SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All *necessary* working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section **A** (Questions 1 - 3),
  Section **B** (Questions 4 - 6) and
  Section **C** (Questions 7 - 8).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 90 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 90
Attempt Questions 1 - 8
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (8 marks)	**Marks**

(a) Given $z = \dfrac{3-i}{2+i}$ find

(i) $z\bar{z}$ — 2

(ii) $\tan(\arg z)$ — 1

(b) (i) If $z = 1 + \cos\theta + i\sin\theta$, for $0 < \theta < \pi$, then show that — 3

$$z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

(ii) Hence, find $|z|$ and $\arg z$ in terms of θ — 2

Question 2 (9 marks)

(a) If α, β and γ are the roots of $x^3 - 2x + 5 = 0$, find the cubic polynomial that has roots

(i) $2-\alpha, 2-\beta$ and $2-\gamma$. — 2

(ii) $\alpha^2+\beta^2, \beta^2+\gamma^2$ and $\gamma^2+\alpha^2$ — 2

(b) Given that p and q are real and also that $1-4i$ is a root of the equation

$$x^2 + (p+i)x + (q-5i) = 0$$

(i) Find the values of p and q. — 3

(ii) Find the other root of the equation. — 2

Question 3 (13 marks) **Marks**

(a) Let $\omega = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$

(i) Express ω^2 and $\frac{1}{\omega}$ in modulus-argument form. 2

(ii) Given that $\omega^2 + \frac{5}{\omega} - 2$ is a purely imaginary number, show that $2\cos^2\theta + 5\cos\theta - 3 = 0$. 3

(iii) Hence, or otherwise, evaluate θ and express ω in modulus-argument form. 2

(b) A and B are two distinct points in an Argand diagram representing two distinct, non-zero numbers z_1 and z_2 respectively.
Suppose that $z_2 = \omega z_1$, where ω is the number found in (a) above.

(i) Find $\left|\frac{z_2}{z_1}\right|$ and $\arg\left(\frac{z_2}{z_1}\right)$ 3

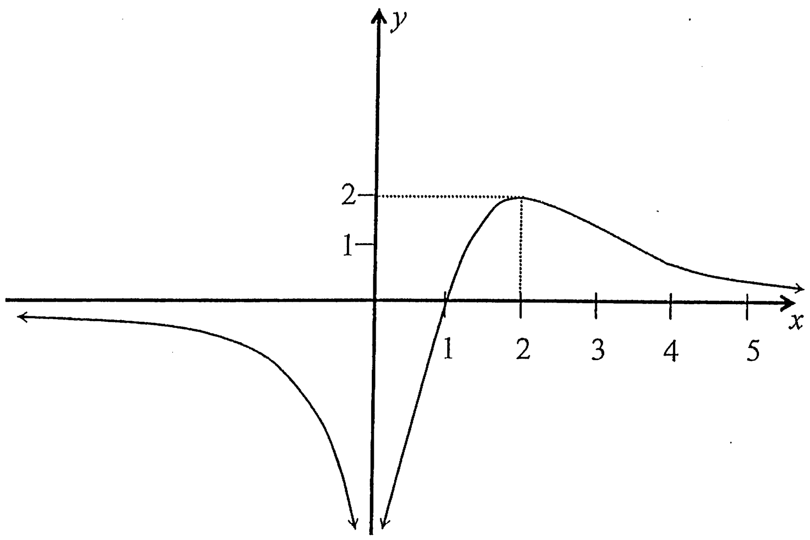
(ii) Show that ΔAOB is an equilateral triangle, where O represents the number 0 in the Argand diagram. 3

End of Section A

SECTION B (Use a SEPARATE writing booklet)

Question 4 (12 marks) **Marks**

The sketch of $y = f(x)$ is shown below



You are given that the x – intercept is $x = 1$;
$x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote.
As well, $(2,2)$ is a stationary point and $(4,1)$ is a point of inflexion.

On the Answer Sheet provided draw separate graphs for

(i) $y^2 = f(x)$ **2**

(ii) $y = \dfrac{1}{[f(x)]^2}$ **3**

(iii) $y = f'(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(v) Draw a possible diagram for the primitive function of $f(x)$ **3**

Question 5 (10 marks) **Marks**

(a) Given that C is the curve defined by

$$f(x)=\frac{\sin 2x}{2-\cos 2x}$$

where $0\le x\le\pi$

(i) Find the x and y intercepts of C. 2

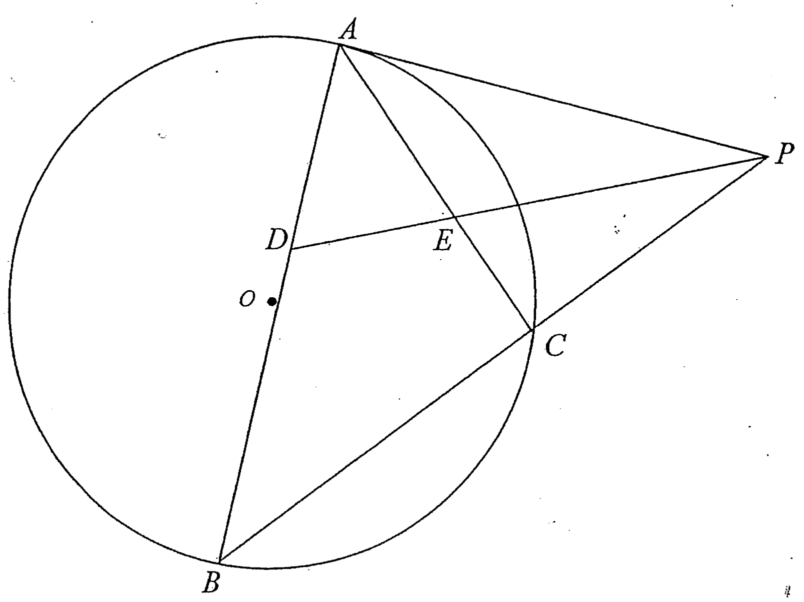
(ii) Find the turning point(s) of C and determine their nature. 3

(iii) Show that $f(x)=-f(\pi-x)$. 2

(iv) Sketch the curve $y=\dfrac{|\sin 2x|}{2-\cos 2x}$ for $0\le x\le\pi$ 3

Question 6 (8 marks)

(a) In the diagram below, PA is the tangent to the circle at A, whose centre is O.
Line PCB cuts the circle at B and C.
The angle bisector of $\angle APB$ meets AB at D and AC at E.



Prove that $\dfrac{DB}{AB}+\dfrac{EC}{AC}=1$ 6

(b) How many ways are there to pick a man and a woman who are not husband and wife from a group of n married couples? 2

End of Section B

SECTION C (Use a SEPARATE writing booklet)

Question 7 (21 marks) **Marks**

(a) (i) Find $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **2**

(ii) Find $\int \frac{3x^2-6x+1}{(x-3)(x^2+1)}\,dx$ **2**

(iii) Find $\int \frac{1}{1-\cos x}\,dx$ **2**

(iv) Find $\int x^6 \ln x\,dx$ **2**

(b) (i) Show that $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$ **3**

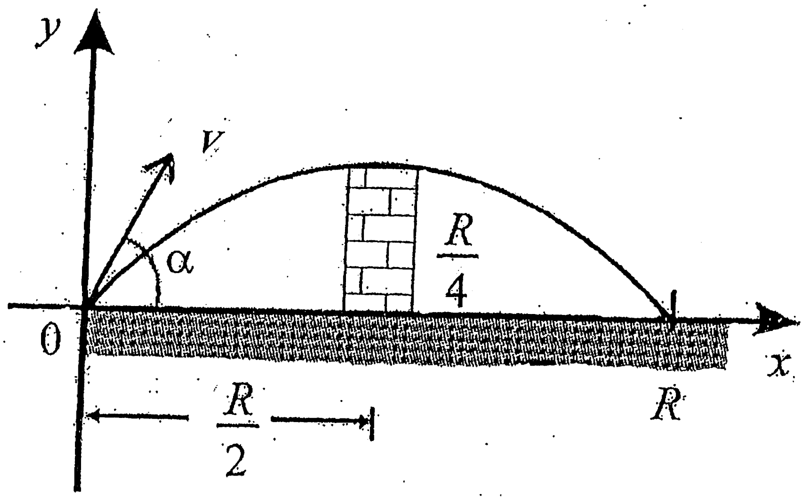
(ii) Hence show that $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{e^x \sin^2 3x}{1+e^x}\,dx = \frac{\pi}{6}$ **3**

(c) (i) Let α and β be the roots of the equation $x^2 - ax + b = 0$. **2**
Find the monic quadratic equation that has roots α^2 and β^2.

(ii) Hence, show that the equation whose roots are the eighth powers of the roots of the equation $x^2 - x - 1 = 0$ is $x^2 - 47x + 1 = 0$. **3**

(iii) Using the results of (ii) above show that $\left(\frac{47+21\sqrt{5}}{2}\right)^{\frac{1}{8}} = \frac{1+\sqrt{5}}{2}$ **2**

Question 8 (9 marks)



The diagram above shows a particle projected from a point O on horizontal ground with speed V m/s and at an angle of elevation, $\alpha°$, where $0° < \alpha° < 90°$.

(i) Show that the *maximum* horizontal range of the particle is R where $R = \frac{V^2}{g}$. **4**

You may assume that $\ddot{x} = 0$ and $\ddot{y} = -g$, where g is the acceleration due to gravity.
x and y are the respective horizontal and vertical distances travelled by the particle.

(ii) Show that the equation of the trajectory of the particle can be written as $y = x\tan\alpha° - \frac{x^2\left(1+\tan^2\alpha°\right)}{2R}$. **1**

You may **assume** that the formula for the trajectory is given by $y = x\tan\alpha° - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha°\right)$.

(iii) The particle just clears a wall, which has horizontal and vertical distances from O are $\frac{R}{2}$ and $\frac{R}{4}$ respectively.

(α) Find two possible value of $\alpha°$. **2**

(β) Find the horizontal range of the particle, in terms of R, for each possible value of $\alpha°$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right) x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

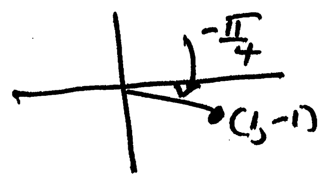
QUESTION 1

(a) $z = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)}$

$= \frac{6-5i-1}{5}$

$= \frac{5-5i}{5}$

$= 1-i$



(i) $\therefore z\bar{z} = |z|^2 = (\sqrt{2})^2$

$= \boxed{2}$ ✓

(ii) $\tan(\arg z) = \tan(-\frac{\pi}{4})$

$= -\tan\frac{\pi}{4}$

$= \boxed{-1}$ ✓

(b) (i) $z = 1+\cos\theta + i\sin\theta$

$= 1 + (2\cos^2\frac{\theta}{2} - 1) + i \cdot 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}$.

$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}$.

$= \boxed{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$ ✓✓✓

(ii) $\therefore |z| = \boxed{2\cos\frac{\theta}{2}}$ ✓

$\arg z = \boxed{\frac{\theta}{2}}$ ✓

(in the form $r(\cos\phi + i\sin\phi)$ where $r = |z| = 2\cos\frac{\theta}{2}$ & $\phi = \frac{\theta}{2}$).

QUESTION 2.

(a)(i) Let $X = 2 - x$ ie $x = 2 - X$

$\therefore x^3 - 2x + 5 = 0$ becomes $(2-X)^3 - 2(2-X) + 5 = 0.$

ie. $8 - 12X + 6X^2 - X^3 - 4 + 2X + 5 = 0$

ie. $\boxed{X^3 - 6X^2 + 10X - 9 = 0}$

(ii) Using the identity $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0^2 - 2 \times -2$

$= 4.$

$\alpha^2 + \beta^2, \beta^2 + \gamma^2$ and $\gamma^2 + \alpha^2 \Rightarrow 4 - \gamma^2, 4 - \alpha^2, 4 - \beta^2$

Using $X = 4 - x^2 \Rightarrow x^2 = 4 - X \Rightarrow x = \sqrt{4 - X}$

$\therefore x^3 - 2x + 5 = 0$ becomes

$(\sqrt{4-X})^3 - 2(\sqrt{4-X}) + 5 = 0$

$(4-X)\sqrt{4-X} - 2\sqrt{4-x} + 5 = 0$

$\sqrt{4-X}(4 - X - 2) = -5$

$\sqrt{4-X}(2-X) = -5$

squaring

$(4-X)(4 - 4X + X^2) = 25.$

$16 - 16X + 4X^2 - 4X + 4X^2 - X^3 = 25$

Simplifying becomes

$\boxed{X^3 - 8X^2 + 20X + 9 = 0.}$

(b) (i) $x^2 + (p+i)x + q - 5i = 0$

$1 - 4i$ is a root.

$\therefore (1-4i)^2 + (p+i)(1-4i) + q - 5i = 0.$

$-15 - 8i + p - 4pi + i + 4 + q - 5i = 0$

$-15 + p + 4 + q = 0$ $\quad p + q = 11$

$-8 - 4p + 1 - 5 = 0$

$4p = -12$

$p = -3$ $\quad q = 14.$

(ii) $\therefore x^2 + (-3+i)x + 14 - 5i = 0$

is the equation.

(CAN'T USE THE CONJUGATE ROOT THEOREM. as co-efficients are not Real.)

$\sum \alpha = -(-3+i) = 3 - i$

$\therefore 1 - 4i + z = 3 - i$

$z = 2 + 3i$

QUESTION 3.

(a)(i) $w = \cos\theta + i\sin\theta.$

$\therefore$ by De Moivre's theorem.

$w^2 = \cos 2\theta + i\sin 2\theta.$

$\& \ \frac{1}{w} = w^{-1} = \cos(-\theta) + i\sin(-\theta)$

OR $\cos\theta - i\sin\theta.$

(ii) $w^2 + 5w^{-1} - 2 = \cos 2\theta + i\sin 2\theta + 5(\cos\theta - i\sin\theta) - 2.$

$= 2\cos^2\theta - 1 + 5\cos\theta - 2 + i(\sin 2\theta - 5\sin\theta)$

If this is purely imaginary the Real part is zero.

ie. $2\cos^2\theta + 5\cos\theta - 3 = 0.$ (A)

(iii) Solving (A) $\cos\theta = \frac{-5 \pm \sqrt{25+24}}{4}$ OR.

$= -3, \frac{1}{2}.$

$(2\cos\theta - 1)(\cos\theta + 3) = 0$

$\cos\theta = \frac{1}{2}, -3.$

$\therefore \cos\theta = \frac{1}{2} \quad (\cos\theta \neq -3)$

$\theta = \frac{\pi}{3}.$

$\therefore \ w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

(B)

(b) (i) now $z_2 = w z_1$

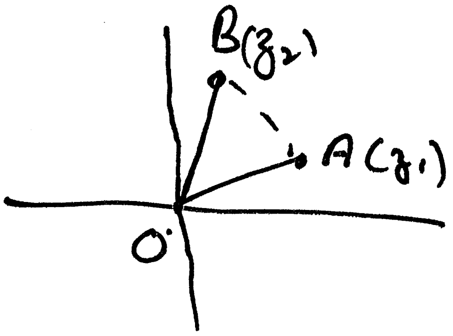
$$\therefore \quad \frac{z_2}{z_1} = w.$$

now $\left|\frac{z_2}{z_1}\right| = |w| = \left|\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right|$

$$\therefore \left|\frac{z_2}{z_1}\right| = 1. \quad (C)$$

$$\arg\frac{z_2}{z_1} = \arg w = \frac{\pi}{3}.$$ (from a (iii)).
ie. (B).

(ii)



now

$$\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} = 1 \quad \text{(from (C))}$$

$$\therefore |z_2| = |z_1|$$

$\therefore \Delta AOB$ is isosceles

Furthermore $\arg\frac{z_2}{z_1} = \arg z_2 - \arg z_1$

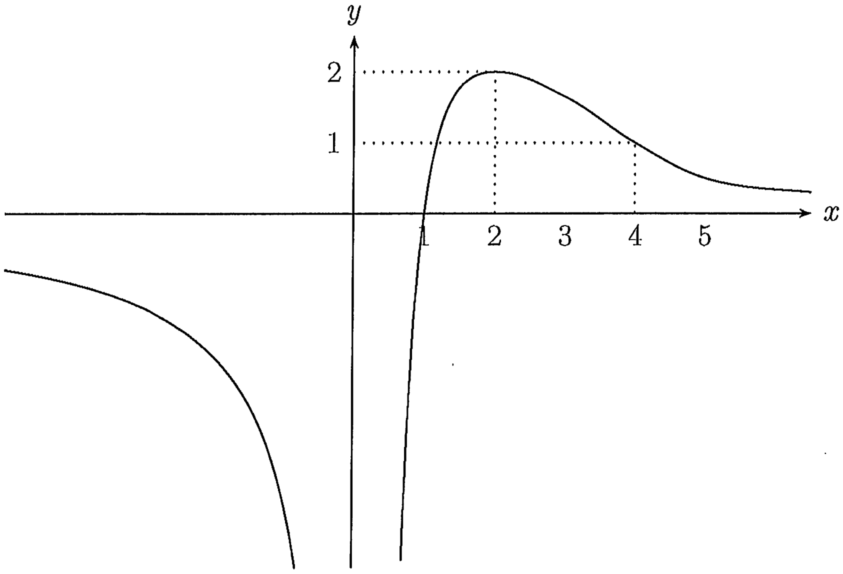
$$= \angle AOB$$

$$= \frac{\pi}{3}.$$

($\therefore \Delta AOB$ is equilateral as it is isosceles and contains a 60° angle!).

2006 Mathematics Extension 2 Assessment 2: **Solutions Part B**

4. (a) The sketch of $y = f(x)$ is shown below.

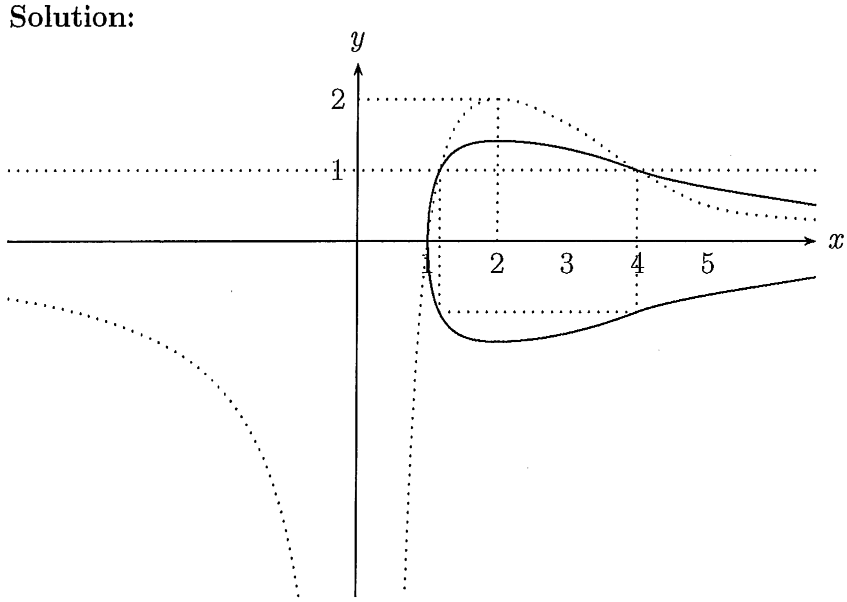


You are given the x-intercept is $x = 1$,
$x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote.
As well, (2, 2) is a stationary point and (4, 1) is a point of inflexion.

On the answer sheet provided draw separate graphs for

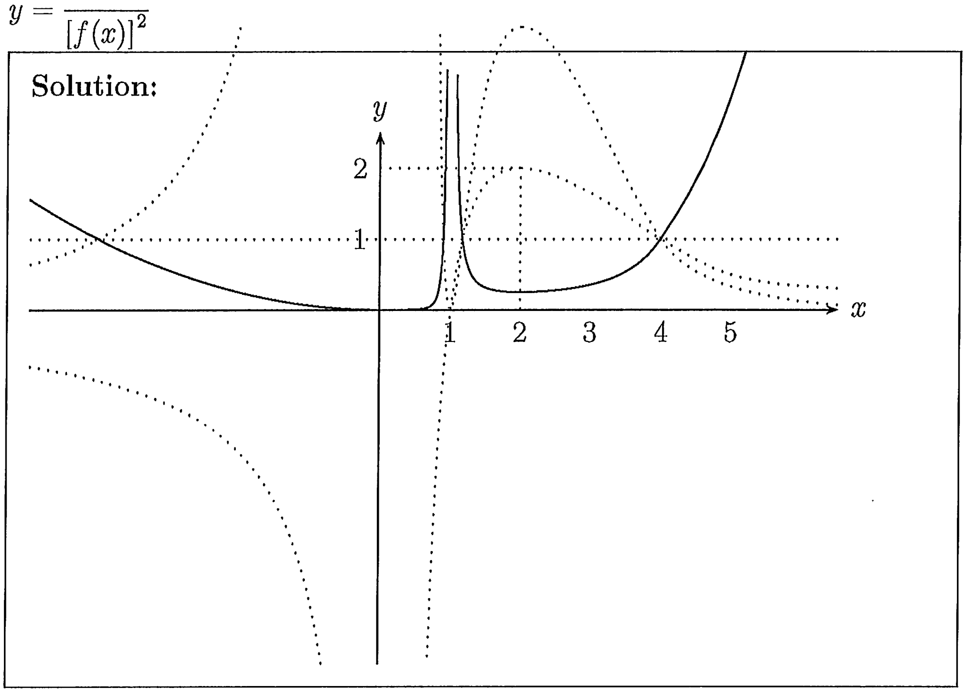
(i) $y^2 = f(x)$ **2**

Solution:



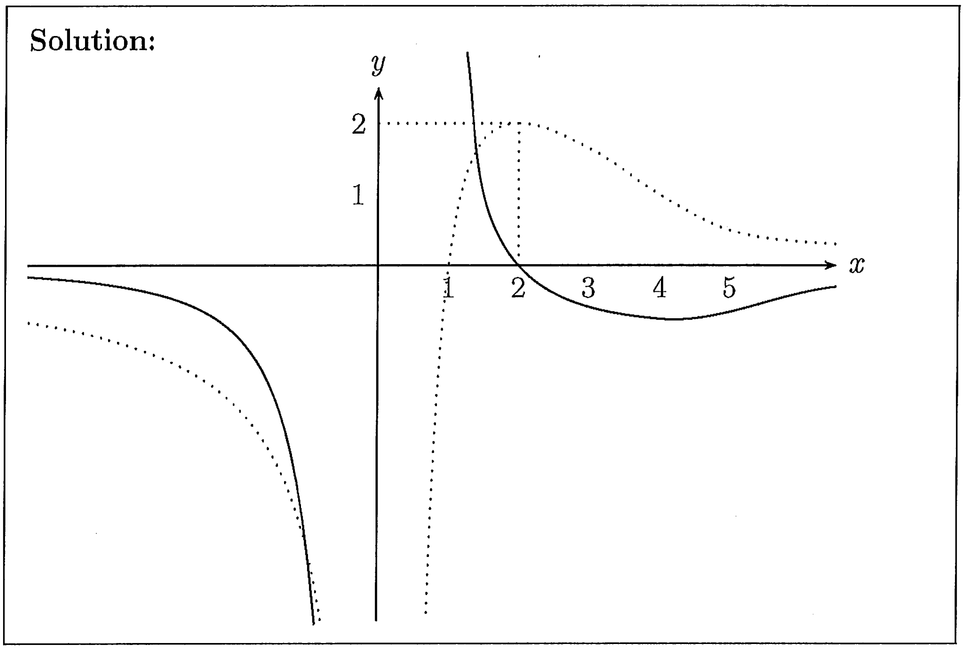
(ii) $y = \dfrac{1}{[f(x)]^2}$ 3

Solution:



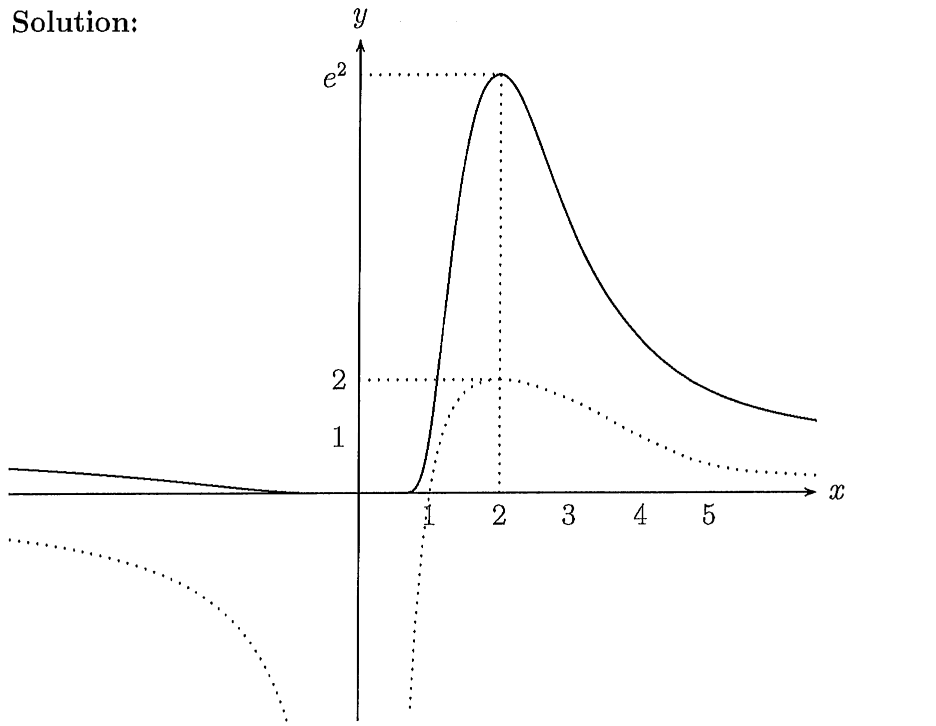
(iii) $y = f'(x)$ 2

Solution:



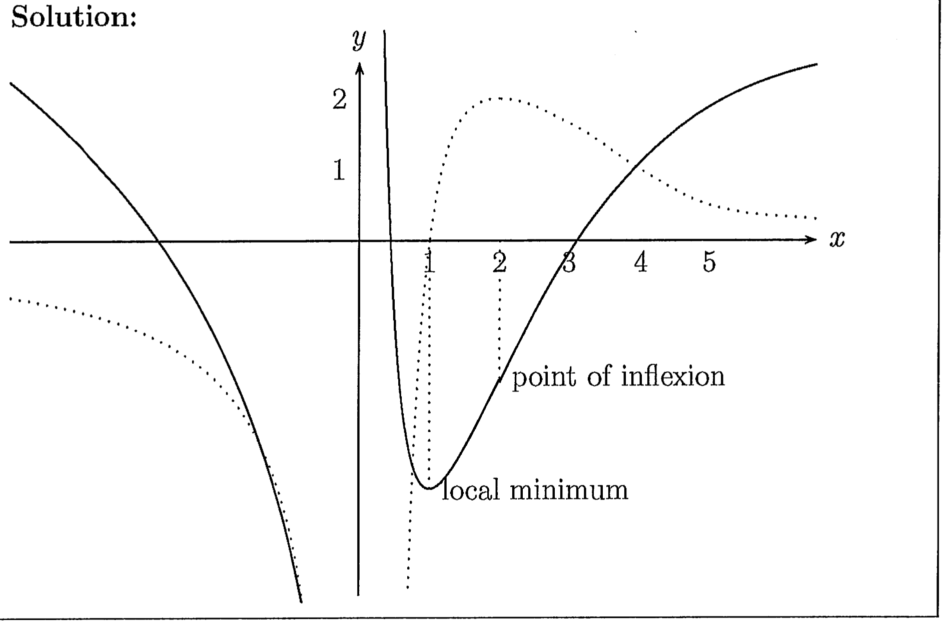
(iv) $y = e^{f(x)}$ 2

Solution:



(v) Draw a possible diagram for the primitive function of $f(x)$. 3

Solution:



5. (a) Given that C is the curve defined by

$$f(x) = \frac{\sin 2x}{2 - \cos 2x}$$

where $0 \le x \le \pi$:

(i) Find the x and y intercepts of C. [2]

Solution: We note that $2 - \cos 2x > 0 \ \forall \ x$
and $\sin 2x = 0$ when $2x = 0, \ \pi, \ 2\pi,$
i.e., $x = 0, \ \frac{\pi}{2}, \ \pi.$
So x-intercepts are at $0, \ \frac{\pi}{2}, \ \pi$
and the y-intercept is 0.

(ii) Find the turning point(s) of C and determine their nature. [3]

Solution:
$$\begin{aligned} f'(x) &= \frac{(2 - 2\cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(2-\cos 2x)^2} \\ &= \frac{4\cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(2-\cos 2x)^2} \\ &= \frac{2(2\cos 2x - 1)}{(2-\cos 2x)^2} \\ &= 0 \text{ when } \cos 2x = \tfrac{1}{2} \\ &\qquad \text{i.e., } 2x = \tfrac{\pi}{3}, \tfrac{5\pi}{3} \\ &\qquad\qquad x = \tfrac{\pi}{6}, \tfrac{5\pi}{6} \end{aligned}$$

Now
$$\begin{aligned} f'(0) &= \frac{2}{1} \\ f'\left(\frac{\pi}{2}\right) &= \frac{-6}{9} = -\frac{2}{3} \\ f'(\pi) &= \frac{2}{1} \end{aligned}$$

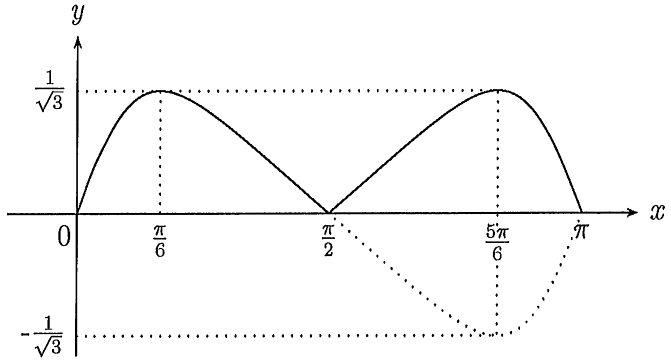
$\therefore$ Maximum turning point at $\left(\frac{\pi}{6}, \frac{1}{\sqrt{3}}\right)$
and minimum turning point at $\left(\frac{5\pi}{6}, -\frac{1}{\sqrt{3}}\right)$.

(iii) Show that $f(x) = -f(\pi - x)$. [2]

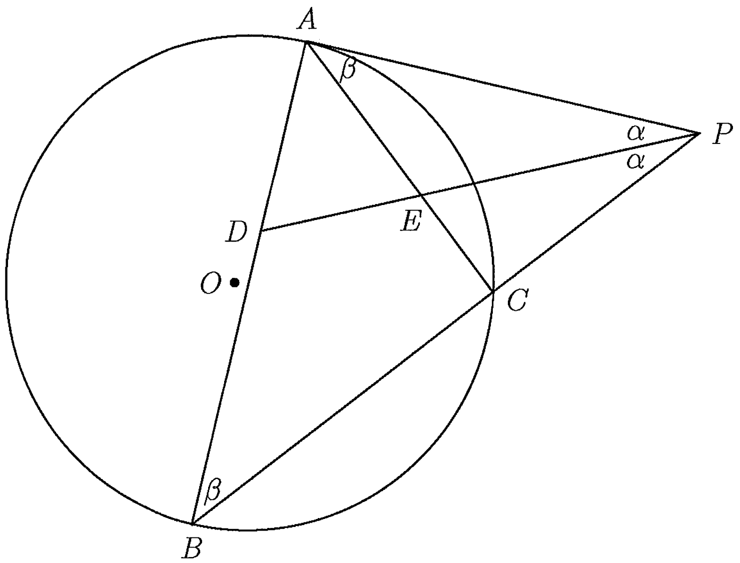
Solution:
$$\begin{aligned} \text{R.H.S.} &= \frac{-\sin(2[\pi - x])}{2 - \cos(2[\pi - x])}, \\ &= \frac{-\sin(2\pi - 2x)}{2 - \cos(2\pi - 2x)}, \\ &= \frac{-\sin(-2x)}{2 - \cos(-2x)}, \\ &= \frac{\sin 2x}{2 - \cos 2x}, \\ &= \text{L.H.S.} \end{aligned}$$

(iv) Sketch the curve $y = \dfrac{|\sin 2x|}{2 - \cos 2x}$ for $0 \leq x \leq \pi$. [3]

Solution: From parts (i) and (ii) we have



6. (a) In the diagram below, PA is the tangent to the circle at A, whose centre is O. [6]
Line PCB cuts the circle at B and C.
The angle bisector of $\angle APB$ meets AB at D and AC at E.



Prove that $\dfrac{DB}{AB} + \dfrac{EC}{AC} = 1$.

Solution:
In $\triangle$s BAP, ACP,

$$\angle APC = \angle BPA \quad \text{(common)}$$
$$\angle PAC = \angle PBA \quad (\angle \text{ in alternate segment})$$

$\therefore\ \triangle BAP \,|||\, \triangle ACP$ (equiangular).

$\dfrac{BP}{AP} = \dfrac{BA}{AC}$ (corresp. sides of similar $\triangle$s) .... [1]

[Solution continues on next page.]

Solution: [Continued from last page.]
Also, in $\triangle$s DBP, EAP,

$$\angle BPD = \angle APE \quad (PE \text{ bisects } \angle APB$$
$$\angle PAC = \angle PBA \quad (\text{ as above})$$

$\therefore\ \triangle DBP \,|||\, \triangle EAP$ (equiangular).

$\dfrac{BP}{AP} = \dfrac{DB}{EA}$ (corresp. sides of similar $\triangle$s) .... [2]

So, from [1] and [2] we have:

$$\frac{AC}{AB} = \frac{AE}{DB},$$
$$\frac{DB}{AB} = \frac{AE}{AC},$$
$$\frac{DB}{AB} = \frac{AC - EC}{AC},$$
$$\frac{DB}{AB} = 1 - \frac{EC}{AC},$$

i.e., $\dfrac{DB}{AB} + \dfrac{EC}{AC} = 1.$

(b) How many ways are there to pick a man and a woman who are not husband and wife from a group of n married couples? [2]

Solution: Method 1—
Ways of choosing the man, $\binom{n}{1} = {}^{n}\mathrm{C}_1 = n$.
Ways of choosing a woman who is not his wife $= n - 1$.
$\therefore$ Total number of ways $= n(n-1)$.

Solution: Method 2—
Ways of choosing a couple without restriction $= n^2$.
Ways of choosing a married couple $= n$.
$\therefore$ Total number of ways $= n^2 - n$.

Question 7

(a) (i) $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ let $u = e^x$ ½

$du = e^x dx$

$\therefore \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$

$= \sin^{-1}(e^x) + c$ 1

(ii) $I = \int \frac{3x^2-6x+1}{x^3-3x^2+x-3} dx = \log_e (x^3-3x^2+x-3) + c$ 2

OR

$3x^2 - 6x + 1 = A(x^2+1) + (Bx+c)(x-3)$

$\Rightarrow A+B=3$; $C-3B=-6$; $A-3C=1$

$\therefore A=1 \quad B=2 \quad C=0$ 1

$\Rightarrow I = \int \left(\frac{1}{x-3} + \frac{2x}{x^2+1}\right) dx$

$= \ln(x-3) + \ln(x^2+1)$

$= \ln(x-3)(x^2+1)$ 1

$= \ln(x^3-3x^2+x-3) + c$

(iii) $\int \frac{1}{1-\cos x} dx$ by "t results"

$\Rightarrow t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2} dx$

$dt = \frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]dx$

$\frac{2dt}{1+t^2} = dx$

$= \int \frac{\frac{2}{1+t^2}\, dt}{1 - \frac{1-t^2}{1+t^2}}$

$= \int \frac{2}{2t^2} dt$

$= \int t^{-2} dt$

$= -\frac{1}{t}$

$= -\frac{1}{\tan\frac{x}{2}}$ or

$= -\cot\frac{x}{2} + c$

(iv) $\int x^6 \ln x \, dx$

$= \int \ln x \cdot \frac{d}{dx}\left(\frac{x^7}{7}\right) dx$

$= \frac{x^7}{7}\ln x - \int \frac{x^7}{7}\cdot\frac{1}{x} dx$

$= \frac{x^7}{7}\ln x - \frac{1}{7}\int x^6$

$= \frac{x^7}{7}\ln x - \frac{1}{49}x^7 + c$

(b)

Q⑦ (b) (i)

$$LHS = \int_{-a}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

(let $x = -u$ etc.)

$$= \int_{a}^{0} f(-u).-du + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]\,dx$$

$= RHS$

(ii)

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{e^x \sin^2 3x}{1+e^x}\,dx = \int_{0}^{\pi/3} \left(\frac{e^x \sin^2 3x}{1+e^x} + \frac{e^{-x}\sin^2(-3x)}{1+e^{-x}} \right) dx$$

$$= \int_{0}^{\pi/3} \left(\frac{e^x \sin^2 3x}{1+e^x} + \frac{\sin^2 3x}{1+e^x} \right) dx$$

NOTE: $\sin^2(-3x) = [\sin(-3x)]^2$
$= [-\sin 3x]^2$
$= \sin^2 3x$

$$= \int_{0}^{\pi/3} \frac{\sin^2 3x.(e^x+1)}{(e^x+1)}\,dx$$

$\left(1 + e^{-x} = 1 + \frac{1}{e^x}\right)$

$$= \int_{0}^{\pi/3} \sin^2 3x\,dx \quad 1\tfrac{1}{2}$$

$$= \int_{0}^{\pi/3} \tfrac{1}{2}(1-\cos 6x)\,dx \quad \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[x - \tfrac{1}{6}\sin 6x\right]_0^{\pi/3} \quad \tfrac{1}{2}$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} - 0\right) - (0-0)\right] = \frac{\pi}{6} \quad \tfrac{1}{2}$$

(c) (i) $x^2 - ax + b = 0$

$\alpha + \beta = -\frac{(-a)}{1} = a$ $\frac{1}{2}$

$\alpha\beta = b$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$\alpha^2 + \beta^2 = a^2 - 2b$ $\frac{1}{2}$

Required eq^n $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

ie $x^2 - [a^2 - 2b]x + b^2 = 0$ 1

(ii) $x^2 - x - 1 = 0 \Rightarrow a = 1 \quad b = -1$ from (i)

$\therefore$ Eq^n with roots $\alpha^2, \beta^2 \Rightarrow x^2 - [1 - 2(-1)]x + (-1)^2 = 0$

ie $x^2 - 3x + 1 = 0$ 1

Eq^n with roots $\alpha^4, \beta^4 \Rightarrow x^2 - [(-3)^2 - 2(1)]x + 1 = 0$ 1

ie $x^2 - 7x + 1 = 0$

Eq^n with roots $\alpha^8, \beta^8 \Rightarrow x^2 - [(7)^2 - 2]x + 1 = 0$ 1

ie $x^2 - 47x + 1 = 0$

(ii) $x^2 - 47x + 1 = 0$

$x = \frac{47 \pm \sqrt{2205}}{2}$

$x = \frac{47 \pm \sqrt{21}}{2}$ $\frac{1}{2}$

$\therefore x = \frac{47 + \sqrt{21}}{2}$ applies

From $x^2 - x - 1 = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$ $\frac{1}{2}$

$\therefore x = \frac{1+\sqrt{5}}{2} > 0$ applies

1

SINCE $\frac{47 + 21\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^8 \Rightarrow \frac{1+\sqrt{5}}{2} = \left(\frac{47+21\sqrt{5}}{2}\right)^{\frac{1}{8}}$

Question 8

(i) $t=0$

$\ddot{x}=0 \quad \ddot{y}=-g \quad x=0 \quad y=0 \quad \dot{x}=V\cos\alpha \quad \dot{y}=V\sin\theta$

$\dot{x}=C_1 \quad \dot{y}=-gt+C_2$

$\boxed{\dot{x}=V\cos\alpha}$ $\quad C_2=V\sin\alpha$

$\boxed{\dot{y}=V\sin\alpha-gt}$

$x=Vt\cos\alpha+C_3$

$C_3=0 \quad y=Vt\sin\alpha-\frac{gt^2}{2}+C_4$

$\boxed{x=Vt\cos\alpha}$ (1) $\quad C_4=0$

$\boxed{y=Vt\sin\alpha-\frac{gt^2}{2}}$

For max range let $y=0$

$\Rightarrow 0=t\left(V\sin\theta-\frac{gt}{2}\right)$

$\therefore t=0$ or $t=\frac{2V\sin\alpha}{g}$ ✓ (1)

Range $=Vt\cos\alpha$ ✓

$=V.\frac{2V\sin\alpha}{g}.\cos\alpha$

Range $=\frac{V^2}{g}\sin 2\alpha$ ✓ (1)

Max Range when $\alpha=\frac{\pi}{4}$

ie $\boxed{R=\frac{V^2}{g}}$ (1)

(ii)

$y=x\tan\alpha-\frac{gx^2}{2V^2}(1+\tan^2\alpha)$

Since $R=\frac{V^2}{g}$

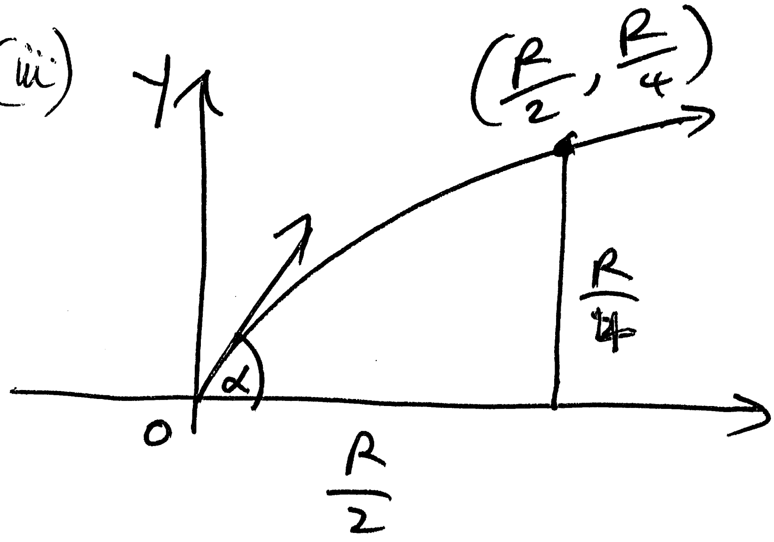
$\Rightarrow V^2=Rg$ (1)

$\therefore$

$y=x\tan\alpha-\frac{gx^2}{2Rg}(1+\tan^2\alpha)$

ie $\boxed{y=x\tan\alpha-\frac{x^2}{2R}(1+\tan^2\alpha)}$ (*)

(iii)



(α) Using ⊛ $\Rightarrow$

$$\frac{R}{4} = \frac{R}{2}\tan\alpha - \frac{\frac{R^2}{4}}{2R}(1+\tan^2\alpha)$$

$$\frac{R}{4} = \frac{R}{2}\tan\alpha - \frac{R}{8}(1+\tan^2\alpha)$$

$$2 = 4\tan\alpha - (1+\tan^2\alpha)$$

ie $\tan^2\alpha - 4\tan\alpha + 3 = 0$

$$(\tan\alpha - 3)(\tan\alpha + 1) = 0$$

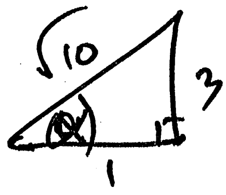
$\tan\alpha = 3 \Rightarrow$ $\boxed{\alpha \doteqdot 71^\circ 34'}$ (1)

$\tan\alpha = 1 \Rightarrow$ $\boxed{\alpha = \frac{\pi}{4}}$ (1)

(β) Range $= \frac{V^2}{g}\sin 2\alpha$

When $\theta = \frac{\pi}{4} \Rightarrow$ $\boxed{\text{Range} = \frac{V^2}{g}\sin\frac{\pi}{2} = \frac{V^2}{g}}$ (1)

When $\theta = \tan^{-1} 3 \Rightarrow$ Range $= \frac{V^2}{g} . 2\sin\alpha\cos\alpha$ (1)



$\boxed{\text{Range} = \frac{V^2}{g} . 2 . \frac{3}{\sqrt{10}} . \frac{1}{\sqrt{10}} = \frac{3}{5}R}$