SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009
YEAR 12
ASSESSMENT TASK 2

Mathematics Extension 2

General Instruction

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answer in simplest exact form unless otherwise instructed

Total Marks – 105

- Attempt questions 1-6
- Hand up in 3 sections clearly marked A, B & C

Examiner: *C. Kourtesis*

Section A – (Start a new Booklet)

Question 1. (22 marks) Marks

(a) Find (i) $\int \frac{dx}{x^2+16}$ 3

(ii) $\int xe^x dx$

(b) Simplify $(15+2i)\overline{(2-i)}$ 2

(c) Find the value of k if k is real and $\frac{3+ki}{4-2i}$ is purely imaginary 2

(d) It is given that $|z|^2 = z + \bar{z}$. On an Argand diagram sketch the locus of the point P representing the complex number z. 2

(e) i) Factorise $z^3 - 1$ over the real numbers 1

ii) Solve $z^3 - 1$ over the complex numbers, expressing the complex roots in the form $a + ib$ where a and b are real. 2

iii) Hence solve $z^6 - 9z^3 + 8 = 0$ over the complex numbers 2

(f) The polynomial $16x^3 - 12x^2 + 1$ has a zero of multiplicity 2. Find all the zeros of the polynomial 3

(g) Find the gradient of the tangent to the curve $x^2 + xy + 2y^2 = 28$ at the point (2, 3) 3

(h) If α, β and γ are non-zero roots of $x^3 + px + q = 0$ find the cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ 2

End of Question 1.

Question 2. (19 marks) Marks

(a) Consider the function $f(x)=\frac{e^x}{x}$

i) Find the coordinates of any stationary points and determine their nature. 2

ii) Find the equations of any asymptotes. 2

iii) Sketch the graph of $y=f(x)$ 1

(b) Consider the function $f(x)=\frac{x^2+x+1}{x}$

i) Find the equation of any asymptotes. 2

ii) Determine whether there are any intercepts with the coordinate axes. 1

iii) Show that there is a minimum turning point at $x = 1$ and a maximum turning point at $x = -1$. 3

iv) Sketch the graph of $y = f(x)$. 1

(c) Consider the function $f(x)=x^{\frac{2}{3}}(x+5)$

i) Find the coordinates of the critical points and determine their nature. 4

ii) Find $f''(x)$ and test for any points of inflexion 2

iii) Sketch the graph of $y=f(x)$ 1

End of Question 2.

Section B – (Start a new Booklet)

Question 3. (16 marks) Marks

(a) i) A particle moves in a straight line. Prove that its acceleration $\ddot{x}$ at any instant is $\ddot{x} = v\dfrac{dv}{dx}$ where x denotes its position coordinate and v its velocity. 2

ii) A particle of mass m is projected vertically upwards. If the air resistance at any instant is given by mkv where v is the velocity and k a positive constant, briefly show with the aid of a diagram why the acceleration $\ddot{x}$ is given by 2

$$\ddot{x} = -(g + kv)$$

where g is the acceleration due to gravity.

4

iii) If the particle is projected vertically upwards with initial speed U prove that the time T taken to reach the maximum height is given by $T = \dfrac{1}{k}\log_e\left[1 + \dfrac{kU}{g}\right]$

4

iv) Prove that the maximum height H of the particle is given by $kH = U - gT$

4

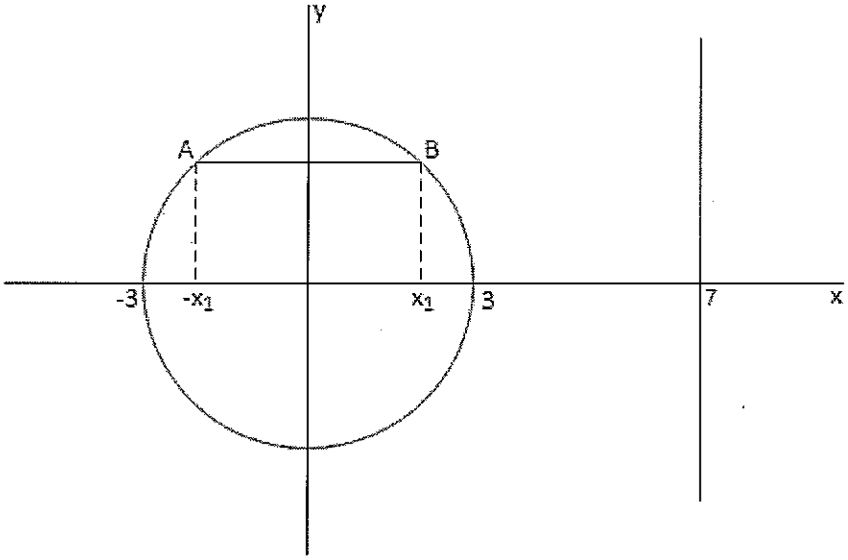
v) Show that the speed W with which the particle returns to its point of porojection is given by $k(U + W) = g\log_e\left[\dfrac{g + kU}{g - kW}\right]$.

End of Question 3.

Question 4. (15 marks) Marks

(a) The base of a certain solid lies in the xy – plane and is bounded by the curve $y = x^2$ and the line $y = 4$. Every cross–section perpendicular to the y axis is a square. Find the volume of the solid. 4

(b)



The circle $x^2 + y^2 = 9$ is rotated through 360° about the line $x = 7$ to form a ring. When the circle is rotated the line segment AB at height y sweeps out an annulus. The x coordinates of the end points of AB are $-x_1$ and x_1 where $x_1 = \sqrt{9 - y^2}$.

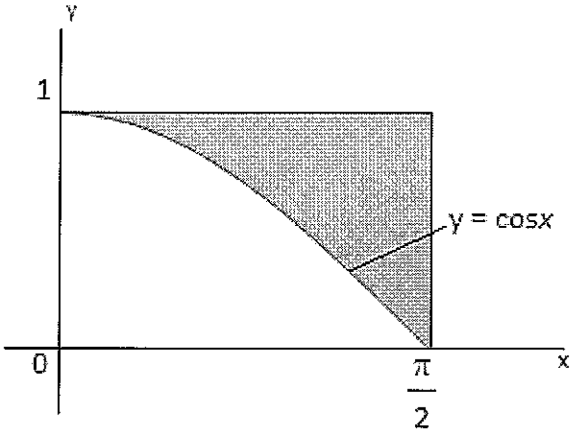
i) Show that the area of the annulus is 3

$$28\pi\sqrt{9 - y^2}$$

ii) Find the volume of the ring. 2

Question 4 continues on the next page.

(c) The region in the first quadrant bounded by the curve $y = \cos x$ and the lines $y = 1$ and $x = \frac{\pi}{2}$ is revolved about the line $x = \frac{\pi}{2}$.



i) Use the method of cylindrical shells to show that the resulting volume V is given by 2

$$V = 2\pi\int_0^{\frac{\pi}{2}}(1-\cos x)(\frac{\pi}{2}-x)\,dx$$

ii) Using the theorem $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ or otherwise show 4

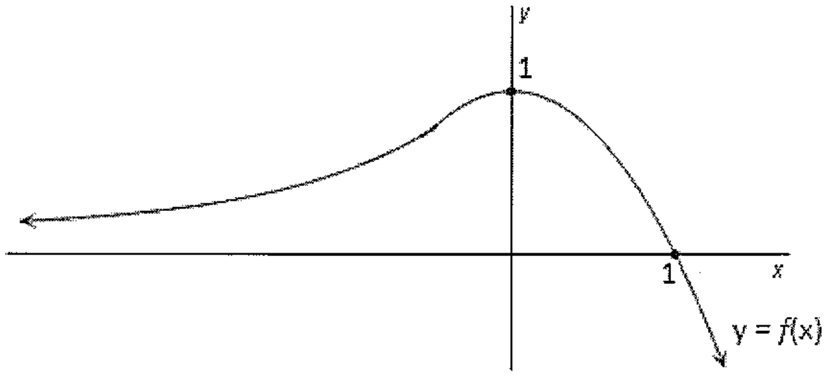
that $V = 2\pi\left(\frac{\pi^2}{8}-1\right)$

End of Question 4.

Section C – (Start a new Booklet)

Question 5. (16 marks) Marks

(a) The graph of y = f(x) is sketched below. There is a stationary point at (0, 1).



Use this graph to sketch the following, showing essential features in each case:

i) $y=-f(x)$ 1

ii) $y=|f(x)|$ 1

iii) $y=f\left(\frac{x}{2}\right)$ 1

iv) $y=\frac{1}{f(x)}$ 2

v) $y=f\left(\frac{1}{x}\right)$ 2

Question 5 continues on the next page.

(b) Let $z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

i) Find z^7 1

ii) Plot, on the Argand diagram, all complex numbers that are solutions of $z^7 = 1$. 2

iii) Show that $z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} = 0$ 2

iv) If k is a positive integer show that $z^k + z^{-k} = 2\cos\frac{2k\pi}{7}$ 2

v) Hence, or otherwise, show that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$ 2

End of Question 5.

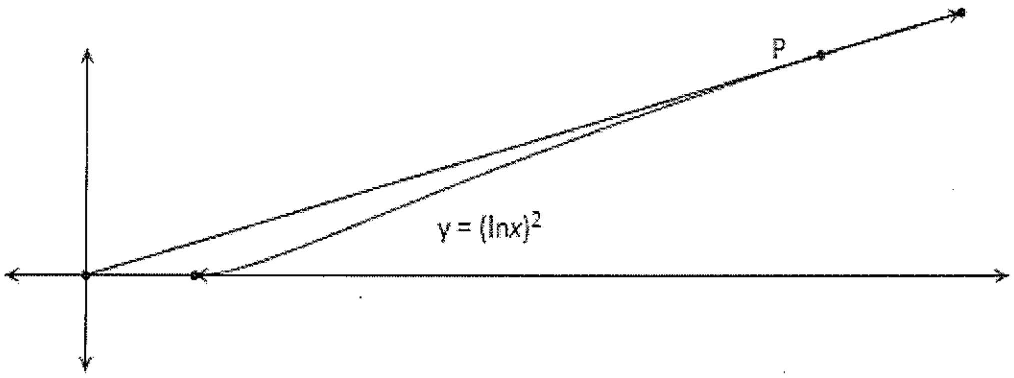
Question 6. (17 marks) Marks

(a) Beach volleyball is played with two teams where each team has two players.

i) In how many ways can four players be grouped in pairs to play a game of beach volleyball. 2

ii) The eight members of a beach volleyball club meet to play two games at the same time on two separate courts. In how many different ways can the club members be selected to play these two games? 3

(b)



The diagram shows the graph of the function $f(x) = (\ln x)^2$ for $x \geq 1$. P is a point on the curve such that the tangent to the curve at P passes through the origin.

i) Find the coordinates of P. 2

ii) Find the set of values of the real number k such that the equation $f(x) = kx$ has two distinct real roots. 1

(c) Prove that the equation $x^5 - 5cx + 1 = 0$ $(c < 0)$ has only one real root which is negative. 3

Question 6 continues on the next page.

(d)

Let $u_n = \int_0^{\frac{1}{2}} \frac{(\tan^{-1} 2x)^n}{1+4x^2} dx$

Where n is a positive integer.

i) Show that $u_n = \left(\frac{\pi}{4}\right)^{n+1} . \frac{1}{2(n+1)}$ 3

ii) Hence or otherwise show that 3

$$u_0 \times u_1 \times u_2 \times u_3 \times \ldots\ldots \times u_{2n-1} = \left(\frac{\pi}{4}\right)^{2n^2+n} . \frac{1}{2^{2n}.(2n)!}$$

End of Question 6.

End of the Examination.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

QUESTION. 1

(a) (i) $\int \frac{dx}{x^2+16}. = \boxed{\frac{1}{4}\tan^{-1}\frac{x}{4} + c.}$ ✓

(ii) $\int x e^x dx = \int x.\frac{d}{dx}(e^x).dx.$

$= xe^x - \int e^x dx.$

$= \boxed{xe^x - e^x + c.}$ ✓✓

(b) $(15+2i)\overline{(2-i)} = (15+2i)(2+i)$

$= 30 + 19i - 2.$

$= \boxed{28+19i}$ ✓✓

(c) $\frac{3+ki}{4-2i} = \frac{3+ki}{4-2i} \times \frac{4+2i}{4+2i}$

$= \frac{12+6i+4ki-2k.}{16+4}$

$=. \frac{12-2k+i(6+4k)}{20.}$

now if Real part is zero.

$\frac{12-2k}{20} = 0$

$\boxed{k=6.}$ ✓✓

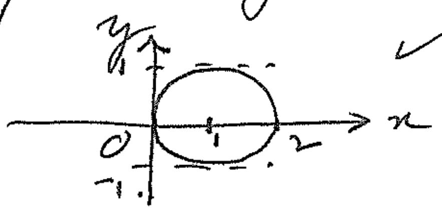
(d) $|z^2| = z+\bar{z}$ Ⓐ

Let $z = x+iy$.

Ⓐ becomes $x^2+y^2 = 2x.$

$x^2-2x+y^2=0$

$(x-1)^2+y^2=1.$ ✓✓



(e) (i) $z^3-1 = \boxed{(z-1)(z^2+z+1)}$ ✓

(ii) $z^3-1=0$

$z = 1, \frac{-1\pm\sqrt{1-4}}{2}$

$= 1, \frac{-1\pm i\sqrt{3}}{2}$

$\therefore \boxed{z = 1+0i, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}}$ ✓✓

(iii) $z^6 - 9z^3 + 8 = 0$

$(z^3-8)(z^3-1) = 0$

CONSIDER

$z^3-8=0$

$(z-2)(z^2+2z+4)=0$

$z = 2, \frac{-2\pm\sqrt{4-16}}{2}$

$= 2, \frac{-2\pm 2i\sqrt{3}}{2}$

$= 2, -1\pm i\sqrt{3}.$

$\therefore \boxed{z = 1, \frac{-1\pm i\sqrt{3}}{2}, 2, -1\pm i\sqrt{3}.}$ ✓✓

(f) $P(x) = 16x^3-12x^2+1$

$P'(x) = 48x^2-24x.$

$= 24x(2x-1)$

now if $P'(x) = 0$

$x = 0, \frac{1}{2}.$

$P(0) \neq 0$ $\quad P(\frac{1}{2}) = \frac{16}{8} - \frac{12}{4} + 1$

$= 2 - 3 + 1$

$= 0.$

$\therefore P(\frac{1}{2}) = P'(\frac{1}{2}) = 0$

$\therefore (2x-1)^2$ is a factor.

OR

$\sum$ Roots are

$\frac{1}{2} + \frac{1}{2} + \alpha = \frac{12}{16}.$

$= \frac{3}{4}$

$\boxed{\alpha = -\frac{1}{4}}$

(f) CONTD.

$\therefore$ zeros are ✓✓✓✓

$$\boxed{\frac{1}{2}, \frac{1}{2} \text{ and } -\frac{1}{4}.}$$

(g) given $x^2 + xy + 2y^2 = 28.$

differentiate both sides.

$$2x + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dt}(x + 4y) = -(y + 2x)$$

$$\frac{dy}{dx} = -\frac{(y+2x)}{x+4y}.$$

✓✓

at $(2, 3)$

$$\text{slope} = -\frac{(3+4)}{2+12}.$$

$$= -\frac{7}{14}$$

✓

$$= \boxed{-\frac{1}{2}.}$$

(h) given $x^3 + px + q = 0.$ Ⓐ

Let $X = \frac{1}{x} \Rightarrow x = \frac{1}{X}.$

$\therefore$ Ⓐ becomes $\left(\frac{1}{X}\right)^3 + P\left(\frac{1}{X}\right) + q = 0$

$$\boxed{1 + pX^2 + qX^3 = 0}$$

OR $\boxed{qx^3 + px^2 + 1 = 0}$ ✓✓

QUESTION 2.

(a) (i) $f(x) = \frac{e^x}{x}$.

$$f'(x) = \frac{xe^x - e^x}{x^2}$$

$$f''(x) = \frac{x^2[xe^x + e^x - e^x] - 2x(xe^x - e^x)}{x^4}$$

$$= \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4}$$

$$= \frac{xe^x[x^2 - 2x + 2]}{x^4}$$

$$= \frac{e^x(x^2 - 2x + 2)}{x^3}$$

For stationary points $f'(x) = 0$.

$$\frac{e^x(x-1)}{x^2} = 0.$$

$x = 1$, $y = e$

Testing

x	$\frac{1}{2}$	1	2
y'	$\frac{-e^{\frac{1}{2}}}{1/4}$	0	$\frac{e^2}{4}$

-ve +ve.

\ — /

✓ ✓ $\therefore (1, e)$ is a minimum

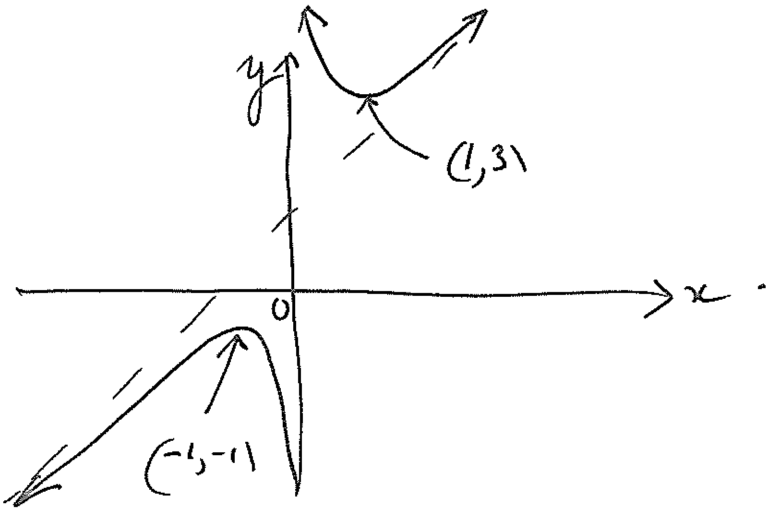
(ii) Undefined at $x = 0$ y-axis or $x = 0$ is vertical asymptote. ✓

as $x \to -\infty$ $y \to 0$. $\therefore$ x-axis or $y = 0, x < 0$ is a horizontal asymptote ✓

at $(1,3)$ $f''(1) = 2$ $\therefore$ MIN. ✓ ✓

at $(-1,-1)$ $f''(-1) = -2$ $\therefore$ MAX. ✓

(IV)



✓

(c) (i). $f(x) = x^{\frac{2}{3}}(x+5)$

$$= x^{5/3} + 5x^{2/3}$$

$$f'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-\frac{1}{3}}$$

$$= \frac{5}{3}\left(x^{2/3} + 2x^{-\frac{1}{3}}\right)$$

$$= \frac{5}{3}\left(\frac{x+2}{x^{\frac{1}{3}}}\right)$$

✓

$\therefore$ Critical points at $x = -2$ ie $(-2, +3\sqrt[3]{4})$

and $x = 0$ $(0,0)$ ✓

Consider $(-2, 3\sqrt[3]{4})$

x	-3	-2	-1
y'	$\cdot$	0	$-5/3$
	/	—	\

$\therefore$ REL MAX. ✓

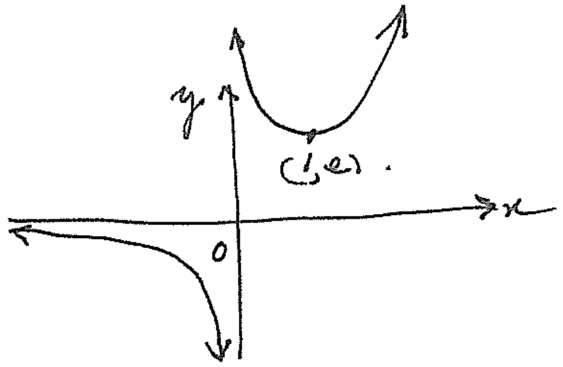
Consider $(0,0)$

x	-1	0	1
y'	$-5/3$	∞	5

$\therefore$ REL MIN. ✓

NB This is a cusp (NOT DIFF'BLE at $x=0$)

(iii)



✓

(b) $f(x) = \frac{x^2+x+1}{x}$ OR $x+1+\frac{1}{x}$.

(i) $f(x) \to \infty$ as $x \to 0^+$.
$f(x) \to -\infty$ as $x \to 0^-$ $\therefore$ $\boxed{x=0}$ OR y-axis is vertical asymptote ✓

Also. as $x \to \infty$. $f(x) \to x+1$ $\therefore$ $\boxed{f(x) = x+1}$ is an oblique asymptote. ✓

(ii) Undefined at $x=0$ $\therefore$ NO y-intercept

at $y=0$ $x^2+x+1=0$ → $\boxed{\text{Could use } \Delta = -3.}$ ✓

$x = \frac{-1 \pm \sqrt{1-4}}{2}$. (NO REAL ROOTS)

$\therefore$ NO x-intercept

(iii) $f'(x) = 1 - \frac{1}{x^2}$
$= \frac{x^2-1}{x^2} = 1 - x^{-2}$.

$\therefore f'(x) = 0$ when $x = \pm 1$,

$f''(x) = \frac{2}{x^3}$

(ii) $f'(x) = \frac{5}{3}\left(x^{\frac{2}{3}} + 2x^{-\frac{1}{3}}\right)$

$$f''(x) = \frac{5}{3}\left(\frac{2}{3}x^{-\frac{1}{3}} - \frac{2}{3}\,x^{-\frac{4}{3}}\right)$$

$$= \frac{10}{9}\,x^{-\frac{4}{3}}(x-1)$$

∴ possible inflexion at $x = 0, 1$.

Test $x = 0$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$.
y''	$\frac{10}{9(\frac{-1}{2})^{-\frac{4}{3}}} \times \frac{-3}{2}$ NEG.	undef.	$\frac{10}{9(\frac{+1}{2})^{\frac{4}{3}}} \times \frac{-1}{2}$. NEG.

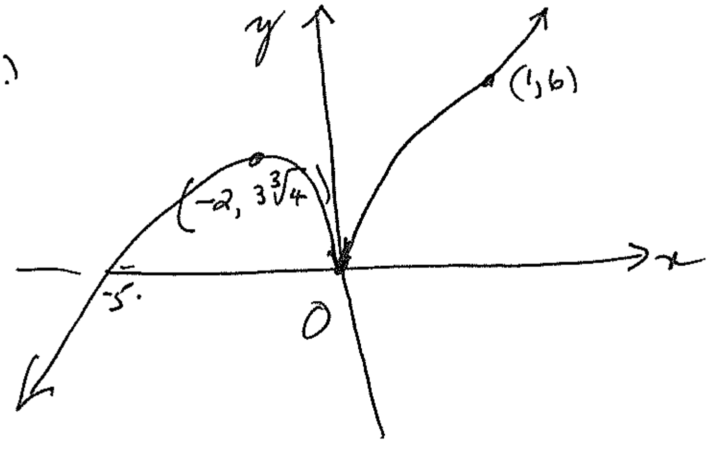
∴ NO CHANGE IN CONCAVITY. NOT AN INFLEXION

Test $x = 1$.

x	$\frac{1}{2}$	1	$1\frac{1}{2}$
y''	$\frac{10}{9}(\frac{1}{2})^{-\frac{4}{3}} \times -\frac{1}{2}$ NEG.	0.	$\frac{10}{9}(\frac{3}{2})^{-\frac{4}{3}} \times \frac{1}{2}$ POS

CHANGE IN CONCAVITY at $(1,6)$ ∴ $(1,6)$ is a point of inflexion

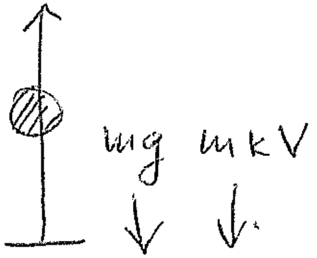
(iii)



Question (3)

(i) $$\ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

(ii)



$$m\ddot{x} = -mg - mkV$$
$$\ddot{x} = -(g+kV)$$

(iii) $$\frac{dv}{dt} = -(g+kv)$$
$$\frac{dt}{dv} = -\frac{1}{g+kv}$$
$$\therefore \int_0^T dt = -\frac{1}{k}\int_U^0 \frac{k\,dv}{g+kv}$$
$$\therefore T = \frac{1}{k}\ln\left|1+\frac{kU}{g}\right|$$

(iv) $$v\frac{dv}{dx} = -(g+kv)$$
$$\therefore \frac{dx}{dv} = -\frac{v}{g+kV}$$
$$\therefore \int dx = \int \frac{-v\,dv}{g+kv}$$
$$= \int\left(-\frac{1}{k} + \frac{g}{k^2}\,\frac{k}{g+kv}\right)dv$$
$$\therefore x = -\frac{v}{k} + \frac{g}{k^2}\ln(g+kv) + C$$

When $x=0$ $v=U$

$$\therefore C = \frac{U}{k} - \frac{g}{k^2}\ln(g+kU)$$
$$\therefore x = \frac{U-v}{k} + \frac{g}{k^2}\ln(g+kU)$$

When $x=H$, $v=0$

$$H = \frac{U}{k} + \frac{g}{k^2}\ln(g+kU)$$
$$= \frac{U}{k} + \frac{g}{k^2}\ln\left[\left(\frac{g+kU}{g}\right)^{-1}\right]$$
$$= \frac{U}{k} - \frac{g}{k^2}\ln\left(1+\frac{k}{g}U\right)$$

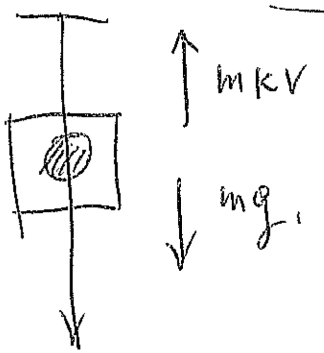
$\therefore$

$$kH = U - \frac{g}{k}\ln\left(1+\frac{kU}{g}\right)$$
$$= U - g\left[\frac{1}{k}\ln\left(1+\frac{kU}{g}\right)\right]$$

i.e $kH = U - gT$.

4 3 2 2 4

(v)



[4]

$$v\frac{dv}{dx} = g - kv$$

$$\therefore \frac{dv}{dx} = \frac{g-kv}{v}$$

$$\therefore dx = \int \frac{v\,dv}{g-kv}$$

$$\boxed{\begin{array}{l} \quad\quad\; -\frac{1}{k} \\ g-kv \,\overline{)\; v \quad\quad\quad} \\ \quad\; -\left(-\frac{g}{k}+v\right) \\ \hline \quad\quad\; g/k \end{array}}$$

$$\therefore x = \int\left[-\frac{1}{k} - \frac{g}{k^2}\left(\frac{-k}{g-kv}\right)\right]dv$$

$$= -\frac{v}{k} - \frac{g}{k^2}\ln(g-kv) + c$$

When $x=0$, $v=0$

$$\therefore 0 = -\frac{g}{k^2}\ln(g) + c$$

$$\Longrightarrow c = \frac{g}{k^2}\ln g$$

$$\therefore \boxed{x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}}$$

When $x = H$, $v = W$

$$\therefore \boxed{H = \frac{g}{k^2}\ln\left[\left(1-\frac{kW}{g}\right)^{-1}\right] - \frac{W}{k}}$$

$$\therefore H = -\frac{W}{k} - \frac{g}{k^2}\ln\left(1-\frac{kW}{g}\right)$$

$$\therefore kH = -W - \frac{g}{k}\ln\left(1-\frac{kW}{g}\right) \quad \text{———} ②$$

Equate ① & ②

$$U - \frac{g}{k}\ln\left(1+\frac{kU}{g}\right) = -W - \frac{g}{k}\ln\left(1-\frac{kW}{g}\right)$$

$$W+U = \frac{g}{k}\left[\ln\left(1+\frac{kU}{g}\right) - \ln\left(1-\frac{kW}{g}\right)\right]$$

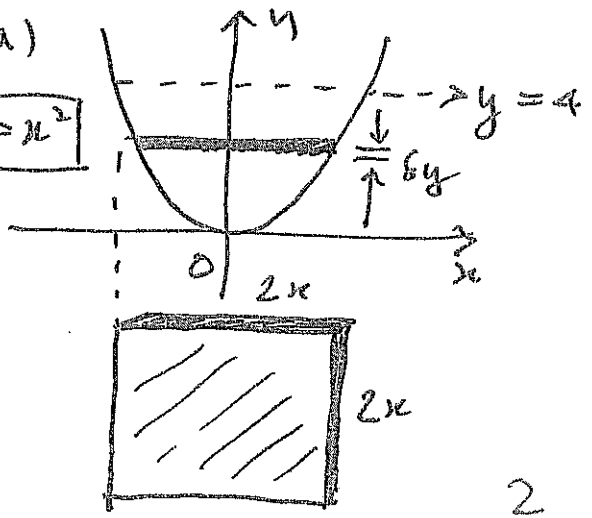
$$W+U = \frac{g}{k}\ln\left|\frac{1+\frac{kU}{g}}{1-\frac{kW}{g}}\right|$$

$$\therefore \boxed{k(W+U) = g\ln\left|\frac{g+kU}{g-kW}\right|} \quad \text{———} ③$$

Question (4)

(a)

$$\boxed{y = x^2}$$



2

$$\delta V = 4x^2\delta y.$$

$$V = \lim_{\delta y \to 0} 4\sum_0^4 x^2\delta y.$$

but $y = x^2$

$$\therefore V = 4\int_0^4 y\,dy.$$

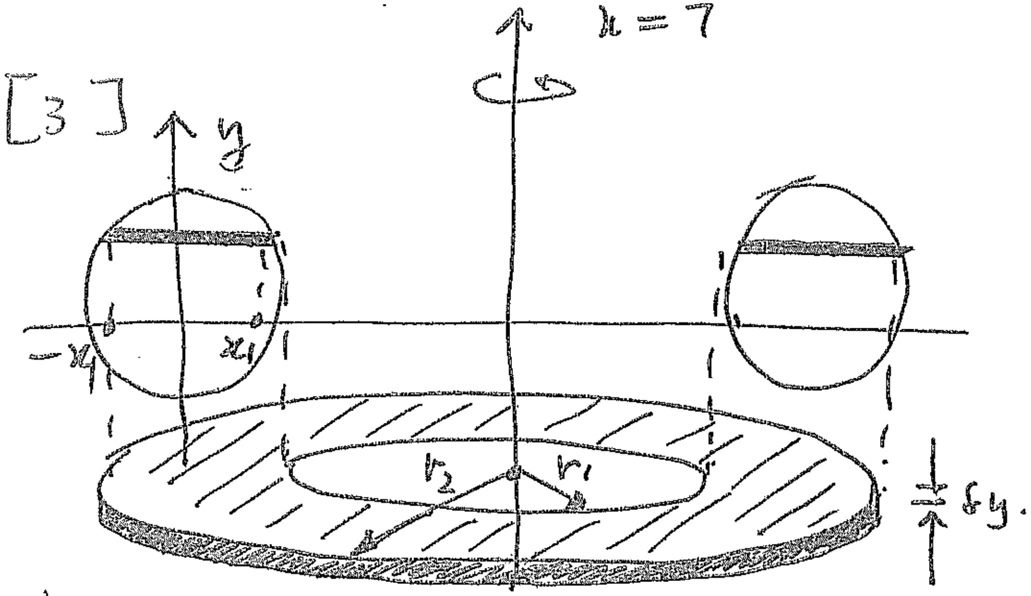
$$= \left[2y^2\right]_0^4 \quad 2$$

$$= \underline{32}.$$

[4]

Question 4(b)

[3]



$r_1 = 7 - x_1 \quad \delta V = \pi\left[r_2^2 - r_1^2\right]\delta y.$

$r_2 = 7 + x_1$

$\delta V = \pi\left[(7+x_1)^2 - (7-x_1)^2\right]\delta y$

$= \pi(28x_1)\delta y. \quad x_1 = \sqrt{9-y^2}$

$= 28\pi\sqrt{9-y^2}\,\delta y.$

$\therefore V = \lim_{\delta y \to 0} 28\pi \sum_{-3}^{3} \sqrt{9-y^2}\,\delta y.$

$= 28\pi \int_{-3}^{3} \sqrt{9-y^2}\,dy.$

[2]

$= 28\pi \times \frac{1}{2}\pi \times 9.$

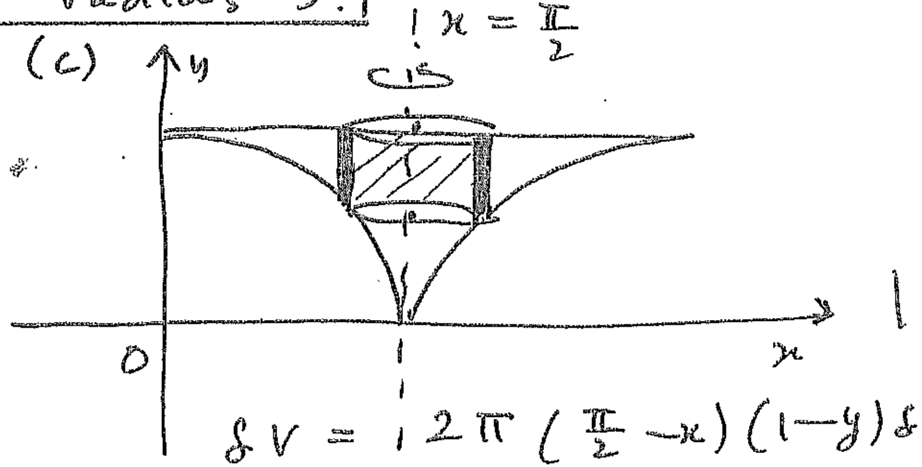
$= 126\pi$ cubic units

where $\int_{-3}^{3}\sqrt{9-y^2}\,dy$ is equivalent

* to find the area of a semi-ci

radius 3.

(c)



$\delta V = 2\pi\left(\frac{\pi}{2} - x\right)(1-y)\delta x$

$\because y = \cos x$

$\therefore V = \lim_{\delta x \to 0} 2\pi \sum_{0}^{\pi/2} (1-\cos x)\left(\frac{\pi}{2} - x\right)dx$

[2] $\therefore V = 2\pi\int_{0}^{\pi/2}(1-\cos x)\left(\frac{\pi}{2} - x\right)dx.$

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore V = 2\pi\int_0^{\pi/2}\left[1-\cos\left(\frac{\pi}{2}-x\right)\right]\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right]$

$= 2\pi\int_0^{\pi/2}(x - x\sin x)\,dx$

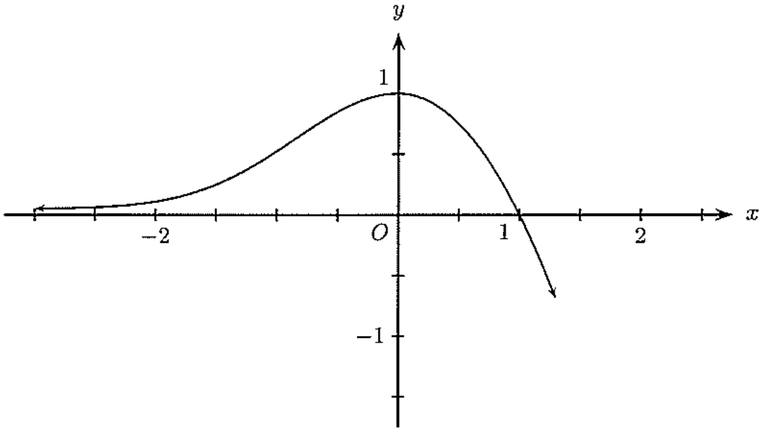
$= \left[\pi x^2\right]_0^{\pi/2} - 2\pi\int_0^{\pi/2} x\frac{d}{dx}(-\cos x)\,dx$

[4] $= \frac{\pi^3}{4} + \left[-2\pi x\cos x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2}\cos x\,dx$

$= \pi^3/4 - 2\pi = \boxed{2\pi\left(\frac{\pi^2}{} - 1\right)}$

2009 Mathematics Extension 2 Assessment 2: **Section C** solutions

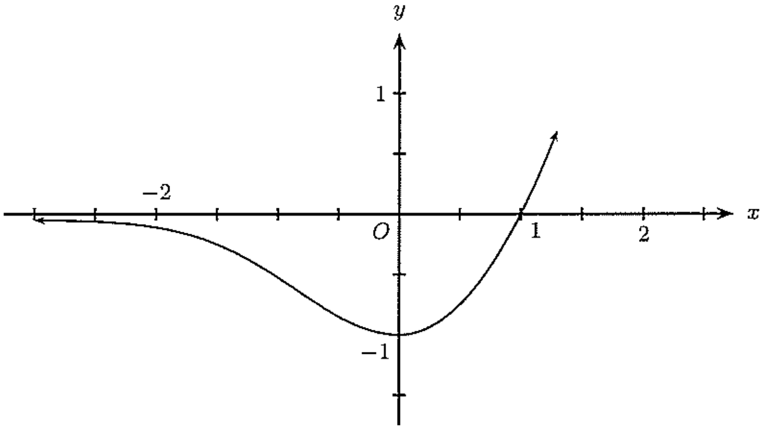
5. (a) The graph of $y = f(x)$ is sketched below. There is a stationary point at (0, 1).



Use this graph to sketch the following, showing essential features in each case:

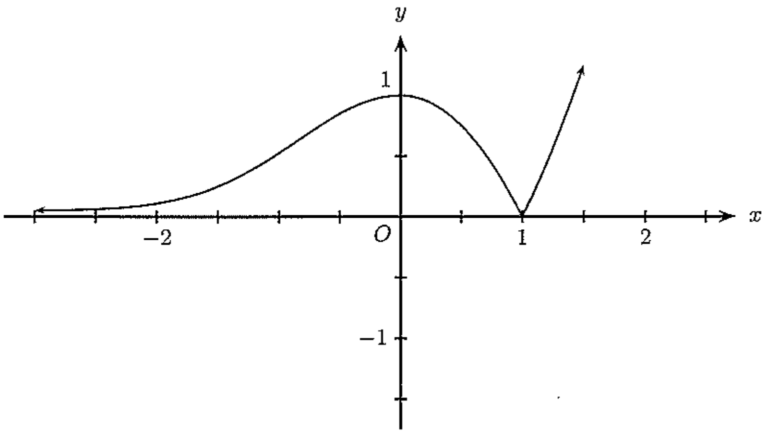
i) $y = -f(x)$ $\boxed{1}$

Solution:



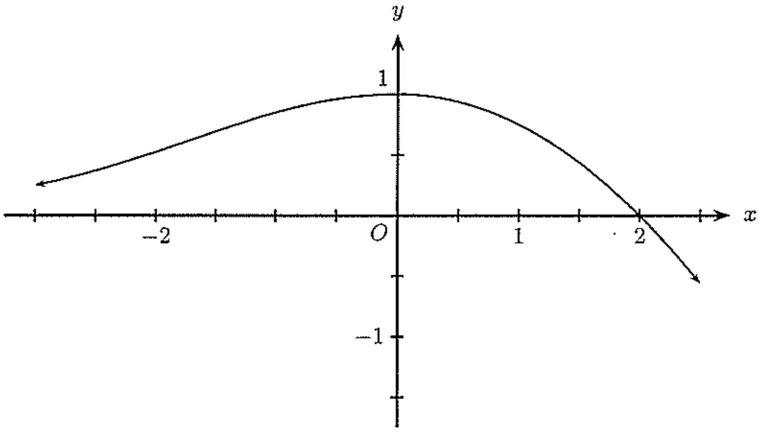
ii) $y = |f(x)|$ $\boxed{1}$

Solution:



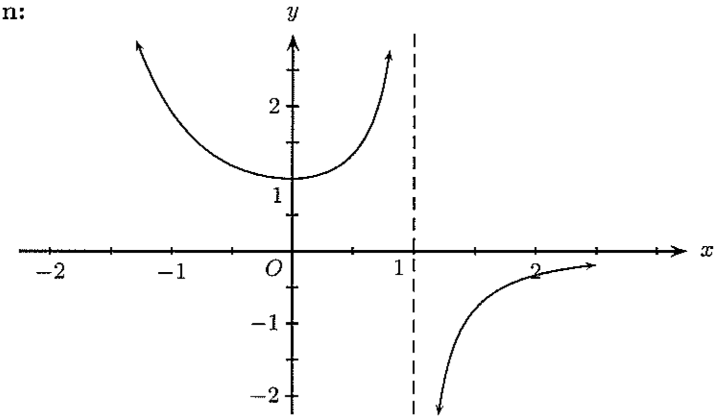
iii) $y = f\left(\frac{x}{2}\right)$ $\boxed{1}$

Solution:



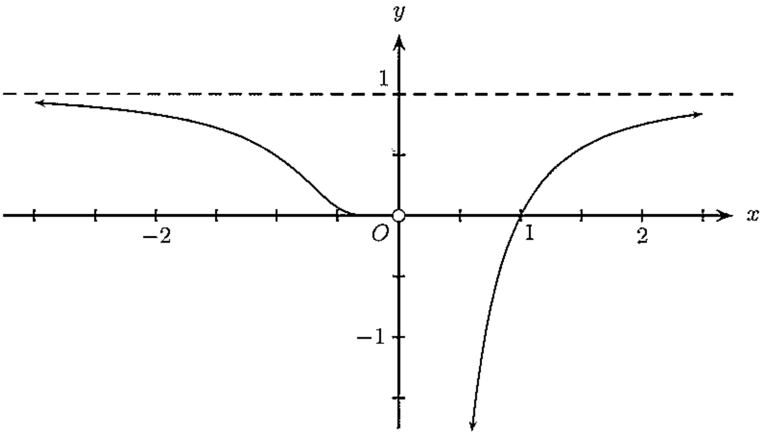
iv) $y = \frac{1}{f(x)}$ $\boxed{2}$

Solution:



v) $y = f\left(\frac{1}{x}\right)$ $\boxed{2}$

Solution:



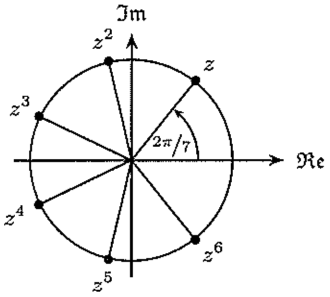
(b) Let $z = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$.

i) Find z^7. $\boxed{1}$

Solution:
$$\begin{aligned} z^7 &= \left(\operatorname{cis}\frac{2\pi}{7}\right)^7, \\ &= \operatorname{cis}2\pi, \text{ by de Moivre's theorem,} \\ &= 1. \end{aligned}$$

ii) Plot, on the Argand diagram, all complex numbers that are solutions of $z^7 = 1$. $\boxed{2}$

Solution:



iii) Show that $z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} = 0$. $\boxed{2}$

Solution:
$$\begin{aligned} z^7 - 1 &= 0, \\ \therefore\ 0 &= (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1), \end{aligned}$$
So $z = 1$ or $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$.
Hence, dividing the complex part by z^3,
$z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} = 0$.

iv) If k is a positive integer, show that $z^k + z^{-k} = 2\cos\dfrac{2k\pi}{7}$ $\boxed{2}$

Solution:
$$\begin{aligned} z^k &= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}. \\ z^{-k} &= \cos\frac{-2k\pi}{7} + i\sin\frac{-2k\pi}{7}, \\ &= \cos\frac{2k\pi}{7} - i\sin\frac{2k\pi}{7}. \end{aligned}$$
Adding, $z^k + z^{-k} = 2\cos\dfrac{2k\pi}{7}$.

v) Hence, or otherwise, show that $\cos\dfrac{\pi}{7} - \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} = \dfrac{1}{2}$. $\boxed{2}$

Solution: From (iii) above:
$$\begin{aligned} z^3 + z^{-3} + z^2 + z^{-2} + z + z^{-1} + 1 &= 0, \\ 2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} + 1 &= 0, \\ -\cos\frac{2\pi}{7} - \cos\frac{4\pi}{7} - \cos\frac{6\pi}{7} &= \frac{1}{2}, \\ -\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} &= \frac{1}{2}, \\ \textit{i.e.}\ \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} &= \frac{1}{2}. \end{aligned}$$

6. (a) Beach volleyball is played with two teams where each team has two players.

i) In how many ways can four players be grouped in pairs to play a game of beach volleyball? $\boxed{2}$

Solution: METHOD 1—
The number of ways of choosing 2 from 4 is ${}^4C_2 = 6$.
If the players are A, B, C and D, then $(A, B) \leftrightarrow (C, D)$ is the same as $(C, D) \leftrightarrow (A, B)$.

$$\therefore\ \tfrac{1}{2} \times {}^4C_2 = \frac{6}{2}, \\ = 3.$$

METHOD 2—
One player can have three possible partners leaving the other two as the other team.
$\therefore$ 3 selections.

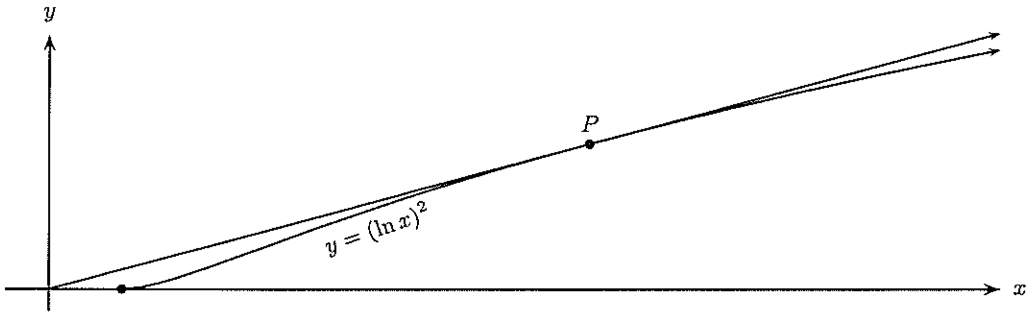
ii) The eight members of a beach volleyball club meet to play two games at the same time on different courts. In how many different ways can the club members be selected to play these two games? $\boxed{3}$

Solution: METHOD 1—
There are 4 groups of 2 to be selected,

$$\therefore \text{ number of combinations} = \frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!}, \\ = 105.$$

Let (A, B), (C, D), (E, F), (G, H) be one set of combinations.
Now (A, B) can play any 3 of the others, leaving the other two pairs to play each other.
$\therefore$ $105 \times 3 = 315$ different selections.
METHOD 2—
One player has seven possible partners for his team, then five for the next and then three.
$\therefore$ $7 \times 5 \times 3 = 105$ possible teams.
One team can play any of three teams leaving the other two for the other court.
$\therefore$ $105 \times 3 = 315$ different selections.

(b)



The diagram shows the graph of the function $f(x) = (\ln x)^2$ for $x \geqslant 1$.
P is a point on the curve such that the tangent at the point P passes through the origin.

i) Find the coördinates of P. $\boxed{2}$

Solution:

$$\begin{aligned} f(x) &= (\ln x)^2, \\ f'(x) &= \frac{2}{x}.\ln x = k \text{ at } P \text{ (intersection of } f(x) \text{ with } kx\text{)}, \\ y - (\ln x_p)^2 &= \frac{2}{x_p}.\ln x_p(x - x_p) \text{ is the tangent at } P. \end{aligned}$$

This passes through the origin when

$$(\ln x_p)^2 - 2\ln x_p = 0,$$

i.e., $\ln x_p = 0$ or $\ln x_p = 2$.

Clearly P is not at $(1, 0)$, $\therefore$ P is $(e^2, 4)$.

ii) Find the set of values of the real number k such that the equation $f(x) = kx$ has two distinct real roots. $\boxed{1}$

Solution: At P, $k = \dfrac{4}{e^2}$; at $(1, 0)$, $k = 0$,

$\therefore\ 0 < k < \dfrac{4}{e^2}$.

(c) Prove that the equation $x^5 - 5cx + 1 = 0$ $(c < 0)$ has only one real root which is negative. $\boxed{3}$

Solution:

$$\begin{aligned} P(x) &= x^5 - 5cx + 1 = 0, \\ P'(x) &= 5x^4 - 5cx, \\ &= 5(x^4 - c), \\ &> 0 \text{ as } c < 0. \end{aligned}$$

$\therefore$ Slope is always positive so the equation has only one real root.

METHOD 1—

Now, if the root is positive, $x^5 > 0$, $-5cx > 0$,

$\therefore$ $x^5 - 5cx + 1 > 0$ for all $x \geqslant 0$.

i.e., there is no root where $x \geqslant 0$.

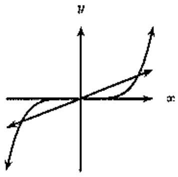
Hence the root must be negative.

METHOD 2—

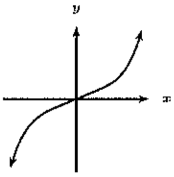
When $x = 0$, $P(x) = 1$, and as the slope is always positive (see above), the root must be negative.

One student (unsuccessfully) attempted an interesting graphical approach:

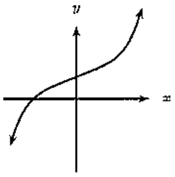
Plot $y = x^5$ and $y = -cx$ for some arbitrary negative c,



then
add
them



and
raise
by 1



It is clear that the sole real root is negative.

(d) Let $u_n = \int_0^{\frac{1}{2}} \frac{(\tan^{-1} 2x)^n}{1+4x^2}\,dx$, where n is a positive integer.

i) Show that $u_n = \left(\frac{\pi}{4}\right)^{n+1} . \frac{1}{2(n+1)}$. $\boxed{3}$

Solution: METHOD 1 (simple substitution)—

$$\begin{aligned} u_n &= \int_0^{\frac{1}{2}} \frac{(\tan^{-1} 2x)^n}{1+4x^2}\,dx, \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} y^n\,dy, \\ &= \frac{1}{2}\left[\frac{y^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}}, \\ &= \left(\frac{\pi}{4}\right)^{n+1} . \frac{1}{2(n+1)}. \end{aligned}$$

$$\begin{aligned} \text{Put } y &= \tan^{-1} 2x, \\ dy &= \frac{2\,dx}{1+4x^2}. \\ \text{When } x = 0, \quad y &= 0, \\ x = \tfrac{1}{2}, \quad y &= \tfrac{\pi}{4}. \end{aligned}$$

METHOD 2 (integration by parts)—

$$\begin{aligned} u_n &= \left.\frac{(\tan^{-1} 2x)^{n+1}}{2}\right]_0^{\frac{1}{2}} - n\int_0^{\frac{1}{2}} \frac{(\tan^{-1} 2x)^n}{1+4x^2}\,dx, \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right)^{n+1} - n.u_n, \\ (n+1)u_n &= \frac{1}{2}\left(\frac{\pi}{4}\right)^{n+1}, \\ \therefore\ u_n &= \left(\frac{\pi}{4}\right)^{n+1} . \frac{1}{2(n+1)}. \end{aligned}$$

$$\begin{aligned} \text{Put } u &= (\tan^{-1} 2x)^n, \\ u' &= \frac{2n(\tan^{-1} 2x)^{n-1}\,dx}{1+4x^2}. \\ v' &= \frac{dx}{1+4x^2}, \\ v &= \tfrac{1}{2}\tan^{-1} 2x. \end{aligned}$$

ii) Hence or otherwise show that $\boxed{3}$

$u_0 \times u_1 \times u_2 \times u_3 \times \cdots \times u_{2n-1} = \left(\frac{\pi}{4}\right)^{2n^2+n} . \frac{1}{2^{2n}.(2n)!}$.

Solution:

$$\begin{aligned} \text{L.H.S.} &= \frac{\pi}{4}.\frac{1}{2} \times \left(\frac{\pi}{4}\right)^2.\frac{1}{4} \times \left(\frac{\pi}{4}\right)^3.\frac{1}{6} \times \left(\frac{\pi}{4}\right)^4.\frac{1}{8} \times \cdots \times \left(\frac{\pi}{4}\right)^{2n}.\left(\frac{1}{2\times 2n}\right), \\ &= \left(\frac{\pi}{4}\right)^{1+2+3+4+\cdots+2n} \times \left(\frac{1}{2}\right)^{2n} \times \frac{1}{1.2.3.4 \times \cdots \times 2n}. \end{aligned}$$

$$\begin{aligned} \text{Now} \quad 1+2+3+4+\cdots+2n &= \frac{2n}{2}(2n+1), \\ &= 2n^2+n. \end{aligned}$$

$$\therefore\ \text{L.H.S.} = \left(\frac{\pi}{4}\right)^{2n^2+n} \times \frac{1}{2^{2n}.(2n)!} = \text{R.H.S.}$$