SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 84

- Attempt questions 1 – 6
- Board approved calculators maybe used.
- Full marks may not be awarded for careless or badly arranged work.

Examiner: *A.M.Gainford*

Section A
(Start a new answer sheet.)

Question 1. (14 marks)

Marks

(a) Given the complex number $z = -1 + \sqrt{3}i$: **3**

(i) Express z in modulus-argument form.

(ii) Hence evaluate z^9.

(b) Sketch the region in the Argand diagram which simultaneously satisfies: **3**

$$\frac{\pi}{4} \leq \arg(z - i) \leq \frac{3\pi}{4} \qquad \text{and} \qquad |z - i| < 2 .$$

(c) Find the square roots of $8 - 6i$. **2**

(d) Find the two complex numbers which satisfy: **3**

$$3z\overline{z} + 2\left(z - \overline{z}\right) = 39 + 12i$$

(e) If $z = \cos\theta + i\sin\theta$, show that $1 + z = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. **3**

Question 2. (14 marks)

Marks

(a) Evaluate: **2**

$$\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$$

(b) Use the substitution $u^6 = x$ to evaluate: **3**

$$\int_0^1 \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$$

(c) Use integration by parts to find: **3**

$$\int e^x \cos x\,dx$$

(d) Find: **3**

$$\int \frac{dx}{x^2-x-6}$$

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$$

Give your answer in simplest exact form.

Section B
(Start a new answer sheet.)

Question 3. (14 marks)

Marks

(a) The equation $x^3 + kx + 2 = 0$ has roots α, β and γ. **5**

(i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, in terms of *k*.

(ii) Find the monic cubic equation with roots $\alpha^2, \beta^2, \gamma^2$, in terms of *k*.

(iii) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of *k*.

(b) (i) Prove that if α is a double root of the polynomial equation $P(x) = 0$, then $P'(\alpha) = 0$. **2**

(ii) Find all the roots of the equation **3**

$$18x^3 + 3x^2 - 28x + 12 = 0$$

given that two of the roots are equal.

(c) Given that $f(x) = \dfrac{11 - x}{x^2 - x - 2}$: **4**

(i) Graph $y = f(x)$, clearly marking its asymptotes and stationary points, if any. (Use approximate values.)

(ii) Hence, or otherwise, graph $y = \dfrac{1}{f(x)}$.

Question 4 (14 marks)

(a) Let $OABC$ be a square on the Argand diagram, where O is the origin, and C is to the left of A. The point A represents the complex number $z = 2 + 3i$. **4**

(i) Find the complex numbers represented by B and C in the form $a + ib$.

(ii) The square is now rotated about O through 45^0 in the anticlockwise direction to $OA'B'C'$. Find the complex numbers represented by the points A', B', C', in the form $a + ib$.

(b) (i) Use the substitution $x = a - t$, where a is a constant, to prove that: **2**

$$\int_0^a f(x)\,dx = \int_0^a f(a-t)\,dt$$

(ii) Hence, or otherwise, show that $\int_0^2 x^2 (2-x)^9 \, dx = 6\tfrac{34}{165}$. **2**

(c) A particle moves in a straight line with velocity given by $v^2 = 36 - 4x^2$, where x is the displacement in metres from a fixed point O and t is the time in seconds. **2**

(i) Show that the motion is simple harmonic.

(ii) Find the period of the motion.

(d) Express the five fifth roots of unity in modulus-argument form, and clearly show them on an Argand diagram. **4**

Section C
(Start a new answer booklet)

Question 5 (14 marks)

(a) Solve for z : **3**

$$z^2+(1+2i)z-(2-4i)=0$$

Give your answers in the form $a+ib$.

(b) Let $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx$ where n is a positive integer. **5**

(i) Find the value of I_1.

(ii) Using integration, show that $I_n+I_{n-2}=\dfrac{1}{n-1}$.

(iii) Evaluate $J=\int_0^{\frac{\pi}{4}}\tan^{11} x\,dx$.

(c) Use de Moivre's theorem to express $\cos 4\theta$ in terms of $\cos\theta$. **2**

(d) Given that $2+i$ is a root of the equation $x^3-x^2-7x+15=0$, find all roots of the equation. **4**

Question 6 (14 marks)

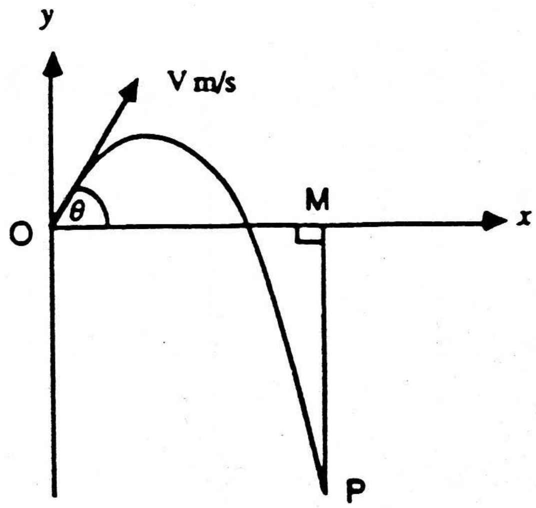
(a) Consider the function $y = e^{-x^2}$: **6**

(i) Sketch the graph of the function in the domain $-2 \le x \le 2$.

(ii) Show, using your graph, that $1 < \int_{-2}^{2} e^{-x^2}\,dx < 4$.

(iii) Approximate $\int_{-2}^{2} e^{-x^2}\,dx$ correct to two decimal places using Simpson's Rule with five x-values.

(c)



A projectile is fired from a point O with initial speed V m/s at an angle of elevation of θ. It is subject only to the force of gravity, resulting in a constant downward acceleration of magnitude g m/s^2. **8**

(i) Find functions for its horizontal (x) and vertical (y) displacements from O after t seconds.

(ii) The projectile falls to a point P, below the level of O, such that $PM=OM$.

Prove that the time taken to reach P is $\dfrac{2V(\sin\theta + \cos\theta)}{g}$ seconds.

(iii) Show that the distance OM is $\dfrac{V^2(\sin 2\theta + \cos 2\theta + 1)}{g}$ metres.

(iv) If the horizontal range of the projectile level with O is r m and the distance OM is $\dfrac{4r}{3}$, prove that $\sin 2\theta - 3\cos 2\theta = 3$, and find the value of $\tan\theta$.

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

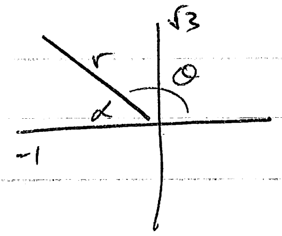
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Question 1

(a) i) $z = -1 + \sqrt{3}i$



$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

$r^2 = (1)^2 + (\sqrt{3})^2$

$r^2 = 4$

$r = 2$

$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

ii) $z^9 = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^9$

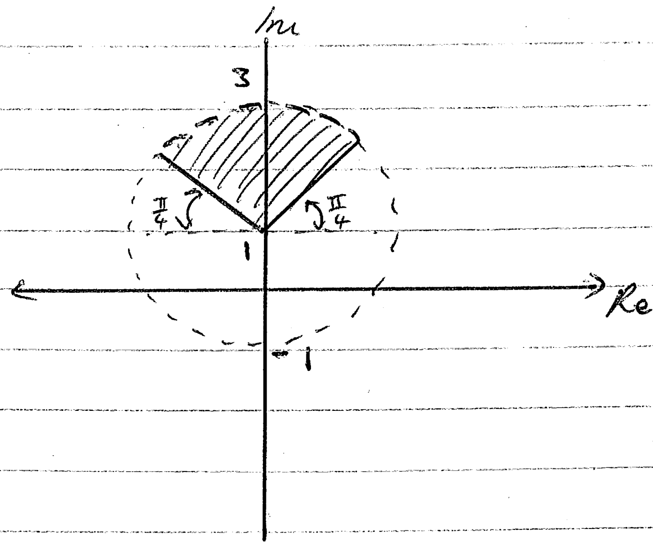
$= 2^9(\cos 6\pi + i\sin 6\pi)$

$= 2^9(1 + i(0))$

$= 2^9$

$= 512$

(b)



(c) let $a + ib = \sqrt{8 - 6i}$

$a^2 - b^2 + 2abi = 8 - 6i$

$a^2 - b^2 = 8$ ——— (1)

$2ab = -6$

$ab = -3$ ——— (2)

$(a^2+b^2)^2 = (a^2-b^2)^2 - 4(ab)^2$

$= 64 + 4(-3)^2$

$= 100$

$a^2 + b^2 = 10$ ——— (3)

(1) + (3)

$2a^2 = 18$

$$= \frac{\pi}{12}$$

(b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

$u^6 = x$

$\frac{dx}{du} = 6u^5$

$dx = 6u^5\,du$

when $x=1$, $u=1$
$x=0$, $u=0$

$$= \int_0^1 \frac{6u^5\,du}{u^3+u^2}$$

$$= 6\int_0^1 \frac{u^3\,du}{1+u}$$

$$= 6\int_0^1 \frac{u^3+1}{1+u}\,du - 6\int_0^1 \frac{du}{1+u}$$

$$= 6\int_0^1 \frac{\cancel{(u+1)}(u^2-u+1)}{\cancel{u+1}}\,du - 6\int_0^1 \frac{du}{u+1}$$

$$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_0^1$$

$$= 6\left[\frac{1}{3} - \frac{1}{2} + 1 - \ln 2 - 0\right]$$

$$= 5 - 6\ln 2$$

(c) $I = \displaystyle\int e^x \cos x\,dx$

$u = \cos x \quad v' = e^x$
$u' = -\sin x \quad v = e^x$

$$I = e^x \cos x + \int e^x \sin x\,dx + C$$

$u = \sin x \quad v' = e^x$
$u' = \cos x \quad v = e^x$

$$I = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx + k$$

$$I = e^x(\cos x + \sin x) - I + k$$

$$2I = e^x(\cos x + \sin x) + k$$

$$I = \frac{e^x}{2}(\cos x + \sin x) + K$$, where k is a constant.

$$a^2 = 9$$
$$a = \pm 3$$

when $a = 3$ $\quad$ $a = -3$
$b = -1$ $\quad$ $b = 1$

$3 - i, \quad -3 + i$

(d) $z = x + iy$
$$\bar{z} = x - iy$$
$$z\bar{z} = x^2 + y^2$$
$$3(x^2+y^2) + 2(2iy) = 39 + 12i$$
$$3(x^2+y^2) = 39 \qquad 4y = 12$$
$$x^2+y^2 = 13 \qquad y = 3$$
$$x^2 + (3)^2 = 13$$
$$x^2 = 4$$
$$x = \pm 2$$
$$z = 2 + 3i, \quad -2 + 3i$$

(e) $z = \cos\theta + i\sin\theta$
$$1 + z = 1 + \cos\theta + i\sin\theta$$
since $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ & $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$$= 2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, i$$
$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Question 2

(a) $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$

$$= \frac{1}{2}\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{4} - x^2}}, \quad a = \frac{1}{2}$$

$$= \frac{1}{2}\left[\sin^{-1} 2x\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{1}{2} - \cancel{\sin^{-1} 0}\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

(d) $\displaystyle\int \frac{dx}{x^2 - x - 6}$

$$= \int \frac{dx}{(x-3)(x+2)}$$

$$\frac{1}{(x-3)(x+2)} \equiv \frac{a}{x-3} + \frac{b}{x+2}$$

$$1 \equiv a(x+2) + b(x-3)$$

let $x = 3$ let $x = -2$

$1 = 5a$ $1 = -5b$

$a = \frac{1}{5}$ $b = -\frac{1}{5}$

$$\therefore \int \frac{dx}{x^2 - x - 6} = \frac{1}{5}\int \frac{dx}{x-3} - \frac{1}{5}\int \frac{dx}{x+2}$$

$$= \frac{1}{5}\ln(x-3) - \frac{1}{5}\ln(x+2) + C$$

$$= \frac{1}{5}\ln\left(\frac{x-3}{x+2}\right) + C$$

(e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x + \sin x}$

$t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

when $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

$$= \int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{dt}{1+t^2+1-t^2+2t}$$

$$= 2\int_0^1 \frac{dt}{2(1+t)}$$

$$= \int_0^1 \frac{dt}{1+t}$$

$$= \Big[\ln(1+t)\Big]_0^1$$

$$= \ln 2 - \cancel{\ln 1} = \ln 2$$

QUESTION 3

(a) given $x^3+kx+2=0$ (A) with roots α, β & γ.

(i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} = \frac{k}{-2} = -\frac{1}{2}k.$ ✓

[NB $\sum\alpha\beta = +\frac{c}{a} = k.$
& $\alpha\beta\gamma = -\frac{d}{a} = -2.$]

(ii) Let $X = x^2$

$\Rightarrow x = \pm\sqrt{X}$. in the above (A)

$\therefore (\pm\sqrt{X})^3 + k(\pm\sqrt{X}) + 2 = 0.$

$\pm X\sqrt{X} \pm k\sqrt{X} = -2.$

$\pm\sqrt{X}.(X+k) = -2.$

$\therefore X(X+k)^2 = 4.$

$X(X^2+2kX+k^2) = 4.$

$\boxed{X^3+2kX^2+k^2X-4=0.}$ ✓✓

(iii) From (A) $\alpha^3+k\alpha+2=0$

$\beta^3+k\beta+2=0$

& $\gamma^3+k\gamma+2=0$

Adding we get:

$\alpha^3+\beta^3+\gamma^3+k(\alpha+\beta+\gamma)+6=0.$

$\therefore \alpha^3+\beta^3+\gamma^3+k\times 0+6 = 0$

$\boxed{\alpha^3+\beta^3+\gamma^3=-6.}$ ✓✓

(b) (i) Let $P(x) = (x-\alpha)^2 Q(x)$.

$$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x).$$

ie $P'(x) = (x-\alpha)(2Q(x) - (x-\alpha)Q'(x))$

$= (x-\alpha) R(x)$

$\therefore P'(\alpha) = (\alpha-\alpha) \cdot R(\alpha)$

$= 0 \cdot R(\alpha)$

$= 0.$

(ii). Let $P(x) = 18x^3 + 3x^2 - 28x + 12 = 0$

$P'(x) = 54x^2 + 6x - 28. = 0$

$P'(x) = 2(27x^2 + 3x - 14) = 0$

$= 2(9x+7)(3x-2) = 0$

$\therefore \quad x = \frac{2}{3}, -\frac{7}{9}$

Consider $x = \frac{2}{3}$

$P(\frac{2}{3}) = 18.(\frac{2}{3})^3 + 3.(\frac{2}{3})^2 - 28 \times \frac{2}{3} + 12.$

$= 18 \times \frac{8}{27} + 3 \times \frac{4}{9} - \frac{56}{3} + 12.$

$= \frac{16}{3} + \frac{4}{3} - \frac{56}{3} + 12.$

$= \frac{16+4-56+36}{3}$

$= 0.$

ie $P(\frac{2}{3}) = P'(\frac{2}{3}) = 0$

$\therefore$ Roots are $\frac{2}{3}, \frac{2}{3}$ and α.

where $\sum$ roots $= \frac{2}{3} + \frac{2}{3} + \alpha = -\frac{3}{18} = \frac{-1}{6}$.

$\therefore \frac{4}{3} + \alpha = \frac{-1}{6}$

$\alpha = -\frac{9}{6} = -1\frac{1}{2}$

(c) $$f(x) = \frac{11-x}{x^2-x-2} = \frac{11-x}{(x-2)(x+1)}$$

$$f'(x) = \frac{(x^2-x-2)(-1) - (11-x)(2x-1)}{(x-2)^2(x+1)^2}$$

$$= \frac{-x^2+x+2 - (22x - 11 + x - 2x^2)}{(x-2)^2(x+1)^2}$$

$$= \frac{-x^2+x+2 - (23x - 11 - 2x^2)}{(x-2)^2(x+1)^2}$$

$$= \frac{x^2 - 22x + 13}{(x-2)^2(x+1)^2}$$

For $f'(x) = 0$

$$x = \frac{22 \pm \sqrt{484 - 52}}{2}$$

$$= \frac{22 \pm \sqrt{432}}{2}$$

$$= \frac{22 \pm 12\sqrt{3}}{2}$$

$$= 11 \pm 6\sqrt{3}.$$

$$\doteqdot 21.4,\ 0.6.$$

$\therefore$ st. pts at $(0.6, -4.6)$ & $(21.4, -0.02)$

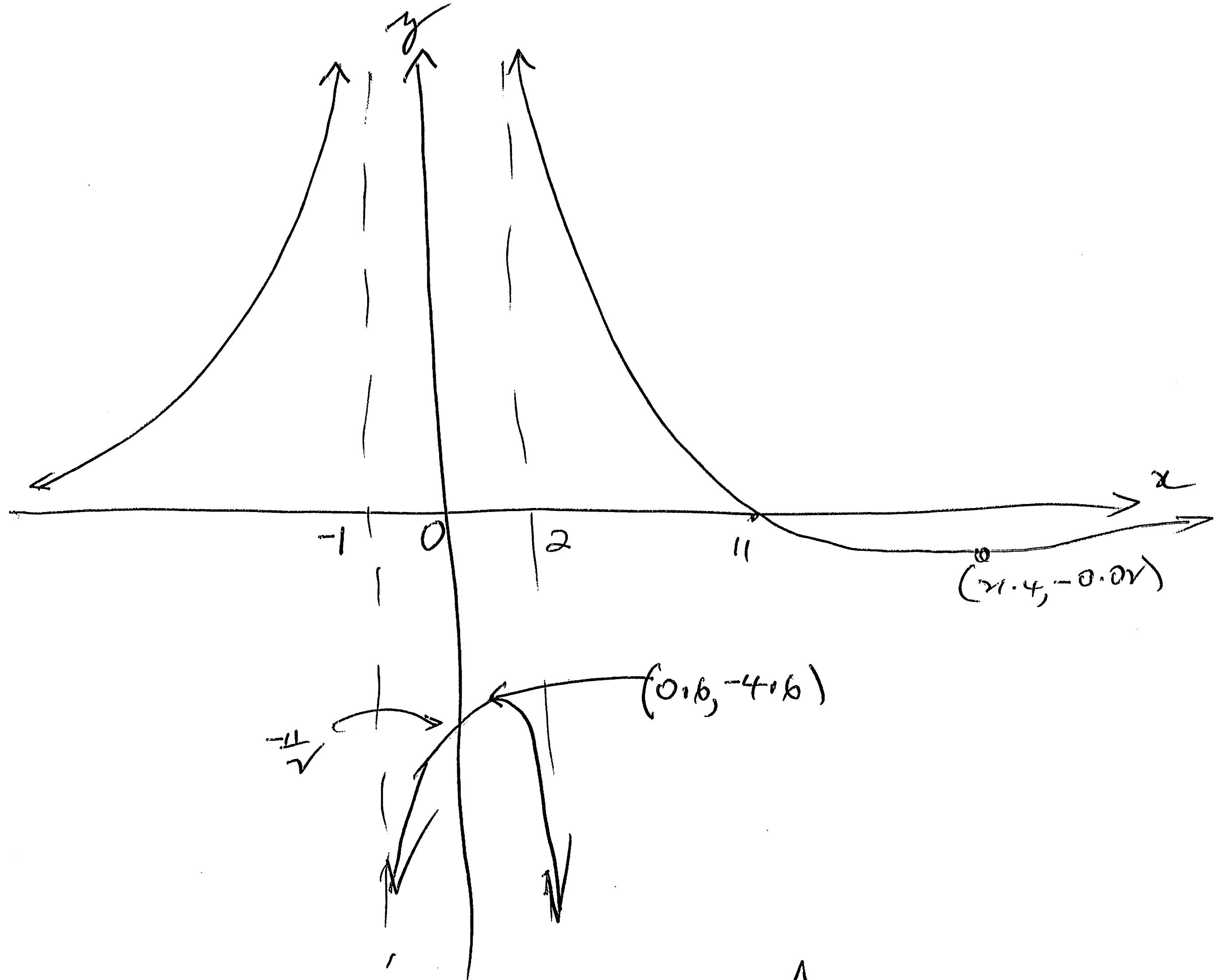
<u>Testing</u>

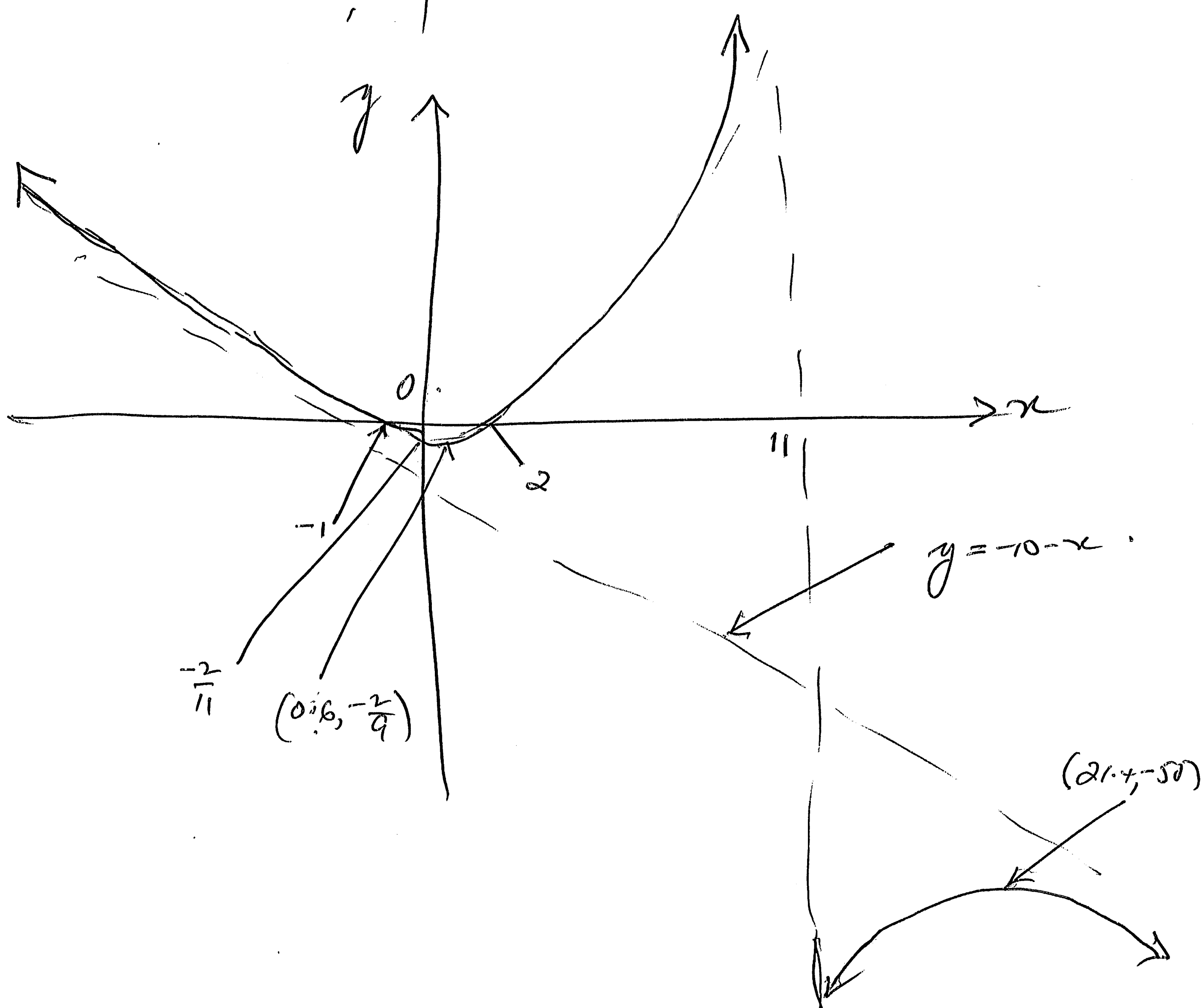
x	0	0.6	1
y'	$\frac{13}{4}$	0	$-2.$
	/	—	\

$\therefore$ REL. MAX at $(0.6, -4.6)$

x	21	21.4	22
y'	$-\cancel{.0009}$ -4.6×10^{-5}	0	6.14×10^{-5}
	\	—	/

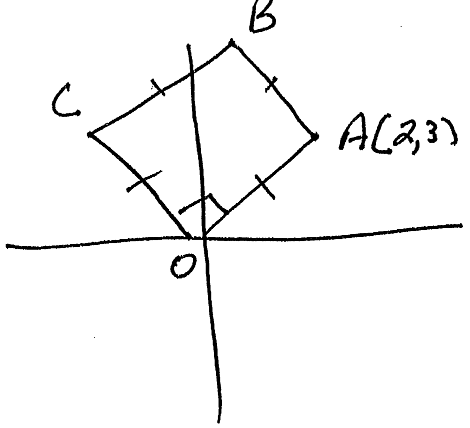
REL MIN at $(21.4, -0.02)$


y
x
-1
0
2
11
(21.4, -0.02)
(0.6, -4.6)
-11/2


y
x
0
11
-1
2
-2/11
(0.6, -2/9)
y = -10 - x
(21.4, -50)

QUESTION 4.

(a)



(i) C is given by.

$i(2+3i) = \boxed{-3+2i}$

(ii) B is given by.

$-3+2i+2+3i = \boxed{-1+5i}$

(iii) $A' = \text{cis}\,45^\circ(2+3i)$

$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(2+3i)$

$= \frac{1}{\sqrt{2}}(1+i)(2+3i)$

$= \frac{1}{\sqrt{2}}(-1+5i)$

$= \boxed{-\frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i. \quad \text{OR } \left(-\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i\right)}$

$B' = \frac{1}{\sqrt{2}}(1+i)(-1+5i)$

$= \frac{1}{\sqrt{2}}(-6+4i)$

$= \boxed{\frac{-6}{\sqrt{2}} + \frac{4}{\sqrt{2}}i. \quad \text{OR } (-3\sqrt{2}+2\sqrt{2}i)}$

$C' = \frac{1}{\sqrt{2}}(1+i)(-3+2i)$

$= \frac{1}{\sqrt{2}}(-5 - i)$

$= \boxed{-\frac{5}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \quad \text{OR } \left(-\frac{5\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)}$

(b) $\int_0^a f(x)\,dx = \int_a^0 f(a-t).-dt$ let $x = a-t$, $dx = -dt$.

$= -\int_a^0 f(a-t)\,dt$

$= \int_0^a f(a-t)\,dt$

$= \int_0^a f(a-x)\,dx$

(ii) $\int_0^2 x^2(2-x)^9\,dx = \int_0^2 (2-x)^2 \cdot x^9\,dx$

$= \int_0^2 (4-4x+x^2)x^9\,dx$

$= \int_0^2 (4x^9-4x^{10}+x^{11})\,dx$

$= \left[\frac{4x^{10}}{10} - \frac{4x^{11}}{11} + \frac{x^{12}}{12}\right]_0^2$

$= \frac{2}{5}\cdot 2^{10} - \frac{4}{11}2^{11} + \frac{2^{12}}{12}$

$= 2^{10}\left[\frac{2}{5} - \frac{8}{11} + \frac{1}{3}\right]$

$= 2^{10} \times \frac{66-120+55}{165}$

$= 2^{10} \times \frac{121-120}{165}$

$= \frac{2^{10}}{165}$

$= \frac{1024}{165}$

$= 6\frac{34}{165}$ ✓✓

(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\frac{1}{2}(36-4x^2)$

$= \frac{d}{dx}(18-2x^2)$

$= -4x$

$\therefore \ddot{x} \propto x \quad \therefore$ S.H.M.

is of the form $\ddot{x} = -n^2x$ ✓

(ii) $T = \frac{2\pi}{n}$

where $n = 2$ ✓

$T = \frac{2\pi}{2}$

$= \pi$ secs ✓

(d) Let $z^5 = 1$.

$(\cos\theta + i\sin\theta)^5 = 1$.

$\cos 5\theta + i\sin 5\theta = 1 + 0i$

$\therefore \cos 5\theta = 1$

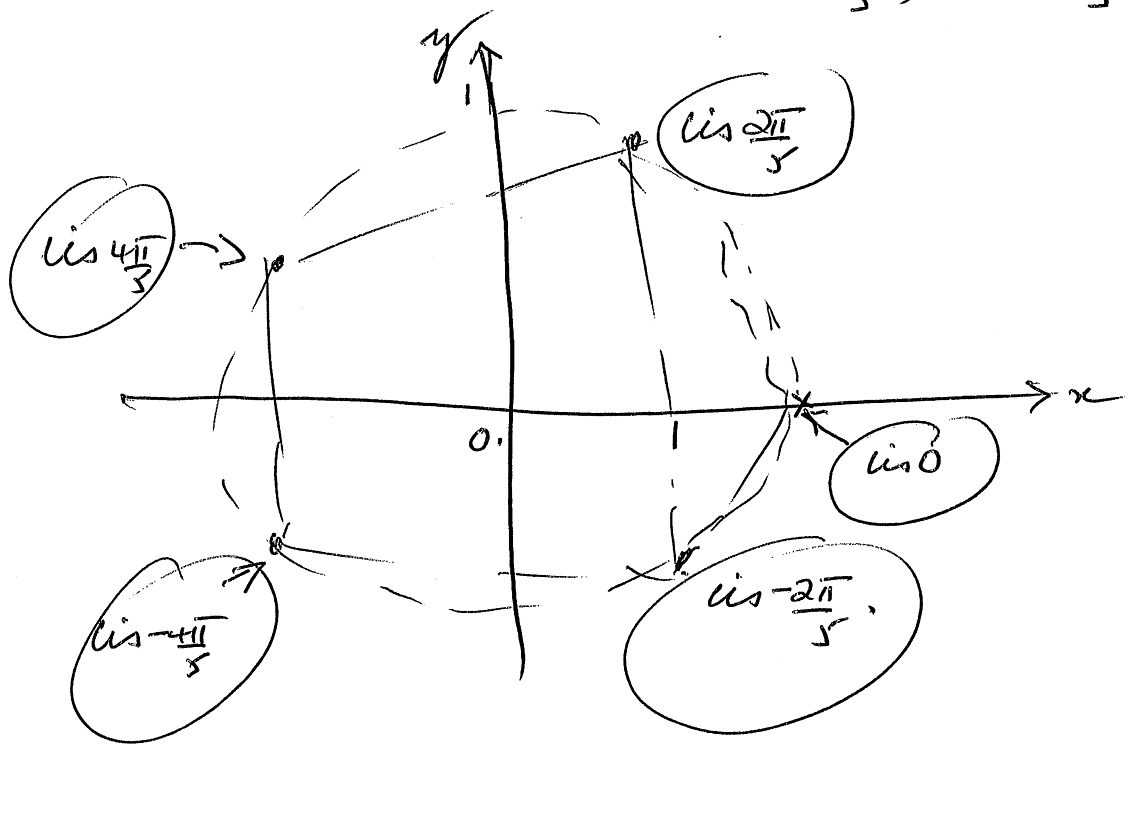
$5\theta = 2k\pi \quad , \quad k \in \mathbb{Z}$.

$\theta = \frac{2k\pi}{5}$.

$= 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$.

[where
$z = r(\cos\theta + i\sin\theta)$.
now $|z| = r$.
$|z^5| = |z|^5 = 1$
$\therefore |z| = r$
$= 1$.]

$\therefore$ roots are $\text{cis}\, 0$, $\text{cis} \pm\frac{2\pi}{5}$, $\text{cis} \pm\frac{4\pi}{5}$.



SECTION C

QUESTION 5

(a) $z^2 + (1+2i)z - (2-4i) = 0$

$$z = \frac{-(1+2i) \pm \sqrt{5-12i}}{2}$$

let $5 - 12i = (a+ib)^2$

$5 - 12i = (a^2 - b^2) + 2abi$

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2+4) = 0$

$a = \pm 3 \Rightarrow b = \mp 2$

3

$$\therefore \quad z = \frac{-(1+2i) \pm (\pm 3 \mp 2i)}{2}$$

$\Rightarrow z = 1 - 2i$ or $z = -2$

(b) $I_n = \int_0^{\pi/4} \tan^n x \, dx$

(i) $\therefore I_1 = \int_0^{\pi/4} \tan x \, dx$ 1

$= \Big[\ln(\sec x)\Big]_0^{\pi/4}$

$= \ln\sqrt{2}$ or $\frac{1}{2}\ln 2$

(ii) $I_n = \int_0^{\pi/4} \tan^n x \, dx$ 2

$= \int_0^{\pi/4} \tan^{n-2} x . \tan^2 x \, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$

$I_n = \frac{1}{n-1}\Big[\tan^{n-1} x\Big]_0^{\pi/4} - \Big[I_{n-2}\Big]$

$\therefore I_n = \frac{1}{n-1} - I_{n-2}$

$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$

(iii) $J = \int_0^{\pi/4} \tan^{11} x \, dx$

$I_n = \frac{1}{n-1} - I_{n-2}$

$\therefore I_{11} = \frac{1}{10} - I_9$

$= \frac{1}{10} - \left[\frac{1}{8} - I_7\right]$

$= \frac{1}{10} - \frac{1}{8} + \frac{1}{6} - I_5$ 2

$= \frac{1}{10} - \frac{1}{8} + \frac{1}{6} - \left[\frac{1}{4} - I_3\right]$

$= \frac{1}{10} - \frac{1}{8} + \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - I_1$

$= \frac{47}{120} - \frac{1}{2}\ln 2$

(d) Since real pol, if $2+i$ is a root $\Rightarrow$ $2-i$ also a root

Product of roots $= -15$

$\therefore (2-i)(2+i)\gamma = -15$

$5\gamma = -15$ 4

$\gamma = -3$

Roots $2+i, \ 2-i, \ -3$

(c)

$\cos 4\theta + i \sin 4\theta = (c + is)^4$ 2

$\cos 4\theta = \mathrm{Re}\left[(c+is)^4\right]$

$= \mathrm{Re}\left[c^4 + 4c^3 si - 6c^2 s^2 - 4cs^3 i + s^4\right]$

$= c^4 - 6c^2 s^2 + s^4$

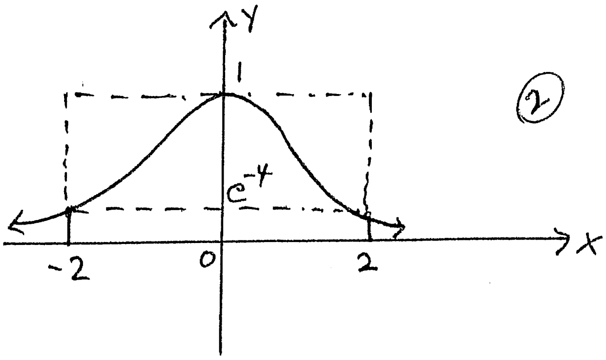
$\cos 4\theta = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + (1-c^2)^2$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

QUESTION 6

(a) $y = e^{-x^2}$

(i) 2



(ii) 2

$$\text{Area small Rect} < \int_{-2}^{2} e^{-x^2}\,dx < \text{Area large Rect}$$

$$4(e)^{-4} < \int_{-2}^{2} e^{-x^2} < 4(1)$$

$$\Rightarrow \quad 1 < \int_{-2}^{2} e^{-x^2} < 4$$

[NOTE: $4e^{-1} > 1$]

(iii) 2

-2 -1 0 1 2

$$\int_{-2}^{2} e^{-x^2}\,dx \doteqdot \frac{1}{3}\left[f(-2) + 4f(-1) + f(0)\right] + \frac{1}{3}\left[f(0) + 4f(1) + f(2)\right]$$

ie $\doteqdot 1.66$

(b) (i) $x = Vt\cos\theta$ 1 $y = Vt\sin\theta - \frac{gt^2}{2}$

(ii) Particle at P $\Rightarrow$ $x = -y$

since OM = MP 1

$$\Rightarrow Vt\cos\theta = \frac{gt^2}{2} - Vt\sin\theta$$

$$\therefore t = \frac{2V(\sin\theta + \cos\theta)}{g}$$

(iii) $OM = x = Vt\cos\theta$

$$= V\cos\theta \cdot \left[\frac{2V(\sin\theta + \cos\theta)}{g}\right]$$

$$= \frac{V^2}{g}\left[2(\sin\theta + \cos\theta)\cos\theta\right]$$

2

$$= \frac{V^2}{g}\left[\sin 2\theta + 2\cos^2\theta\right]$$

$$= \frac{V^2}{g}\left[\sin 2\theta + \cos 2\theta + 1\right]$$

(iv) Range $r = \frac{V^2}{g}\sin 2\theta$ 2

$$OM = \frac{4}{3}r = \frac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1)$$

$$\Rightarrow \frac{4}{3}\left(\frac{V^2}{g}\sin 2\theta\right) = \frac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1)$$

$$\therefore 4\sin 2\theta = 3\sin 2\theta + 3\cos 2\theta + 3$$

ie $\boxed{\sin 2\theta - 3\cos 2\theta = 3}$

Now $2\sin\theta\cos\theta = 3(1 + \cos 2\theta)$

$2\sin\theta\cos\theta = 3(2\cos^2\theta)$

$(\div \cos^2\theta)$

$$\Rightarrow \frac{2\sin\theta}{\cos\theta} = 6$$ 2

$$\therefore \boxed{\tan\theta = 3}$$