SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section A (Questions 1 - 3), Section B (Questions 4 - 5) and Section C (Questions 6 - 7).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 75 Marks
- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 78
Attempt Questions 1 – 7
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (15 marks) **Marks**

(a) Evaluate **3**

$$\int_0^3 \frac{x\,dx}{\sqrt{16+x^2}}$$

(b) By completing the square first, find **2**

$$\int \frac{dx}{x^2+6x+13}$$

(c) Use integration by parts to find **2**

$$\int xe^{-x}\,dx$$

(d) Find **3**

$$\int \cos^3\theta\; d\theta$$

(e) (i) Find real numbers A, B, and C such that **3**

$$\frac{x^2-4x-1}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$$

(ii) Hence find $\displaystyle\int \frac{x^2-4x-1}{(1+2x)(1+x^2)}\,dx$ **2**

Question 2 (10 marks) **Marks**

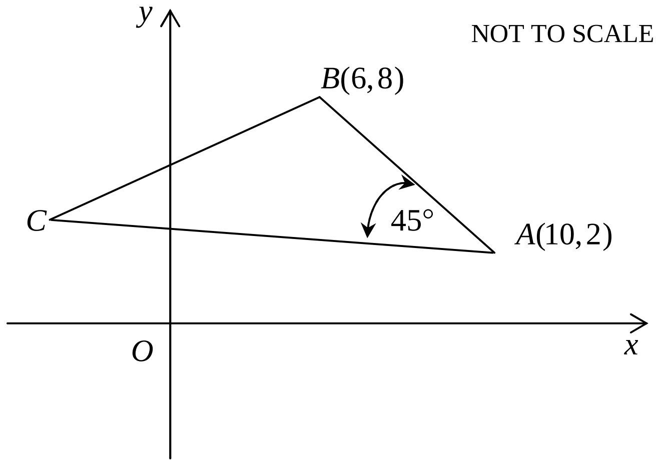
(a) On separate Argand diagrams, sketch the locus defined by:

(i) $2|z| = z + \overline{z} + 2$ **2**

(ii) $\left|z^2 - (\overline{z})^2\right| \geq 4$ **2**

(iii) $\arg(z-1) - \arg(z+1) = -\pi/3$ **2**

(b)



ΔABC is drawn in the Argand diagram above where $\angle BAC = 45°$, A and B are the points $(10,2)$ and $(6,8)$ respectively.
The length of side AC is twice the length of side AB.

Find:

(i) the complex number that the vector $\overrightarrow{AB}$ represents the complex number $-4+6i$; **2**

(ii) the complex number that the point C represents. **2**

Question 3 (12 marks)

(a) The quadratic equation $x^2 - x + K = 0$, where K is a real number, has two distinct positive real roots α and β.

(i) Show that $0 < K < \dfrac{1}{4}$ **1**

(ii) Show that $\alpha^2 + \beta^2 = 1 - 2K$ and deduce that $\alpha^2 + \beta^2 > \dfrac{1}{2}$ **2**

(iii) Show that $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} > 8$ **2**

SECTION A continued

Question 3 continued **Marks**

(b) (i) Show, using De Moivre's Theorem, that $z = \omega$, where $\omega = \sqrt{2} + i\sqrt{2}$ satisfies $z^4 = -16$. **3**
Hence write down, in the form $x + iy$ where x and y are real, all the other solutions of $z^4 = -16$.

(ii) Hence write $z^4 + 16$ as a product of two quadratic factors with real coefficients. **2**

(iii) Show that $\omega + \dfrac{\omega^3}{4} + \dfrac{\omega^5}{16} + \dfrac{\omega^7}{64} = 0$ **2**

SECTION B (Use a SEPARATE writing booklet)

Question 4 (8 marks) **Marks**

(a) Evaluate

(i) $\displaystyle\int_0^a x\sqrt{a-x}\,dx$ **2**

(ii) $\displaystyle\int_0^1 \frac{\sin^{-1}x}{\sqrt{1+x}}\,dx$ **2**

(b) (i) Using the substitution $t=\tan\dfrac{x}{2}$ show that **2**

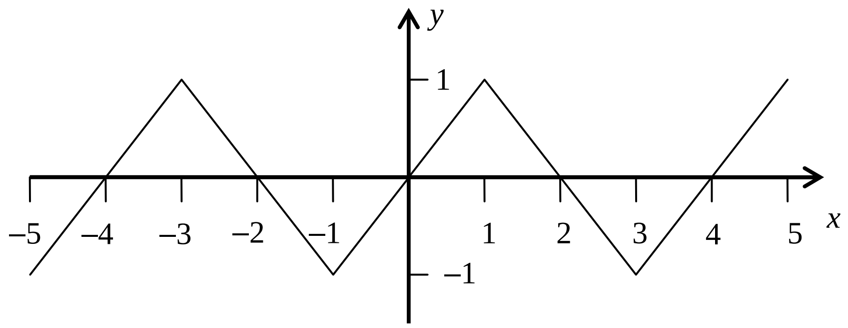
$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}=\ln 2$$

(ii) Hence, by substituting $u=\dfrac{\pi}{2}-x$ evaluate **2**

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$$

Question 5 (11 marks)

(a)



The diagram is a sketch of the function $y=h(x)$ for $-5\le x\le 5$.
On separate diagrams sketch each of the following:

(i) $y=h(x+1)$ **1**

(ii) $y=\dfrac{1}{h(x)}$ **2**

(iii) $y=h\left(|x|\right)$ **1**

(iv) $y=\sqrt{h(x)}$ **2**

(v) $y=h\left(\sqrt{x}\right)$ **2**

(b) Sketch the curve $9y^2=x(x-3)^2$ showing clearly the coordinates of any turning point. **3**

SECTION C (Use a SEPARATE writing booklet)

Question 6 (9 marks) **Marks**

A firework missile of mass $0 \cdot 2$ kg is projected vertically upwards from rest by means of a force that decreases uniformly in 2 seconds from $2g$ newtons to zero and thereafter ceases. Assume no air resistance and that g is the acceleration due to gravity.

(i) If the missile has an acceleration of a m/s^2 at time t seconds, show that **3**

$$a = \begin{cases} g(9-5t) & t \le 2 \\ -g & t > 2 \end{cases}$$

[Hint: Draw a diagram showing the forces on the missile.]

(ii) Hence find:

(α) the maximum speed of the missile; **3**

(β) the maximum height reached by the missile. **3**

Question 7 (10 marks) **Marks**

A particle of mass 1 kg is projected from a point O with a velocity u m/s along a smooth horizontal table in a medium whose resistance is Rv^2 newtons when the particle has velocity v m/s. R is a constant, with $R > 0$.

(i) Show that the equation of motion governing the particle is given by **1**

$$\ddot{x} = -Rv^2$$

where x is the horizontal distance travelled from O.

(ii) Hence show that the velocity, v m/s, after t seconds is given by **3**

$$t = \frac{1}{R}\left(\frac{1}{v} - \frac{1}{u}\right)$$

An equal particle is projected from O simultaneously with the first particle, but vertically upwards with velocity u m/s in the SAME medium.

(iii) Show that the equation of motion governing the second particle is given by **1**

$$\ddot{y} = -(g + Rv^2)$$

where g m/s^2 is the acceleration due to gravity and y represents the vertical distance from O where the particle has a velocity of v m/s.

(iv) Hence show that the velocity V m/s of the first particle when the second one is momentarily at rest is given by **5**

$$\frac{1}{V} = \frac{1}{u} + \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right), \text{ where } Ra^2 = g$$

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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Mathematics Extension 2

Sample Solutions

Section	Marker
A	Mr Hespe
B	Mr Parker
C	Mr Kourtesis

$\boxed{3}$ 1. (a) Method 1:

$$\begin{aligned} \mathrm{I} &= \int_0^3 \frac{xdx}{\sqrt{16+x^2}}, \\ &= \tfrac{1}{2}\int_{16}^{25} u^{-\frac{1}{2}}du, \\ &= \tfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_{16}^{25}, \\ &= 5-4, \\ &= 1. \end{aligned} \qquad \begin{aligned} \text{put } u &= 16+x^2 \\ du &= 2xdx \\ \text{when } x &= 0, & u &= 16 \\ x &= 3, & u &= 25 \end{aligned}$$

Method 2:

$$\begin{aligned} \mathrm{I} &= \int_4^5 \frac{udu}{u}, \\ &= u\Big]_4^5, \\ &= 1. \end{aligned} \qquad \begin{aligned} \text{put } u^2 &= 16+x^2 \\ 2udu &= 2xdx \\ \text{when } x &= 0, & u &= 4 \\ x &= 3, & u &= 5 \end{aligned}$$

Method 3:

$$\begin{aligned} \mathrm{I} &= \int_0^{\tan^{-1}\frac{3}{4}} \frac{4\tan\theta . 4\sec^2\theta d\theta}{4\sec\theta}, \\ &= 4\sec\theta\Big]_0^{\tan^{-1}\frac{3}{4}}, \\ &= 4\left\{\tfrac{5}{4}-1\right\}, \\ &= 1. \end{aligned} \qquad \begin{aligned} \text{put } x &= 4\tan\theta \\ dx &= 4\sec^2\theta d\theta \\ \text{when } x &= 0, & \theta &= 0 \\ x &= 3, & \theta &= \tan^{-1}\tfrac{3}{4} \end{aligned}$$

Method 4:

$$\begin{aligned} \mathrm{I} &= \tfrac{1}{2}\int_0^9 \frac{du}{\sqrt{16+u}}, \\ &= \left[\tfrac{1}{2}\times 2\times\sqrt{16+u}\ \right]_0^9, \\ &= 5-4, \\ &= 1. \end{aligned} \qquad \begin{aligned} \text{put } u &= x^2 \\ du &= 2xdx \\ \text{when } x &= 0, & u &= 0 \\ x &= 3, & u &= 9 \end{aligned}$$

Method 5:

$$\begin{aligned} \mathrm{I} &= \tfrac{1}{2}\int_0^3 \frac{d(x^2)}{\sqrt{16+x^2}}, \\ &= \left[\tfrac{1}{2}\times 2\times\sqrt{16+x^2}\ \right]_0^3, \\ &= 5-4, \\ &= 1. \end{aligned}$$

$\boxed{2}$ (b)
$$\begin{aligned}
\mathrm{I} &= \int \frac{dx}{(x^2+6x+9)+13-9},\\
&= \int \frac{dx}{(x+3)^2+4},\\
&= \tfrac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+\mathrm{C}.
\end{aligned}$$

$\boxed{2}$ (c)
$$\begin{aligned}
\mathrm{I} &= \int xe^{-x}dx, & u &= x, & v' &= e^{-x}\\
& & u' &= 1, & v &= -e^{-x}\\
&= -xe^{-x}+\int e^{-x}dx,\\
&= -xe^{-x}-e^{-x}+\mathrm{C}.
\end{aligned}$$

$\boxed{3}$ (d) Method 1:
$$\begin{aligned}
\mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta, & \text{put } \sin\theta &= u\\
& & \cos\theta d\theta &= du\\
&= \int (1-\sin^2\theta).\cos\theta d\theta,\\
&= \int (1-u^2)du,\\
&= u-\tfrac{1}{3}u^3+\mathrm{C},\\
&= \sin\theta-\tfrac{1}{3}\sin^3\theta+\mathrm{C}.
\end{aligned}$$

Method 2:
$$\begin{aligned}
\mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta,\\
&= \int (1-\sin^2\theta).d\sin\theta,\\
&= \sin\theta-\tfrac{1}{3}\sin^3\theta+\mathrm{C}.
\end{aligned}$$

Method 3:
$$\begin{aligned}
\mathrm{I} &= \int \cos^2\theta.\cos\theta d\theta, & u &= \cos^2\theta, & v' &= \cos\theta\\
& & u' &= -2\sin\theta\cos\theta, & v &= \sin\theta\\
&= \cos^2\theta\sin\theta+2\int \sin^2\theta\cos\theta d\theta,\\
&= \cos^2\theta\sin\theta+2\int \cos\theta(1-\cos^2\theta)d\theta,\\
&= \cos^2\theta\sin\theta+2\sin\theta-2\int \cos^3\theta d\theta,\\
3\mathrm{I} &= \cos^2\theta\sin\theta+2\sin\theta+c,\\
\mathrm{I} &= \tfrac{1}{3}\{\cos^2\theta\sin\theta+2\sin\theta\}+\mathrm{C}.
\end{aligned}$$

$\boxed{3}$ (e) (i) Method 1:

$$\begin{aligned}
x^2-4x-1 &= A(1+x^2)+(Bx+C)(1+2x). \\
\text{Put } x &= -\tfrac{1}{2}, \\
\tfrac{1}{4}+2-1 &= A(1\tfrac{1}{4}), \\
A &= 1. \\
\text{Also, } x^2-4x-1 &= x^2(A+2B)+x(B+2C)+(A+C), \\
\text{so } A+2B &= 1, \\
1+2B &= 1, \\
2B &= 0, \\
B &= 0. \\
\text{And } B+2C &= -4, \\
2C &= -4, \\
C &= -2.
\end{aligned}$$

Method 2:

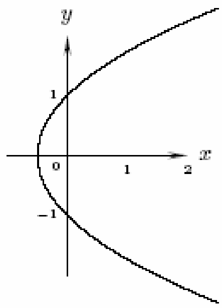
$$\begin{aligned}
\frac{x^2-4x-1}{(1+2x)(1+x^2)} &= \frac{A}{1+2x}+\frac{Bx+C}{x^2+1}. \\
\lim_{x\to-\frac{1}{2}}\left\{\frac{x^2-4x-1}{1+x^2}\right\} &= \lim_{x\to-\frac{1}{2}}\left\{A+\left(\frac{Bx+C}{1+x^2}\right)\times(1+2x)\right\}, \\
\frac{\frac{1}{4}+2-1}{1+\frac{1}{4}} &= A, \\
A &= 1. \\
\lim_{x\to i}\left\{\frac{x^2-4x-1}{1+2x}\right\} &= \lim_{x\to i}\left\{\left(\frac{A}{1+2x}\right)\times(1+x^2)+Bx+C\right\}, \\
\frac{-1-4i-1}{1+2i} &= Bi+C, \\
-2-4i &= Bi+C-2B+2iC, \\
\therefore\ C-2B &= -2, \\
B+2C &= -4, \\
2C-4B &= -4, \\
5B &= 0, \\
B &= 0, \\
C &= -2.
\end{aligned}$$

$\boxed{2}$ (ii)

$$\begin{aligned}
\mathrm{I} &= \tfrac{1}{2}\int\frac{2x\,dx}{1+2x}-2\int\frac{dx}{1+x^2}, \\
&= \tfrac{1}{2}\ln(1+2x)-2\tan^{-1}x+\mathrm{C}.
\end{aligned}$$

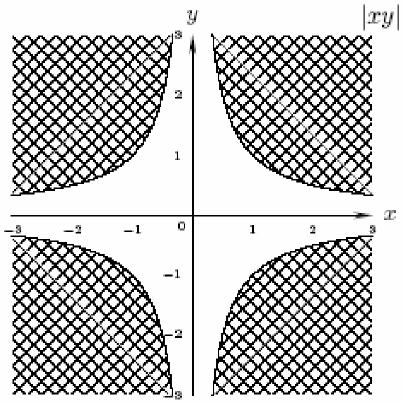
$\boxed{2}$ 2. (a) (i)

$$\begin{aligned} 2\sqrt{x^2+y^2} &= x+iy+x-iy+2, \\ x^2+y^2 &= x^2+2x+1, \\ y^2 &= 4\left(\tfrac{1}{2}\right)\left(x+\tfrac{1}{2}\right). \end{aligned}$$

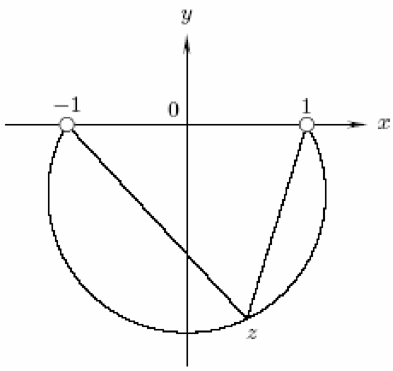


$\boxed{2}$ (ii)

$$\begin{aligned} |(x+iy)^2-(x-iy)^2| &\geq 4, \\ |x^2+2ixy-y^2-(x^2-2ixy-y^2)| &\geq 4, \\ |ixy| &\geq 1, \\ |i||xy| &\geq 1, \\ |xy| &\geq 1. \end{aligned}$$



$\boxed{2}$ (iii)



$\boxed{2}$ (b) (i) $\overrightarrow{BA} = (10-6) + i(2-8),$
$= 4 - 6i.$

Note that the question was in error: what was meant was $\overrightarrow{AB}$. Both answers were accepted, $4-6i$ or $-4+6i$.

$\boxed{2}$ (ii) Method 1:

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{AB} \times 2 \times \text{cis}\, \tfrac{\pi}{4}, \\
&= -(4-6i) \times 2 \times \left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right), \\
&= -\left(4\sqrt{2} + 4\sqrt{2}i - 6\sqrt{2}i + 6\sqrt{2}\right), \\
&= -10\sqrt{2} + 2\sqrt{2}i. \\
\therefore\ C &= (10+2i) + \left(-10\sqrt{2} + 2\sqrt{2}i\right), \\
&= 10\left(1-\sqrt{2}\right) + 2\left(1+\sqrt{2}\right)i.
\end{aligned}$$

Method2:

$$\begin{aligned}
\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AB} \times 2 \times \text{cis}\, \tfrac{\pi}{4}, \\
&= (10+2i) - (4-6i) \times 2 \times \left(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}\right), \\
&= (10+2i) - \left(4\sqrt{2} + 4\sqrt{2}i - 6\sqrt{2}i + 6\sqrt{2}\right), \\
&= 10 + 2i - 10\sqrt{2} + 2\sqrt{2}i. \\
&= 10\left(1-\sqrt{2}\right) + 2\left(1+\sqrt{2}\right)i.
\end{aligned}$$

$\boxed{1}$ 3. (a) (i) Roots $\alpha,\ \beta$ positive implies $\alpha\beta > 0$,
i.e. $\frac{c}{a} > 0$ or $K > 0$.
Also, for distinct real roots,

$$\begin{aligned}
\Delta &= 1 - 4K > 0, \\
1 &> 4K, \\
K &< \tfrac{1}{4}.
\end{aligned}$$

So, $0 < K < \frac{1}{4}$.

$\boxed{2}$ (ii)

$$\begin{aligned}
\alpha + \beta &= 1, \\
\alpha\beta &= K, \\
\alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta, \\
&= 1 - 2K.
\end{aligned}$$

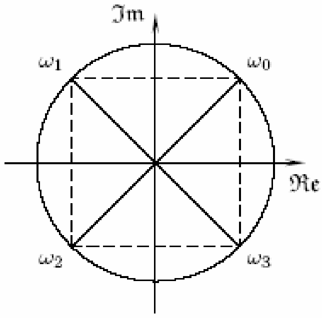
Method 1:
$K < \frac{1}{4}$,
$\therefore\ \alpha^2 + \beta^2 > 1 - 2\left(\frac{1}{4}\right)$ ("greater than" as we are subtracting "less than"),
i.e. $\alpha^2 + \beta^2 > \frac{1}{2}$.

Method 2:

$$\begin{aligned}
2K &= 1 - (\alpha^2 + \beta^2), \\
K &= \frac{1 - (\alpha^2 + \beta^2)}{2} < \frac{1}{4}, \\
-(\alpha^2 + \beta^2) &< -\frac{1}{2}, \\
\therefore\ \alpha^2 + \beta^2 &> \tfrac{1}{2}.
\end{aligned}$$

$\boxed{2}$ (iii) $$\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2},$$
$$= \frac{1-2K}{K^2}, \text{ (from above).}$$
Now, also from above,
$$\alpha^2+\beta^2 > \tfrac{1}{2},$$
$$\alpha^2\beta^2 < \left(\tfrac{1}{4}\right)^2,$$
$$\frac{1}{\alpha^2\beta^2} > 16.$$
So $$\frac{1}{\alpha^2}+\frac{1}{\beta^2} > 16\times\tfrac{1}{2},$$
$$\therefore\ \frac{1}{\alpha^2}+\frac{1}{\beta^2} > 8.$$

$\boxed{3}$ (b) (i) $$\omega = 2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right),$$
$$= 2\text{cis}\,\tfrac{\pi}{4}.$$
$$\omega^4 = \left(2\text{cis}\,\tfrac{\pi}{4}\right)^4,$$
$$= 16\text{cis}\,\pi,$$
$$= -16+16\times 0,$$
$$= -16.$$
$$\therefore\ \omega_0 = \sqrt{2}+\sqrt{2}i,$$
$$\omega_1 = -\sqrt{2}+\sqrt{2}i,$$
$$\omega_2 = -\sqrt{2}-\sqrt{2}i,$$
$$\omega_3 = \sqrt{2}-\sqrt{2}i.$$



$\boxed{2}$ (ii) $$z^4+16 = \left(z-\sqrt{2}-\sqrt{2}i\right)\left(z-\sqrt{2}+\sqrt{2}i\right)\left(z+\sqrt{2}+\sqrt{2}i\right)\left(z+\sqrt{2}-\sqrt{2}i\right),$$
$$= \left(z^2-2\sqrt{2}z+4\right)\left(z^2+2\sqrt{2}z+4\right).$$

$\boxed{2}$ (iii) Method 1:
$$\omega = 2\text{cis}\,\tfrac{\pi}{4},$$
$$\omega^3 = 2\text{cis}\,\tfrac{3\pi}{4},$$
$$\omega^5 = 32\text{cis}\,\tfrac{5\pi}{4},$$
$$\omega^7 = 128\text{cis}\,\tfrac{7\pi}{4}.$$
$$\omega+\frac{\omega^3}{4}+\frac{\omega^5}{16}+\frac{\omega^7}{64} = 2\text{cis}\,\tfrac{\pi}{4}+2\text{cis}\,\tfrac{3\pi}{4}+2\text{cis}\,\tfrac{5\pi}{4}+2\text{cis}\,\tfrac{7\pi}{4},$$
$$= \omega_0+\omega_1+\omega_2+\omega_3, \text{ the sum of the roots,}$$
$$= 0.$$
Method 2:
$$\frac{64\omega+16\omega^3+4\omega^5+\omega^7}{64} = \left(16\omega\left(4+\omega^2\right)+\omega^5\left(4+\omega^2\right)\right)\times\tfrac{1}{64},$$
$$= \omega\left(16+\omega^4\right)\left(4+\omega^2\right)\times\tfrac{1}{64}.$$
But $16+\omega^4=0$,
$$\therefore\ \omega+\frac{\omega^3}{4}+\frac{\omega^5}{16}+\frac{\omega^7}{64}=0.$$

Method 3:
$\omega^4 = -16$,

$$
\begin{aligned}
\omega + \frac{\omega^3}{4} + \frac{\omega^5}{16} + \frac{\omega^7}{64} &= \omega + \frac{\omega^3}{4} + \frac{-16\omega}{16} + \frac{-16\omega^3}{64}, \\
&= \omega - \omega + \frac{\omega^3}{4} - \frac{\omega^3}{4}, \\
&= 0.
\end{aligned}
$$

Method 4:
In the geometric series given, $a = \omega$ and $r = \dfrac{\omega^2}{4}$.

$$
\begin{aligned}
S_4 &= \frac{\omega\left(1 - \left(\frac{\omega^2}{4}\right)^4\right)}{1 - \frac{\omega^2}{4}}, \text{ note that } \frac{\omega^8}{256} = \frac{(-16)^2}{256} = 1, \\
&= \frac{\omega(1-1)}{1 - \frac{\omega^2}{4}}, \\
&= 0.
\end{aligned}
$$

(4)(a)(i) $\int_0^a x\sqrt{a-x}\,dx$

$$= \int_a^0 (a-u)\sqrt{u}\,(-du)$$
$$= \int_0^a (a-u)u^{1/2}\ \ du$$
$$= \int_0^a (au^{1/2} - u^{3/2})du$$
$$= \left[\frac{2a}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^a$$
$$= \left(\frac{2a^2}{3} - \frac{2a^2}{5}\right)\sqrt{a}$$
$$= \frac{4a^2}{15}\sqrt{a} = \frac{4a^{5/2}}{15}$$

$u = a - x \Rightarrow x = a - u; dx = -du$
$x = 0 \Rightarrow u = a$
$x = a \Rightarrow u = 0$

(ii) $\displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}}\,dx$

$$= \int_0^1 \left((1+x)^{-1/2} \times \sin^{-1} x\right) dx$$
$$= 2\sqrt{1+x}\sin^{-1} x\Big|_0^1 - \int_0^1 \frac{2\sqrt{1+x}}{\sqrt{1-x^2}}\,dx$$
$$= \sqrt{2}\pi - 2\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$$
$$= \sqrt{2}\pi + 2\int_0^1 -(1-x)^{-1/2}\,dx$$
$$= \sqrt{2}\pi + 2\times 2\sqrt{1-x}\Big|_0^1$$
$$= \sqrt{2}\pi - 4$$

(b)(i) $\displaystyle\int_0^1 \frac{\left(2dt/1+t^2\right)}{1+\left(1-t^2/1+t^2\right)+\left(2t/1+t^2\right)}$

$$= \int_0^1 \frac{2dt}{1+t^2+1-t^2+2t}$$
$$= \int_0^1 \frac{2dt}{2+2t}$$
$$= \int_0^1 \frac{dt}{1+t}$$
$$= \left[\ln|1+t|\right]_0^1$$
$$= \ln 2$$

$t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

4 (b) (ii)

$$I = \int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\sin x+\cos x}$$

$$= \int_{\frac{\pi}{2}}^{0} \frac{-\left(\frac{\pi}{2}-u\right)du}{1+\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)}$$

$$\boxed{\begin{array}{l} u=\frac{\pi}{2}-x \Rightarrow x=\frac{\pi}{2}-u;\ dx=-du \\ x=0 \Rightarrow u=\frac{\pi}{2};\ x=\frac{\pi}{2} \Rightarrow u=0 \end{array}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-u\right)du}{1+\cos u+\sin u}$$

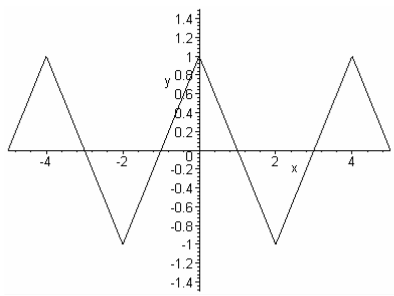
$$\therefore 2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}du}{1+\cos u+\sin u} = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{du}{1+\cos u+\sin u} = \frac{\pi}{2}\times \ln 2$$

$$\therefore I = \frac{\pi \ln 2}{4}$$

5 (a)

(i) $y = h(x+1)$

Move the curve 1 unit to the left



(ii) $y = \dfrac{1}{h(x)}$

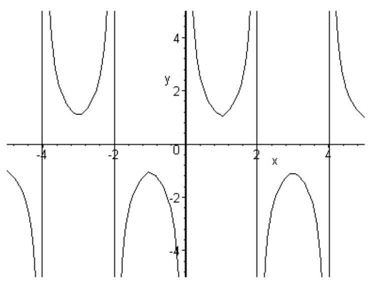
Where $h(x)=0$ there are vertical asymptotes.

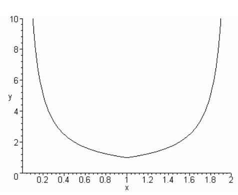
Where $h \to 0^+, y \to \infty$

Where $h \to 0^-, y \to -\infty$

Where h = 1, the reciprocal is *pointed* ie not smooth.

(See the bottom diagram on the right.)

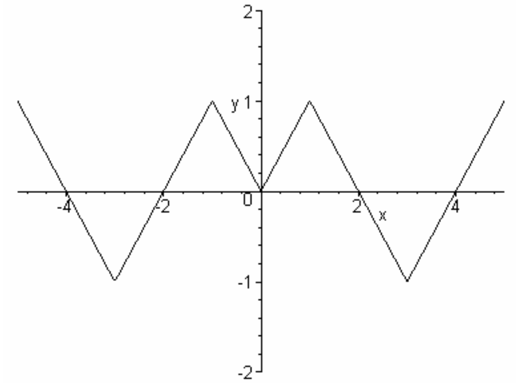




5 (a)

(iii) $y = h(|x|)$.

Erase the LHS of h and then reflect the RHS, so that the result is an even function.



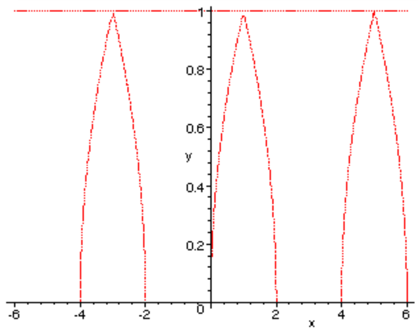
(iv) $y = \sqrt{h(x)}$

First erase the graph where $h < 0$.

Where $0 < h < 1 \Rightarrow \sqrt{h} > h$

Where $y = 1$, the graph is *pointed*, ie not smooth.

Where $y = 0$, vertical tangents.



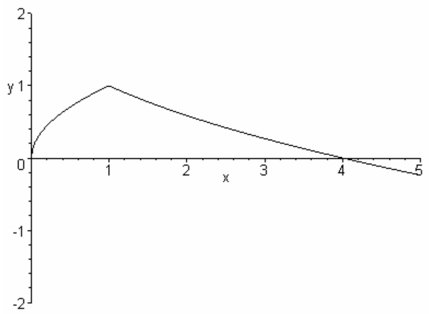
(v) $y = h(\sqrt{x})$

Domain: $x \geq 0$

Note that $0 \leq x \leq 4 \Rightarrow 0 \leq \sqrt{x} \leq 2$

So $h\left(\sqrt{4}\right) = h(2) = 0$

The graph for $0 \leq x \leq 4$ will be the same y values for h over $0 \leq x \leq 2$.



5 (b)

First draw $9y = x(x-3)^2$

Clearly x intercepts are at x = 0 and x = 3 **with** x = 3 is a double root.

$$y = x(x-3)^2/9 \Rightarrow z' = \tfrac{1}{9}\left(2x(x-3)+(x-3)^2\right) = \frac{(x-3)}{9}(2x+x-3)$$

$$\therefore y' = \tfrac{1}{9}(x-3)(3x-3) = \tfrac{1}{3}(x-1)(x-3) = \tfrac{1}{3}(x^2-4x+3)$$

$$\therefore y'' = \tfrac{1}{3}(2x-4)$$

Stationary points when $y'=0 \Rightarrow x=1,3$ ie $\left(1,\tfrac{4}{9}\right) \& (3,0)$

At x = 1, $y''<0 \Rightarrow$ $\left(1,\tfrac{4}{9}\right)$ is a maximum.

The graph in Fig I is the graph of z. The horizontal line is the line y = 1.

So with $y=\tfrac{1}{3}\sqrt{x(x-3)^2}$, the maximum turning point remains the same except it is now $\left(1,\tfrac{2}{3}\right)$.

Any part of the graph in Fig I below the x – axis is not defined for the square root. Where $0<y<1$ we get $\sqrt{y}>y$ and where $y>1$ we get $\sqrt{y}<y$.

The x = 0 intercept will have a vertical tangent, the x = 3 intercept is not smooth. This is shown in Fig 2.

We need to draw $y=\pm\tfrac{1}{3}\sqrt{x(x-3)^2}$: the $\pm$ means that the top part of the graph will be reflected.

The final graph is Fig 3. With turning points $\left(1,\pm\tfrac{2}{3}\right)$

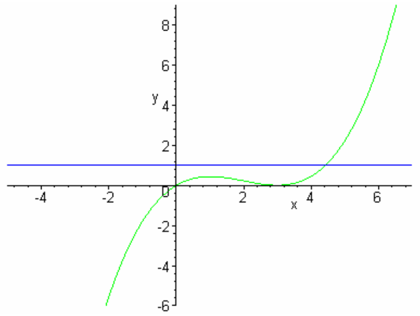


Fig I

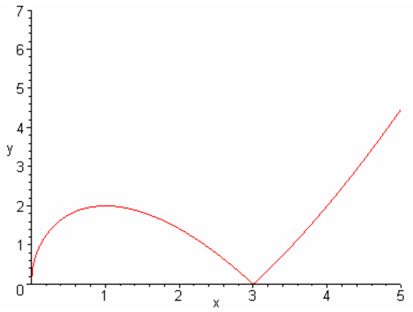


Fig II

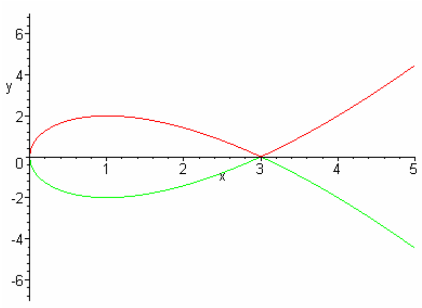
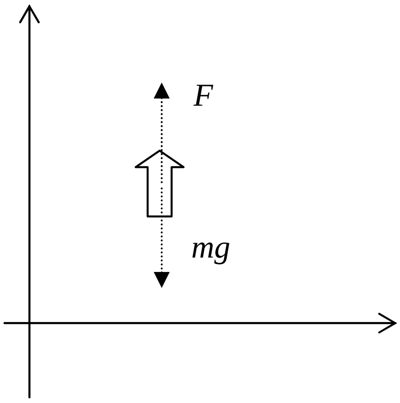


Fig III

Question 6

(i)

$0 \le t \le 2$



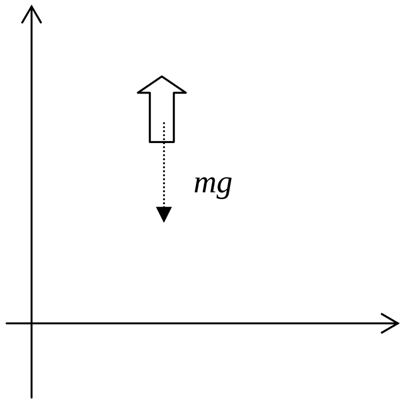
$F = g(2-t)$

$\therefore ma = F - mg = g(2-t) - mg$

$\therefore a = \dfrac{g(2-t)}{0 \cdot 2} - g$

$\therefore a = 5g(2-t) - g = g(9-5t)$

$t > 2$



After 2 seconds the only force acting is gravity

$\therefore a = -g$

6(ii) (α) For $0 \le t \le 2$

$$\frac{dv}{dt} = g(9-5t)$$

$$v = 9gt - \frac{5}{2}gt^2 + c_1$$

$$\left.\begin{matrix} t=0 \\ v=0 \\ c_1=0 \end{matrix}\right\} \therefore v = gt\left[9 - \frac{5}{2}t\right] \quad \& \quad x = \frac{9gt^2}{2} - \frac{5}{6}gt^3 + c_3 \quad (*)$$

Max speed when $a = g(9-5t) = 0$

ie when $t = 9/5$

$\therefore$ Max speed $v = \frac{81g}{10}$ m/s

(β) When $t=2$, $v = 8g$

For $t \ge 2$, $a = -g$

$$\frac{dv}{dt} = -g$$

$$v = -gt + c_2$$

$c_2 = 10g$ ie $v = 10g - gt$ — (**)

For max height $v=0 \Rightarrow \underline{t=10}$

From (*) when $t=2$, $x = \frac{34g}{3}$

From (**) $x = 10gt - \frac{gt^2}{2} + c_4$

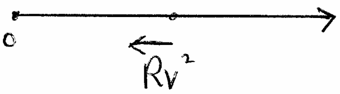
When $t=2$, $x = \frac{34g}{3} \Rightarrow c_4 = -\frac{20g}{3}$

$$\therefore x = -\frac{gt^2}{2} + 10gt - \frac{20g}{3}$$

When $t=10$, $x = \frac{130g}{3}$ m

Question 7

(i)



$m\ddot{x} = -Rv^2$ 1

since $m = 1 \Rightarrow \ddot{x} = -Rv^2$

(ii) $\dfrac{dv}{dt} = -Rv^2$

$\displaystyle\int_u^V \frac{dv}{v^2} = -R\int_0^t dt$

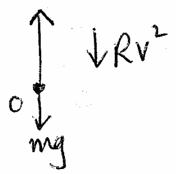
$\left[-\dfrac{1}{v}\right]_u^V = -Rt$ 3

$-\dfrac{1}{V} + \dfrac{1}{u} = -Rt$

$t = \dfrac{1}{R}\left(\dfrac{1}{V} - \dfrac{1}{u}\right)$

(iii)

↑+ve



1

$m\ddot{y} = -mg - Rv^2$

$m = 1 \Rightarrow \ddot{y} = -(g + Rv^2)$

(iv) $\ddot{y} = -(g + Rv^2)$

Since $g = Ra^2$

$\Rightarrow \ddot{y} = -(Ra^2 + Rv^2)$

i.e. $\ddot{y} = -R(a^2 + v^2)$

$\dfrac{dv}{dt} = -R(a^2 + v^2)$

$\displaystyle\int_u^0 \frac{dv}{a^2+v^2} = -R\int_0^t dt$

$\left[\dfrac{1}{a}\tan^{-1}\dfrac{v}{a}\right]_u^0 = -Rt$

$0 - \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} = -Rt$

$\Rightarrow t = \dfrac{1}{Ra}\tan^{-1}\dfrac{u}{a}$ $3\frac{1}{2}$

At this time 2nd particle is at rest.

Subst. into (ii) with $v = V$ $1\frac{1}{2}$

$\Rightarrow \dfrac{1}{R}\left[\dfrac{1}{V} - \dfrac{1}{u}\right] = \dfrac{1}{Ra}\tan^{-1}\dfrac{u}{a}$ 4

$\therefore \dfrac{1}{V} = \dfrac{1}{u} + \dfrac{1}{a}\tan^{-1}\dfrac{u}{a}$

where V = vel. of 1st particle when the second is momentarily at rest