SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JUNE 2007
TASK #2
YEAR 12

Mathematics Extension 2

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A(Questions 1 and 2),
  Section B(Questions 3 and 4),
  Section C(Questions 5 and 6),

Total marks—90 Marks

- Attempt questions 1–6.
- All questions are of equal value.

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section A

Marks

Question 1 (15 marks)

(a) Find $\displaystyle\int x\sec^2(1-x^2)\,dx$. $\boxed{2}$

(b) Find $\displaystyle\int \frac{x^2-4x+5}{x-2}\,dx$. $\boxed{2}$

(c) (i) Find the values of constants A, B, and C if $\boxed{2}$

$$\frac{A}{x+1}+\frac{Bx+C}{x^2+1}=\frac{4x^2-2x}{(x+1)(x^2+1)}.$$

(ii) Hence or otherwise find $\displaystyle\int \frac{4x^2-2x}{(x+1)(x^2+1)}\,dx$. $\boxed{3}$

(d) Given that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x+\cos x}\,dx=\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x+\cos x}\,dx$, $\boxed{3}$

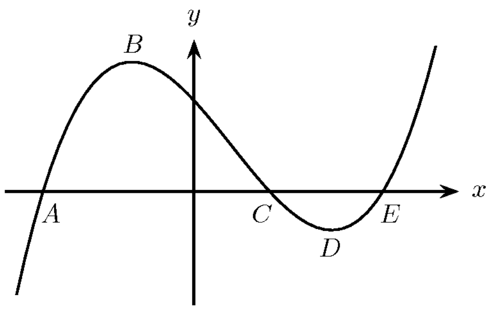
evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x+\cos x}\,dx$.

(e) Use the method of integration by parts to evaluate $\displaystyle\int_0^{\ln 2} xe^{-2x}\,dx$. $\boxed{3}$

Marks

Question 2 (15 marks)

(a) A sketch of $y = f'(x)$ is shown below: 3



Draw a neat sketch of a *possible* representation of $f(x)$. Mark on your diagram the points A', B', C', D', and E' which are related to the points A, B, C, D, and E.

(b) (i) Draw a neat sketch of $f(x) = (x+1)(2-x)(x-4)$ showing the intercepts on the axes. 2

(ii) Without using calculus, use your graph of $f(x) = (x+1)(2-x)(x-4)$ to sketch, on separate axes, the graphs of:

(α) $f(x) = |(x+1)(2-x)(x-4)|$ 1

(β) $f(x) = \dfrac{1}{(x+1)(2-x)(x-4)}$ 2

(γ) $f(x) = \sqrt{(x+1)(2-x)(x-4)}$ 2

(c) Draw a neat sketch of $y = \sin x \sin 2x$ for $-2\pi \leq x \leq 2\pi$ by considering 5

(i) the zeroes of the function,

(ii) the sign of the function in different sections of its domain,

(iii) whether the function is odd or even.

Section B

(Use a separate writing booklet.)

Marks

Question 3 (15 marks)

(a) Given that $z = -2 + 2i$ and $w = -1 - \sqrt{3}i$,

(i) find

(α) $z - w$, $\boxed{1}$

(β) $\mathfrak{Re}(z^2)$, $\boxed{1}$

(γ) $|w|$, $\boxed{1}$

(δ) $z\bar{w}$. $\boxed{1}$

(ii) Write z, $\bar{w}$, and $\dfrac{z}{\bar{w}}$ in modulus and argument form. $\boxed{3}$

(iii) Hence determine the number λ which satisfies $\boxed{2}$

$$\lambda\left(\frac{z}{\bar{w}}\right) = \left(\frac{z}{\bar{w}}\right)^{25}$$

.

(b) If the argument of the complex number $(z-1)/(z+1)$ is $\frac{1}{4}\pi$ show that z lies upon a fixed circle whose centre is at the point which represents i. $\boxed{3}$

(c) Find a complex number z whose argument is $\frac{1}{6}\pi$ such that $\boxed{3}$

$$|z - \sqrt{3} - i| = |z - 2\sqrt{3} - 2i|$$

Marks

Question 4 (15 marks)

(a) Factorise $z^4 + z^2 - 6$ over

(i) $\mathbb{R}$ $\boxed{1}$

(ii) $\mathbb{C}$ $\boxed{1}$

(b) If α, β, γ are roots of the equation $2x^3 - 4x^2 - 6x + 5 = 0$, find the cubic equation with roots $\alpha - 1$, $\beta - 1$, $\gamma - 1$. $\boxed{2}$

(c) It is suspected that $12x^3 - 4x^2 - 5x + 2 = 0$ has a repeated root. Use this to find all the solutions. $\boxed{4}$

(d) (i) The roots of the quadratic polynomial $x^2 + px + q = 0$ (where $q \neq 0$) are α and β, and one root of the quadratic equation $x^2 + p'x + q = 0$ is $k\alpha$. Show that the other root of this equation is $\dfrac{\beta}{k}$. $\boxed{2}$

(ii) Assuming that $k^2 \neq 1$, write down for each equation an expression for the sum of its roots and hence find α and β in terms of p, p', and k. $\boxed{2}$

(iii) Deduce that $k(kp - p')(kp' - p) = \left(k^2 - 1\right)^2 q$. $\boxed{1}$

(iv) For any given values of p, p', and q, show that the sum of the four possible values of k is $\dfrac{pp'}{q}$. $\boxed{2}$

Section C

(Use a separate writing booklet.)

Marks

Question 5 (15 marks)

(a) If no two boys are to sit together, in how many ways can six girls and three boys be arranged

(i) in a row, $\boxed{2}$

(ii) around a circular table? $\boxed{1}$

(b) A pair of integers is selected at random from the set of positive integers $1,\ 2,\ 3, \ldots n$.

(i) In how many ways may the selection be made? $\boxed{1}$

(ii) If n is an odd integer, show that the probability that both integers in the pair selected will be odd is $\dfrac{n+1}{4n}$. $\boxed{2}$

(iii) If n is even, find the probability that both integers in the pair are odd. $\boxed{2}$

(c) Establish a reduction formula for $\mathbf{I}_n = \int \sec^n x\, dx$. $\boxed{2}$

(d) (i) Any integer n can be expressed in one of the forms $4p$, $4p+1$, $4p+2$, $4p+3$, where p is an integer. Find, for each of these forms, the value of the expression $(1+i^n)(1+i^{2n})$ where $i^2=-1$. $\boxed{4}$

(ii) Hence write down a function of n which has the value 366 when n is a multiple of 4 and 365 for all other integral values of n $\boxed{1}$

Marks

Question 6 (15 marks)

(a) $\displaystyle\int_0^2 \sqrt{4+x^2}\,dx$ 2

[NOTE: you may find the result from Q5(c) useful.]

(b) (i) Determine how the number of stationary points on the curve $y = e^{-x^2}\left(x^2+c\right)$ depends on the value of the real constant c. 2

(ii) Find the coördinates of the points of inflexion on the curve for which $c = 2$. 3

(iii) Sketch the curve for which $c = 2$. 2

(c) A rock of mass 5 kg is projected vertically upwards into the air from the ground with initial speed of $12\,\mathrm{ms}^{-1}$. The rock is subject to air resistance of $\dfrac{v^2}{2}$ Newtons in the opposite direction to its velocity, $v\,\mathrm{ms}^{-1}$. The rock is also subject to a downward gravitational force of 50 Newtons.

(i) Find the greatest height reached by the rock. 2

(ii) Find the time taken to reach the highest point. 2

(iii) Find how fast the rock is travelling when it hits the ground. 2

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$
$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$
$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$
$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$
$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$
$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$
$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$
$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$
$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$
$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$
$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION 1

(a) $\int x \sec^2(1-x^2)\,dx$ 2

$= -\frac{1}{2}\tan(1-x^2) + c$

(b) $\int \frac{x^2-4x+5}{x-2}\,dx$ 2

$= \int \left(x - 2 + \frac{1}{x-2}\right) dx$ ①

$= x^2 - 2x + \cancel{\tfrac{4}{2}} \ln|x-2| + c$ ①

(c)

(i) $4x^2 - 2x = A(x^2+1) + (Bx+c)(x+1)$

$A = 3 \quad B = 1 \quad c = -3$ 2

(ii) $\therefore I = \int \left(\frac{3}{x+1} + \frac{x-3}{x^2+1}\right) dx$

3

$= 3\ln|x+1| + \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + c$

(d) $I = \frac{1}{2}\int \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}\,dx$ 3

$= \frac{1}{2}\int \sin^2 x + \cos^2 x - \cancel{1}\sin x\cos x$

$= \frac{1}{2}\int \left(1 - \frac{1}{2}\sin 2x\right) dx$

$= \frac{1}{2}\left(x - \frac{1}{2} \cdot -\cos 2x \cdot \frac{1}{2}\right)$

$\therefore \int_0^{\frac{\pi}{2}} \frac{s^3}{s+c}\,dx = \left[\frac{x}{2} + \frac{1}{8}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi - 1}{4}$

(e) $\int_0^{\ln 2} x e^{-2x}\,dx$

$= \int_0^{\ln 2} x \cdot \frac{d}{dx}\left(\frac{1}{2}e^{-2x}\right) dx$ 3

$= \left[-\frac{1}{2}x \cdot e^{-2x}\right]_0^{\ln 2} - \int_0^{\ln 2} -\frac{1}{2} e^{-2x}\,dx$ 4

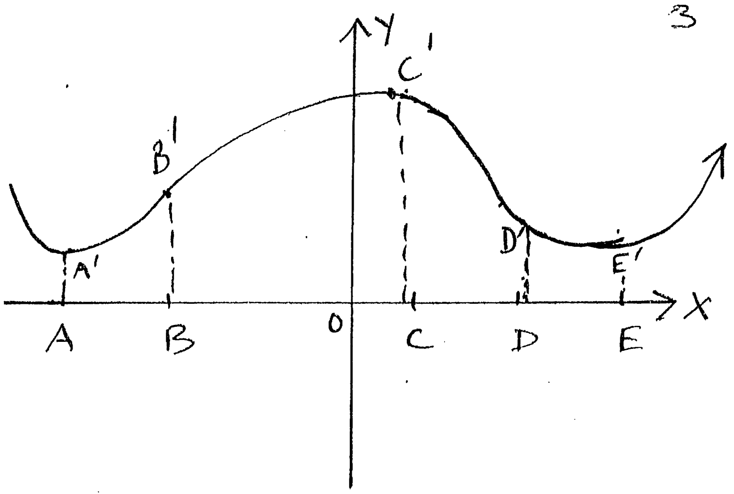
$= \left[-\frac{\ln 2}{2} e^{-2\ln 2}\right] + \left[-\frac{1}{4}\right]\left[e^{-2\ln 2} - e^0\right]$

$= -\frac{\ln 2}{2} \cdot 2^{-2} + \left(-\frac{1}{4}\right)\left(2^{-2} - 1\right)$

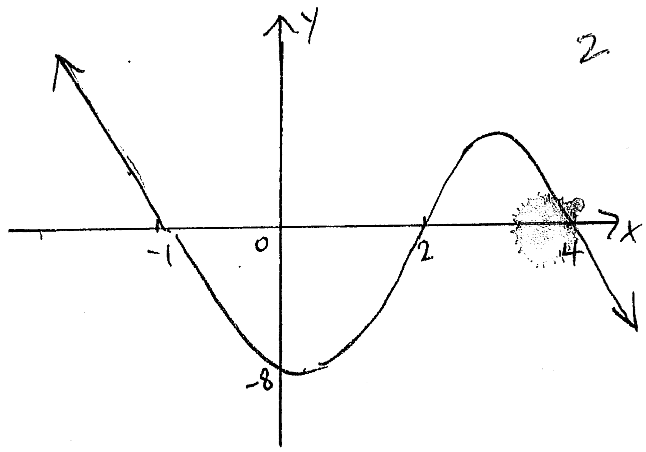
$= -\frac{1}{8}\ln 2 + \frac{3}{16}$

QUESTION 2

(a) 3

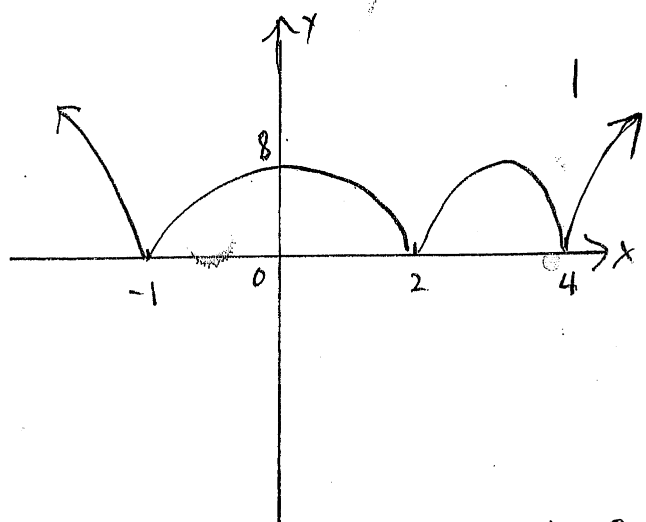


(b) (i) 2

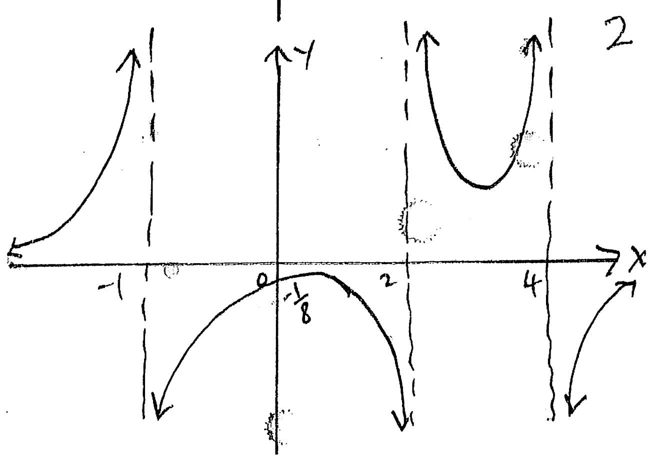


(ii)

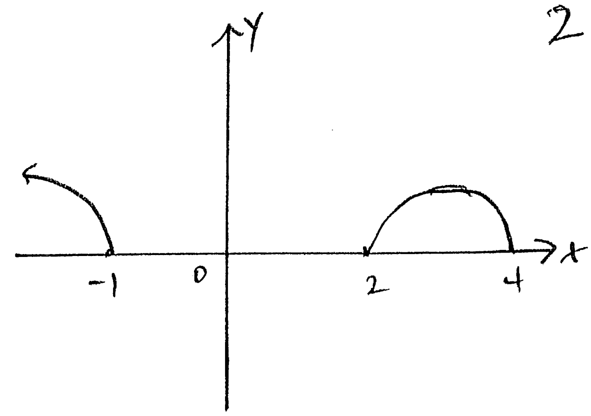
(α) 1



(β) 2



(γ) 2



(c) $\sin x \sin 2x = 0 \Rightarrow$ 5

$\sin x = 0$

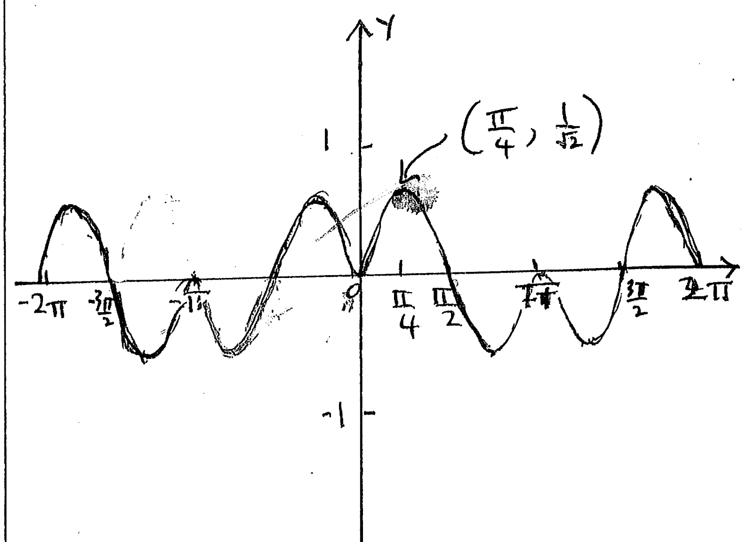
$x = -2\pi, -\pi, 0, \pi, 2\pi$

OR

$\sin 2x = 0$

ie $2x = -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$

$\therefore x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



$f(x) = \sin x \sin 2x$

$f(-x) = \sin(-x) . \sin(-2x)$

$= -\sin x . -\sin 2x$

$= \sin x . \sin 2x$

$= f(x) \quad \therefore f(x)$ <u>even</u>

Section B Question 3

(a) $z = -2+2i \quad w = -1-\sqrt{3}i$

(i) (α) $z - w = (-2+2i)-(-1-\sqrt{3}i)$

$= -1+(2+\sqrt{3})i$ $\boxed{1}$

(β) $Re(z^2) = Re(-2+2i)^2$

$= Re(-8i)$

$= 0$ $\boxed{1}$

(γ) $|w| = \sqrt{1^2+\sqrt{3}^2}$

$= 2$ $\boxed{1}$

(δ) $z\bar{w} = (-2+2i)(-1+\sqrt{3}i)$

$= 2-2\sqrt{3}i-2i-2\sqrt{3}$

$= (2-2\sqrt{3})-(2+2\sqrt{3})i$ $\boxed{1}$

(ii) $z = -2+2i \quad \arg z = \frac{3\pi}{4}$

$|z| = 2\sqrt{2}$

$\therefore z = 2\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$ $\boxed{1}$

$\bar{w} = -1+\sqrt{3}i \quad \arg\bar{w} = \frac{2\pi}{3}$

$|\bar{w}| = 2$

$\therefore \bar{w} = 2\operatorname{cis}\frac{2\pi}{3}$ $\boxed{1}$

$\frac{z}{\bar{w}} = \frac{2\sqrt{2}\operatorname{cis}\frac{3\pi}{4}}{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}$

$= \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{2\pi}{3}\right)$

$= \sqrt{2}\operatorname{cis}\frac{\pi}{12}$ $\boxed{1}$

(iii) $\lambda\left(\frac{z}{\bar{w}}\right) = \left(\frac{z}{\bar{w}}\right)^{25}$

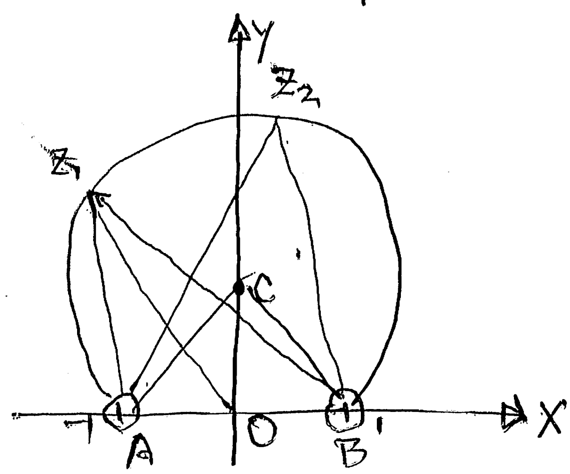
$\therefore \lambda = \left(\frac{z}{\bar{w}}\right)^{24}$

$= (\sqrt{2})^{24}\operatorname{cis}\left(\frac{\pi}{12}\times 24\right)$

$= 4096\operatorname{cis}2\pi$

$= 4096$ $\boxed{2}$

(b) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$



Let Z_1, Z_2 be points representing two values of z. Now $\angle AZ_1B = \angle AZ_2B$

$= \pi/4$

$\therefore$ By the theorem, all z lie on a circle.

Let C be the centre of the circle.

$\therefore \angle ACB = \frac{\pi}{2}$ (angle at the centre twice angle at circumference).

$\therefore$ By symmetry and similarity, $OC = 1$

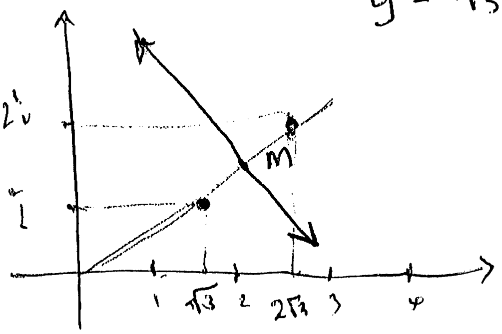
$\therefore$ Centre is i. $\boxed{3}$

(c) $\arg z = \frac{\pi}{6}$

$|z-(\sqrt{3}+i)| = |z-2(\sqrt{3}+i)|$

This is the perp. bisector of the interval joining $\sqrt{3}+i$ and $2\sqrt{3}+2i$

$y = -\sqrt{3}x + 3$



Midpoint M is $\frac{3\sqrt{3}}{2} + \frac{3i}{2}$

We seek z lying on this line with $\arg z = \frac{\pi}{6}$.

But $\operatorname{Arg}(M) = \frac{\pi}{6}$

$\therefore z = M$

$= \frac{3}{2}(\sqrt{3}+i)$

$= 3\operatorname{cis}\frac{\pi}{6}$ $\boxed{3}$

Question 4

(a) z^4+z^2-6

$= (z^2+3)(z^2-2)$ $\boxed{1}$

(i) $= (z^2+3)(z+\sqrt{2})(z-\sqrt{2})$

(ii) $= (z+\sqrt{3}i)(z-\sqrt{3}i)(z+\sqrt{2})(z-\sqrt{2})$ $\boxed{1}$

(b) $2x^3-4x^2-6x+5=0$ has roots α, β, γ.

Let $y = x-1$

$\therefore x = y+1$

Thus $2(y+1)^3-4(y+1)^2-6(y+1)+5=0$ has roots $\alpha-1, \beta-1, \gamma-1$

Simplifying:

$2(y^3+3y^2+3y+1)-4(y^2+2y+1)-6y-6+5=0$

$2y^3+6y^2+6y+2-4y^2-8y-4-6y-1=0$

$2y^3+2y^2-8y-3=0$

Replace y with x:

$2x^3+2x^2-8x-3=0$ has req'd roots. $\boxed{2}$

(c) $12x^3-4x^2-5x+2=0$ — ①

Derived eq'n:

$36x^2-8x-5=0$ — ②

$x = \frac{8 \pm \sqrt{64+4\times180}}{72}$

$= \frac{1}{2}, -\frac{5}{18}$ $\boxed{1}$

Test $x=\frac{1}{2}$ in Eq'n ①

$12(\frac{1}{2})^3-4(\frac{1}{2})^2-5(\frac{1}{2})+2=0$

$\therefore x=\frac{1}{2}$ is a repeated root. $\boxed{1}$

$\therefore (x-\frac{1}{2})^2$ is a factor of ①.

$$\begin{array}{r|l} & 12x+8 \\ x^2-1+\frac{1}{4} & 12x^3-4x^2-5x+2 \\ & 12x^3-12x^2+3x \\ & \qquad 8x^2-8x+2 \\ & \qquad 8x^2-8x+2 \\ & \qquad\qquad 0 \end{array}$$

$\therefore$ Roots are $\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}$ $\boxed{2}$

Q4

(d) $x^2+px+q=0$ (Roots α, β) — ①

$x^2+p'x+q=0$ (Roots $k\alpha$, γ say) — ②

(i) From ① $\alpha\beta = q$

From ② $k\alpha\gamma = q$

$\therefore \alpha\beta = k\alpha\gamma$

$\therefore \gamma = \dfrac{\beta}{k}$ $\boxed{2}$

(ii) For ① $\alpha+\beta = -p$

For ② $k\alpha + \dfrac{\beta}{k} = -p'$

Substitute $\beta = -p-\alpha$

$\therefore k\alpha - \dfrac{p+\alpha}{k} = -p'$

$k^2\alpha - (p+\alpha) = -kp'$

$k^2\alpha - \alpha = p - kp'$

$\alpha(k^2-1) = p-kp'$

$\alpha = \dfrac{p-kp'}{k^2-1}$

Now $\beta = -p-\alpha$

$= -p - \left(\dfrac{p-kp'}{k^2-1}\right)$

$= \dfrac{-p(k^2-1)-(p-kp')}{k^2-1}$

$\therefore \beta = \dfrac{kp'-k^2p}{k^2-1}$ $\boxed{2}$

(iii) Now $\alpha\beta = q$

$\therefore \dfrac{p-kp'}{k^2-1} \cdot \dfrac{k(p'-kp)}{k^2-1} = q$

$\therefore (p-kp')k(p'-kp) = (k^2-1)^2 q$

$\therefore k(kp-p')(kp'-p) = (k^2-1)^2 q$

QED $\boxed{1}$

(iv) Expanding both sides and collecting terms on the left gives the quartic eq'n in k:

$qk^4 - pp'k^3 + \ldots = 0$

$\therefore$ By sum of roots

Sum of all k's $= \dfrac{pp'}{q}$ $\boxed{2}$

QED

Q5 (a) (i)

× × × × × ×

Arrangements of girls $= 6!$

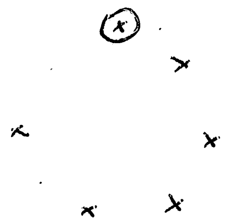
Choose 3 ~~from~~ 7 locations for boys $= {}^7P_3$

$\therefore$ No of arrangements $= 6! \cdot {}^7P_3$

$= 151200$

2

(ii)



Arrangements of girls $= 5!$

Choose 3 from 6 locations for boys $= {}^6P_3$

$\therefore$ No of arrangements $= 5! \cdot {}^6P_3$

$= 14\,400$

1

(b) (i) $\quad {}^nC_2 = \dfrac{n(n-1)}{2}$

1

(ii) $\quad P(\text{odd, odd}) = \dfrac{\frac{n+1}{2}}{n} \cdot \dfrac{\frac{n-1}{2}}{n-1}$

$= \dfrac{n+1}{2n} \cdot \dfrac{1}{2}$

$= \dfrac{n+1}{4n}$

2

$\dfrac{\frac{n}{2}-1}{2n-2}$

$= \dfrac{n-2}{4n-4}$

(iii) $\quad P(\text{odd, odd}) = \dfrac{1}{2} \cdot \dfrac{\frac{n-2}{2}}{n-1}$

$= \dfrac{1}{4}\,\dfrac{n-2}{n-1}$

$= \dfrac{n-2}{4n-4}$

2

$\dfrac{{}^mC_2}{{}^{2m}C_2}$

$= \dfrac{\frac{m(m-1)}{2}}{\frac{2m(2m-1)}{2}}$

$= \dfrac{m-1}{\ldots}$

(c) $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2} x \, . \sec^2 x \, dx$

$= \sec^{n-2} x \, . \tan x$ $\quad u = \sec^{n-2} x \quad v = \tan x$

$- \int (n-2) \sec^{n-3} x \, . \sec x \tan x \, . \tan x \, dx$ $\quad u' = (n-2)\sec^{n-3} x \, . \sec x \tan x$ $\quad v' = \sec^2 x$

$= \sec^{n-2} x \tan x$

$- (n-2) \int \sec^{n-2} x \, . \tan^2 x \, dx$

$= \sec^{n-2} x \tan x$

$- (n-2) \int \sec^{n-2} x \, . (\sec^2 x - 1) \, dx$

$= \sec^{n-2} x \tan x - (n-2) \left\{ \int \sec^n x \, dx - \int \sec^{n-2} x \, dx \right\}$

$\therefore I_n = \sec^{n-2} x \tan x - (n-2) \{ I_n - I_{n-2} \}$.

$\therefore I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\therefore I_n (n-1) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\therefore I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$

2

(d) (i) If $n = 4p$

$(1+i^n)(1+i^{2n})$

$= (1+i^{4p})(1+i^{8p})$

$= 2 \cdot 2$

$= 4$

If $n = 4p+1$

$(1+i^n)(1+i^{2n})$

$= (1+i^{4p+1})(1+i^{8p+2})$

$= (1+i)(1+i^2)$

$= (1+i)(1-1)$

$=$

If $n = 4p+2$

$(1+i^n)(1+i^{2n})$

$= (1+i^{4p+2})(1+i^{8p+4})$

$= (1+i^2)(1+1)$

$= 0$

4

If $n = 4p+3$

$(1+i^{4p+3})(1+i^{8p+6})$

$= (1-i)(1-1)$

$= 0$

1

(ii) Function is $f(n) = 365 + \dfrac{(1+i^n)(1+i^{2n})}{4}$

Q6 (a) $\int_0^2 \sqrt{4+x^2}\, dx$ let $x = 2\tan\theta$

$\therefore dx = 2\sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \sqrt{4+4\tan^2\theta} \cdot 2\sec^2\theta\, d\theta$ If $x=0$, $\theta = 0$

$= \int_0^{\pi/4} 2\sec\theta \cdot 2\sec^2\theta\, d\theta$ If $x=2$ $\theta = \frac{\pi}{4}$

$= 4\int_0^{\pi/4} \sec^3\theta\, d\theta$

$= 4\left\{\left[\frac{1}{2}\sec\theta\tan\theta\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\theta\, d\theta\right\}.$

$= 4\left\{\left[\frac{1}{2}\sec\frac{\pi}{4}\cdot\tan\frac{\pi}{4}\right] - \left[\frac{1}{2}\sec 0\tan 0\right] + \frac{1}{2}\left[\ln(\sec\theta+\tan\theta)\right]_0^{\pi/4}\right\}.$

$= 4\left\{\frac{1}{2}\cdot\sqrt{2}\cdot 1 - 0 + \frac{1}{2}\left[\ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)\right] - \frac{1}{2}\ln(\sec 0+\tan 0)\right\}.$

$= 2\sqrt{2} + 2\ln(\sqrt{2}+1) - \frac{1}{2}\ln 1$

$= 2\sqrt{2} + 2\ln(\sqrt{2}+1)$

2

(b) (i) $y = e^{-x^2}(x^2+c)$

$y' = (x^2+c) \cdot e^{-x^2} \cdot -2x + e^{-x^2} \cdot 2x$

$= 2x e^{-x^2}(-(x^2+c)+1)$

$y' = 0$ if $x = 0$ or $x^2+c-1 = 0$.

$\therefore x^2 = 1-c$

$\therefore 1-c \geq 0$

$\therefore c \leq 1$, for zeros.

If $c=1$ $x=0$

If $c < 1$ the curve has 3 stationary points.

Otherwise the curve has 1 stationary point.

2

(ii) If $c=2$ $y' = -2x e^{-x^2}(x^2+1)$

$= e^{-x^2}(-2x^3-2x)$

$y'' = e^{-x^2} \cdot (-6x^2-2) + -2x(x^2+1) \cdot e^{-x^2} \cdot -2x$

$= e^{-x^2}(-6x^2-2+4x^2(x^2+1))$

$= e^{-x^2}(-6x^2-2+4x^4+4x^2)$

$= e^{-x^2}(4x^4-2x^2-2)$

$= 2e^{-x^2}(2x^4-x^2-1)$

$= 2e^{-x^2}(2x^2+1)(x^2-1)$

$\therefore x^2 = -\frac{1}{2}$ or $x^2 = 1$

$\therefore x = \pm 1$

$\therefore$ Potential point of inflexion are

$$\left(1, \frac{3}{e}\right) \text{ or } \left(-1, \frac{3}{e}\right)$$

If $x = 0.5$ $y'' =$ +ve × +ve × −ve

< 0

If $x = 1.5$ $y'' =$ +ve × +ve × +ve

> 0.

3

$\therefore$ Change of concavity.

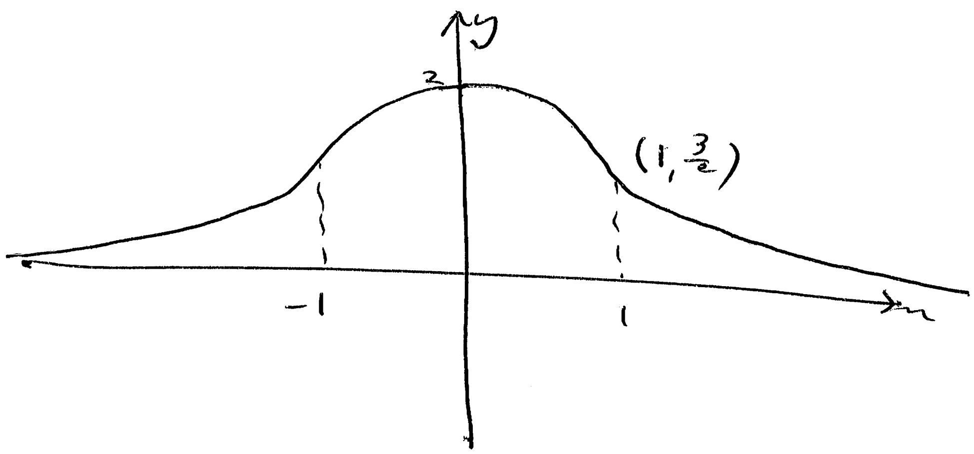
$\left(1, \frac{3}{e}\right)$ is a point of inflexion.

The function is an even function

$\therefore \left(-1, \frac{3}{e}\right)$ is also a point of inflexion

(iii) When $x = 0$ $y = 2$ ~~y~~

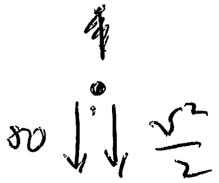
As $x \to \infty$ $y \to 0$



2

(e) (ii)

$$\ddot{x} = -\left(10 + \frac{v^2}{10}\right)$$



$$v\frac{dv}{dx} = -\frac{100+v^2}{10}$$

$$\int \frac{v\,dv}{100+v^2} = -\int \frac{dx}{10}$$

$$\frac{1}{2}\ln(100+v^2) = -\frac{1}{10}x + c$$

when $x = 0$, $v = 12$

$$\therefore \frac{1}{2}\ln 244 = c.$$

$$\therefore -\frac{1}{10}x = \frac{1}{2}\ln(100+v^2) - \frac{1}{2}\ln 244$$

$$\therefore \frac{1}{10}x = \frac{1}{2}\left\{\ln(244) - \ln(100+v^2)\right\}.$$

$$\therefore \frac{1}{5}x = \ln\frac{244}{100+v^2}.$$

Max ht reached when $v = 0$

$$\therefore \frac{1}{5}x = \ln\frac{244}{100}$$

$$\therefore x = 5\ln\frac{244}{100}.$$

$$= 4.4599\ldots \text{ m}$$

2

(ii) $\frac{dv}{dt} = -\left(10 + \frac{v^2}{10}\right)$

$\therefore \frac{dv}{dt} = -\frac{100 + v^2}{10}$

$\therefore \int \frac{dv}{100 + v^2} = \int -\frac{1}{10}\, dt.$

$\therefore \frac{1}{10} \tan^{-1} \frac{v}{10} = -\frac{1}{10} t + c.$

When $t = 0$, $v = 12$

$\frac{1}{10} \tan^{-1} \frac{12}{10} = c.$

$\therefore \frac{1}{10} \tan^{-1} \frac{v}{10} = -\frac{1}{10} t + \frac{1}{10} \tan^{-1} \frac{12}{10}.$

$\therefore \tan^{-1} \frac{v}{10} = -t + \tan^{-1} \frac{12}{10}.$

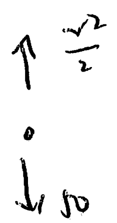
$\therefore t = \tan^{-1} \frac{12}{10} - \tan^{-1} \frac{v}{10}.$

If $v = 0$, $t = \tan^{-1} \frac{12}{10}$

$= 0.8760 \ldots s$

2

(iii)



$$\dot{x} = 10 - \frac{v^2}{10}$$

$$\frac{v\,dv}{dx} = \frac{100 - v^2}{10}.$$

$$\therefore \int \frac{v\,dv}{100 - v^2} = \int \frac{1}{10}\,dx$$

$$-\frac{1}{2}\ln(100 - v^2) = \frac{1}{10}x + c.$$

When $x = 0$, $v = 0$: $-\frac{1}{2}\ln 100 = c.$

$$\therefore -\frac{1}{2}\ln(100 - v^2) = \frac{1}{10}x - \frac{1}{2}\ln 100.$$

$$\frac{1}{10}x = \frac{1}{2}\left(\ln 100 - \ln(100 - v^2)\right)$$

$$x = 5\left(\ln \frac{100}{100 - v^2}\right).$$

If $x = 5\ln\frac{244}{100}$:

$$5\ln\frac{244}{100} = 5\ln\frac{100}{100 - v^2}$$

$$\therefore \frac{244}{100} = \frac{100}{100 - v^2}.$$

$$\therefore 100 - v^2 = \frac{10000}{244}$$

$$v^2 = 100 - \frac{10000}{244}$$

$$v^2 = \frac{24400 - 10000}{244}$$

$$= \frac{14400}{244}$$

$$\therefore v = \frac{120}{\sqrt{244}} = 7.6822 \text{ m/s}.$$

2