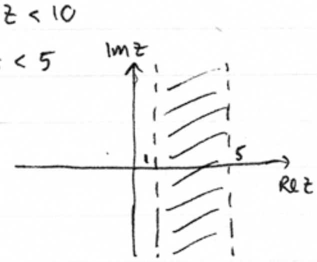
(ii) $|z| < 6$



(iii) $2 < z + \bar{z} < 10$

$\therefore 2 < 2\operatorname{Re} z < 10$

$\therefore 1 < \operatorname{Re} z < 5$

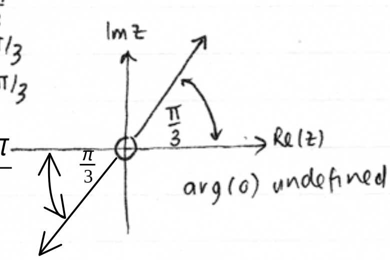


(iv) $\arg(z^2) = \frac{2\pi}{3}$

$\therefore 2\arg z = 2\pi/3$

$\therefore \arg z = \pi/3$

$\arg z = \frac{\pi}{3}, -\frac{2\pi}{3}$



The two solutions
because of z^2

(2) (c) $\binom{10}{3}$ or $\binom{10}{7}$

(d) $x^3 - 2x^2 + 2x - 2 = 0$

$\alpha+\beta+\gamma = 2$

$\alpha\beta+\alpha\gamma+\beta\gamma = 2$

$$\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= 4 - 2(2)$$
$$= 0$$

sum of squares is zero, so non-real roots involved

BUT since the co-efficients are real, the non-real roots

occur in conjugate pairs. So only one real root

e) $P(x)$ is odd ~~polynomial deg 5~~

$\therefore$ $P(x) = ax^5 + bx^3 + cx$

$P(1) = P(2) = 0$ and $P(0) = 0$

$\therefore P(x) = x(x-1)(x-2)\,Q(x)$, $\deg Q(x) = 2$

BUT $P(x)$ is odd $P(1) = 0 \Rightarrow P(-1) = 0$

$P(2) = 0 \Rightarrow P(-2) = 0$

$\therefore P(x) = ax(x+1)(x-1)(x+2)(x-2)$

$P(3) = 240$

$\Rightarrow 3a(4)(2)(5)(1) = 240$

$\Rightarrow 120a = 240$

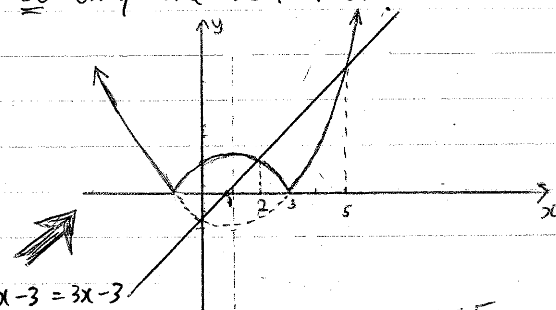
$\Rightarrow a = 2$

$\therefore P(x) = 2x(x+1)(x-1)(x+2)(x-2)$

(3) (a) $|x^2 - 2x - 3| < 3x - 3$

ie $|(x-3)(x+1)| < 3(x-1)$

Find points of intersection



check $-(x^2-2x-3) = 3x-3$

$-x^2 - x + 6 = 0$

$x^2 + x - 6 = 0$

$\boxed{(x+3)(x-2) = 0}$

$x^2 - 2x - 3 = 3x - 3$

$\therefore x^2 - 5x = 0$

$x(x-5) = 0$

$\boxed{x = 5}$

$\therefore$ $2 < x < 5$

(b) $y = x(x-3)^2$

$= x(x^2 - 6x + 9)$

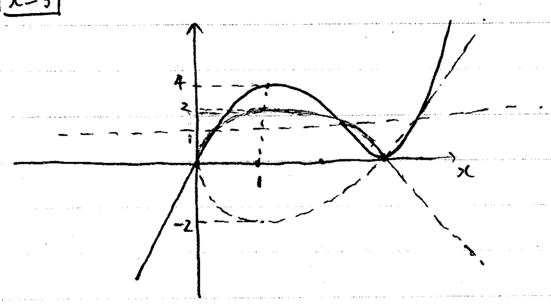
$= x^3 - 6x^2 + 9x$

$y' = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$x = 1$, $y = 4$



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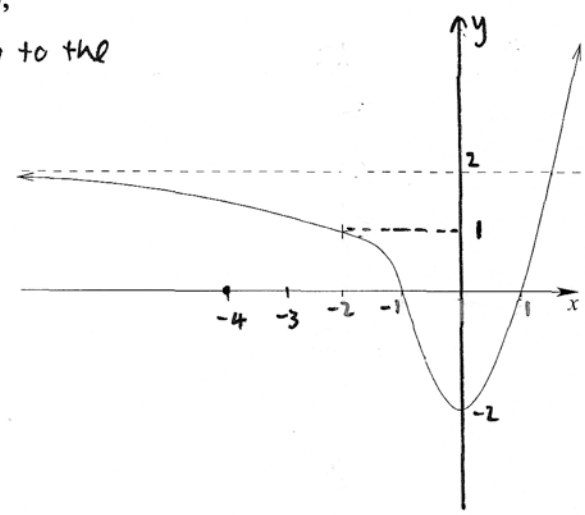
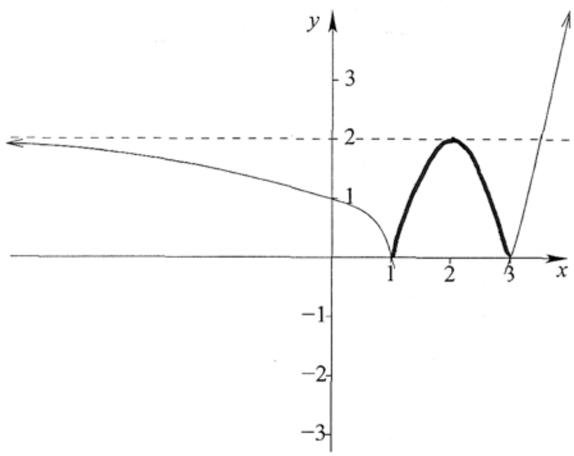
Insert this sheet into your answer booklet for Question 3. **Marl**

(vii) Sketch $y = f(x+2)$, 2

move y-axis 2 units to the right

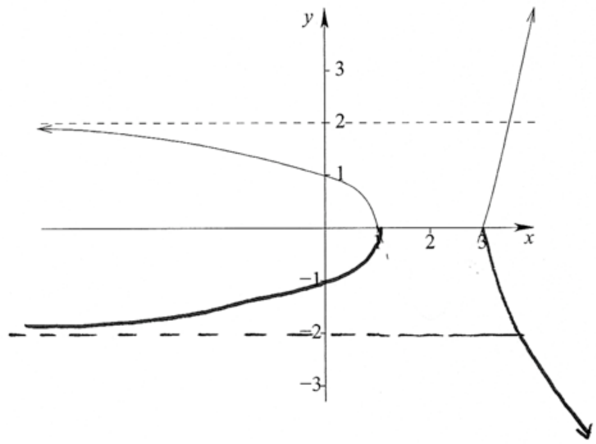


(viii) Sketch $y = |f(x)|$. 2

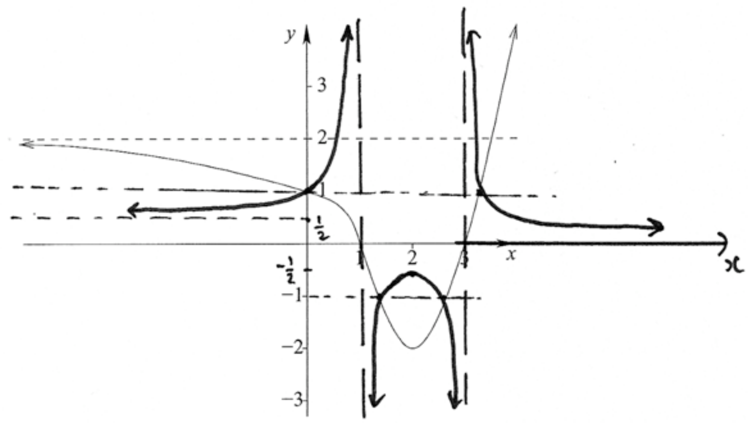


Marks

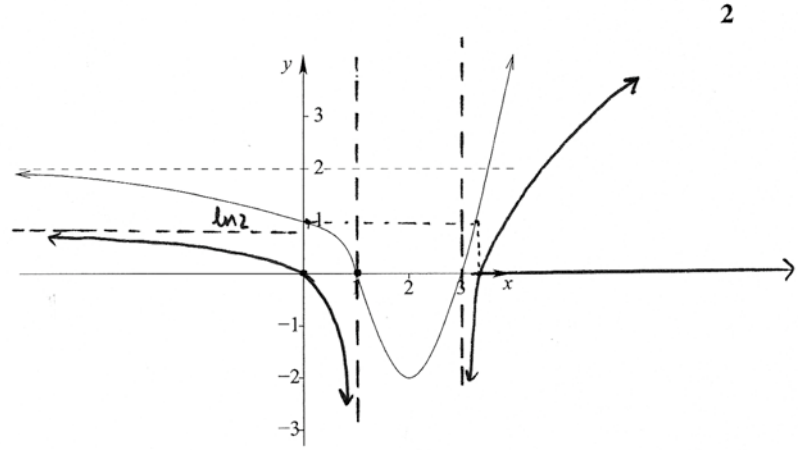
(ix) Sketch $|y| = f(x)$. **2**



(x) Sketch $y = \dfrac{1}{f(x)}$. **2**



(xi) Sketch $y = \ln f(x)$. **2**



(4)

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x^2)^{3/2}}\,dx$

$x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = \frac{1}{2} \Rightarrow \theta = \pi/4$

$$= \int_0^{\pi/4} \frac{\sin\theta \cdot 2\sin\theta\cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}}$$

$$= \int_0^{\pi/4} \frac{2\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$$

$$= 2\int_0^{\pi/4} \tan^2\theta\,d\theta$$

$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta$$

$$= 2\left[\tan\theta - \theta\right]_0^{\pi/4}$$

$$= 2\left[1 - \pi/4\right]$$

$$= 2 - \pi/2$$

* $\sqrt{x} = \sqrt{\sin^2\theta} = |\sin\theta|$

$= \sin\theta$ for $0 \le \theta \le \frac{\pi}{4}$

[N.B. if you chose $\theta = -\pi/4$ above then you need to choose $-\sin\theta$]

$\sqrt{\cos^2\theta} = \cos\theta$ for $0 \le \theta \le \pi/4$

(b) $f(x) = f(a-x)$

(i) $\displaystyle\int_0^a x f(x)\,dx$

let $u = a - x \Rightarrow du = -dx$

$x = 0 \Rightarrow u = a$

$x = a \Rightarrow u = 0$

$x = a - u$

$$= \int_a^0 (a-u) f(a-u) \times -du$$

$$= \int_0^a (a-u) f(a-u)\,du$$

$$\therefore \int_0^a x f(x)\,dx = \int_0^a a f(a-x)\,dx - \int_0^a x f(a-x)\,dx$$

$$= \int_0^a a f(x)\,dx - \int_0^a x f(x)\,dx$$

$$\therefore 2\int_0^a x f(x)\,dx = a\int_0^a f(x)\,dx$$

$$\therefore \int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$$

(ii) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$ $[u = \cos x]$

$$= \frac{\pi}{2}\int_1^{-1} \frac{-du}{1+u^2} = 2 \times \frac{\pi}{2} \times \int_0^1 \frac{du}{1+u^2} \quad \text{(even)}$$

$$= \pi \times \tan^{-1}(1) = \pi \times \pi/4 = \pi^2/4 \quad \text{Q.E.D.}$$

(4) (c) (i) $I_n = \int_0^1 x^n e^{2x}\,dx$

$$= \frac{1}{2}e^{2x} \times x^n \Big]_0^1 - \int_0^1 \left(\frac{1}{2}e^{2x}\right) \times n x^{n-1}\,dx$$

$$= \left(\frac{1}{2}e^2 \times 1\right) - (0) - \frac{n}{2}\int_0^1 x^{n-1} e^{2x}\,dx$$

$$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

$$I_n = \frac{1}{2}\left(e^2 - nI_{n-1}\right)$$

(ii)

$$I_4 = \int_0^1 x^4 e^{2x}\,dx$$

$$I_4 = \frac{1}{2}\left(e^2 - 4I_3\right)$$

$$I_3 = \frac{1}{2}\left(e^2 - 3I_2\right)$$

$$I_2 = \frac{1}{2}\left(e^2 - 2I_1\right)$$

$$I_1 = \frac{1}{2}\left(e^2 - I_0\right)$$

$$I_0 = \int_0^1 e^{2x}\,dx = \frac{1}{2}\ \ e^{2x}\ {}_0^1 = \frac{1}{2}\left(e^2 - 1\right)$$

$$I_1 = \frac{1}{2}\ \ e^2 - \frac{1}{2}\left(e^2 - 1\right)\ \ = \frac{1}{4}\left(e^2 + 1\right)$$

$$I_2 = \frac{1}{2}\ \ e^2 - 2 \times \frac{1}{4}\left(e^2 + 1\right)\ \ = \frac{1}{4}\left(e^2 - 1\right)$$

$$I_3 = \frac{1}{2}\ \ e^2 - 3 \times \frac{1}{4}\left(e^2 - 1\right)\ \ = \frac{1}{8}\left(e^2 + 3\right)$$

$$I_4 = \frac{1}{2}\ \ e^2 - 4 \times \frac{1}{8}\left(e^2 + 3\right)\ \ = \frac{1}{4}\left(e^2 - 3\right)$$

(5)

(i) $y = x^2 - b \qquad y = \frac{k}{x}$

$\therefore x^2 - b = \frac{k}{x}$

$\therefore x^3 - bx = k$

$\therefore x^3 - bx - k = 0$

(ii) A is where they touch ie. a common tangent. Hence the double root, since there must be 3 solutions, so at A they are identical.

(iii) let $f(x) = x^3 - bx - k$

$\therefore f'(x) = 3x^2 - b$

let $x = \alpha$ be the x-coord of A

$\therefore f(\alpha) = f'(\alpha) = 0$

$\therefore 3\alpha^2 = b \qquad (b > 0)$

$f(\alpha) = 0 \Rightarrow \alpha(\alpha^2 - b) = k$

$\therefore \alpha^2(\alpha^2 - b)^2 = k^2$

$\therefore \frac{b}{3}(b/3 - b)^2 = k^2$

$\therefore \frac{b}{3} \times (-\frac{2b}{3})^2 = k^2$

$\therefore \frac{b}{3} \times \frac{4b^2}{9} = k^2$

$\therefore 4b^3 = 27k^2$

(iv) $b = 12 \Rightarrow 4 \times 12^3 = 27k^2$

$\therefore 4 \times 1728 = 27$

$\therefore k^2 = 256$

$k > 0 \Rightarrow k = 16$

$\therefore \alpha + \alpha + \beta = 0 \Rightarrow 2\alpha + \beta = 0$ –① (β is the x-coord of B)

$\alpha^2 + 2\alpha\beta = -b \Rightarrow \alpha^2 + 2\alpha\beta = -12$

$\alpha^2\beta = -(-k) \Rightarrow \alpha^2\beta = 16$ –②

① → ② $\beta = -2\alpha \Rightarrow \alpha^2(-2\alpha) = 16$

$-\alpha^3 = 8 \Rightarrow \alpha = -2 \quad \therefore \beta = 4$

$\therefore A(-2, -8) \quad B(4, 4)$

5(b) 10 R, 10 B, 10 Y

R1, ..., R10, B1, ..., B10, Y1, ..., Y10

(i) 4 cards $\Rightarrow \binom{30}{4} = 27405$

No red card $\Rightarrow \binom{20}{4} = 4845$

preferred solution

$\therefore$ At least one red card $= {}^{30}C_4 - {}^{20}C_4 = 22560$

- - - - - - - - - - - - - - - - -

OR $\binom{10}{1}\binom{20}{3} + \binom{10}{2}\binom{20}{2} + \binom{10}{3}\binom{20}{1} + \binom{10}{4}$

(ii) At least one Red, and at least one of each colour

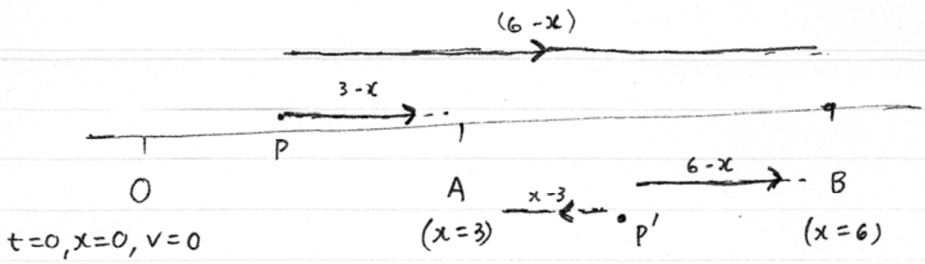
$\therefore$ RRYB, RYYB, RYBB

ie. $\binom{10}{2}\binom{10}{1}\binom{10}{1} \times 3$

$= 13500$

$\therefore$ Prob $= \dfrac{13500}{22560} = \dfrac{225}{376} \doteqdot 59.8\%$

(6)



(arrows indicating direction of acceleration)

(i) · For $0 \le x \le 3$ ie for some point P

$$\ddot{x} = (3-x) + (6-x)^2$$
$$= 3 - x + 36 - 12x + x^2$$
$$= x^2 - 13x + 39$$

For $3 \le x \le 6$ ie for some Point P'

N.B. distance from A is $x-3$ BUT the acceleration is negative
distance from B is $6-x$ BUT acceleration is positive.

$$\therefore \ddot{x} = -(x-3) + (6-x)^2$$
$$= x^2 - 13x + 39$$

(ii) $$\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = x^2 - 13x + 39$$

$$\therefore \frac{1}{2}v^2 = \frac{1}{3}x^3 - \frac{13}{2}x^2 + 39x + c$$

$$v^2 = \frac{2}{3}x^3 - 13x^2 + 78x + k \qquad (x=0, v=0) \Rightarrow k = 0$$

$$\therefore v^2 = \frac{2}{3}x^3 - 13x^2 + 78x$$
$$= \frac{x}{3}(2x^2 - 39x + 234)$$

(iii) $v = 0 \Rightarrow \frac{x}{3} = 0$ or $2x^2 - 39x + 234 = 0$

$\therefore x = 0$ or $2x^2 - 39x + 234 = 0$

BUT $\Delta = -351 < 0$

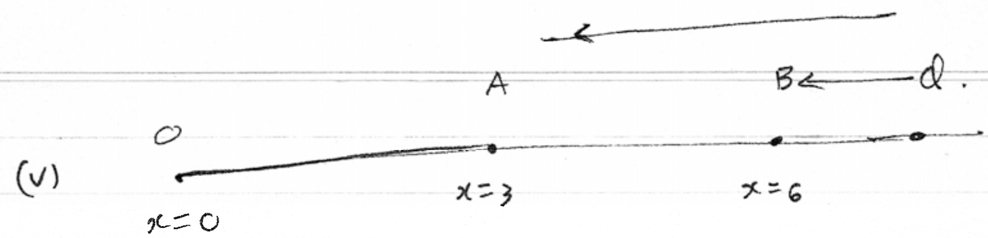
$\therefore$ no real solution

ie. $v \neq 0$ except at $x = 0$ ie. initially

(iv) $x = 6$, $\therefore v^2 = \frac{6}{3}(2 \times 36 - 39 \times 6 + 234)$

$= 2(72) = 144$

$\therefore$ speed $= |v| = 12$



distance from A is $x-3$

distance from B is $x-6$

but acceleration is towards A **and** B

$$\therefore \ddot{x} = -(x-3) - (x-6)^2$$
$$= -x+3 - (x^2-12x+36)$$
$$= -x+3 - x^2+12x-36$$
$$= -x^2+11x-33$$

(vi) $\therefore \dfrac{d(\frac{1}{2}v^2)}{dx} = -x^2+11x-33$

$$\tfrac{1}{2}v^2 = -\tfrac{1}{3}x^3 + \tfrac{11}{2}x^2 - 33x + C$$

$$\therefore v^2 = -\tfrac{2}{3}x^3 + 11x^2 - 66x + R \qquad (x=6,\ v^2=144)$$

$$144 = -144 + 11\times 36 - 66\times 6 + R$$

$$\therefore R = 288$$

N.B. $v^2 = -\tfrac{2}{3}x^3 + 11x^2 - 66x + 288$

at $x=11$: $v^2 = -\tfrac{2}{3}\times 11^3 + 11\times 11^2 - 66\times 11 + 288 = 5\tfrac{2}{3}$

$\therefore$ particle is **IN** motion at $x=11$

at $x=12$: $v^2 = -\tfrac{2}{3}\times 12^3 + 11\times 12^2 - 66\times 12 + 288 = -72$

$\therefore$ it does NOT reach $x=12$

So it MUST stop $11 < x < 12$ Q.E.D.