SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JUNE 2010
TASK #2
YEAR 12

Mathematics Ext 2

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—120 Marks

- Attempt all questions.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A (Questions 1 and 2),
  Section B (Questions 3 and 4),
  Section C (Questions 5 and 6).

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section A

Marks

Question 1 (20 marks)

(a) Find the exact value of $\displaystyle\int_0^{\ln 5} e^x \, dx$. $\boxed{2}$

(b) Integrate

(i) $\cos^4 x \sin x$, $\boxed{2}$

(ii) $\dfrac{x}{x^2+5}$, $\boxed{2}$

(iii) $\dfrac{\sin\theta}{\sqrt{1-\cos\theta}}$. $\boxed{3}$

(c) Evaluate $\displaystyle\int_1^{e} \ln(x^x) \, dx$. $\boxed{4}$

(d) (i) Express $\dfrac{x}{(x-2)(x-1)}$ in partial fractions and hence find $\displaystyle\int \frac{x\,dx}{(x-2)(x-1)}$. $\boxed{4}$

(ii) Evaluate $\displaystyle\int_{-1}^{3} \frac{4\,dx}{x^2-2x+5}$ by completing the square on the denominator. $\boxed{3}$

Marks

Question 2 (20 marks)

(a) (i) Verify by substitution that $1 + 2i$ is a solution of the quadratic equation $x^2 + 5 = 2x$. $\boxed{3}$

(ii) Write down the other solution. $\boxed{1}$

(b) Simplify, leaving in the form $a + ib$:

(i) $-i^{73}$, $\boxed{1}$

(ii) $(3 + i)(5 - i)$, $\boxed{2}$

(iii) $\dfrac{3 - 4i}{4 + 3i}$, $\boxed{2}$

(c) (i) Draw an Argand diagram showing the three vectors $\boxed{3}$

$$\frac{1+2i}{2}, \quad \frac{1-2i}{2}, \quad \frac{1+2i}{1-2i}.$$

(ii) Sketch $|z - 1| \leqslant 2$ on an Argand diagram. $\boxed{2}$

(d) (i) Let $z = \cos\theta + i\sin\theta$, then show that $z^n + z^{-n} = 2\cos n\theta$. $\boxed{2}$

(ii) Hence or otherwise, find $\displaystyle\int \cos^6 x\, dx$. $\boxed{4}$

Section B

(Use a separate writing booklet.)

Marks

Question 3 (20 marks)

(a) (i) If $z = x + iy$, the conjugate of z, $\overline{z} = x - iy$. Prove that

(α) $|z| = |\overline{z}|$, $\boxed{1}$

(β) $z\overline{z} = |z|^2$, $\boxed{1}$

(γ) If $w = u + iv$, then $(\overline{z + w}) = \overline{z} + \overline{w}$. $\boxed{1}$

(ii) Two variable complex numbers z and w are such that $z + \overline{z} = 3$ and $w\overline{w} = 4$.

(α) Show on the same Argand diagram the loci of the points which represent z and w. $\boxed{3}$

(β) Find in the form $a + ib$ the complex numbers represented by the points of intersection of the two loci. $\boxed{2}$

(b) (i) Sketch $h(x) = |x - 1| - 1$, over $[-2, 4]$. $\boxed{2}$

Hence sketch the following:

(ii) $\big(h(x)\big)^2$, $\boxed{3}$

(iii) $\dfrac{1}{h(x)}$, $\boxed{3}$

(iv) $\ln\left(\dfrac{1}{h(x)}\right)$. $\boxed{4}$

Marks

Question 4 (20 marks)

(a) A particle is fired from a point A which is $32\,\text{m}$ above horizontal ground.
The angle of projection above the horizontal is α, where $\tan\alpha = \frac{1}{2}$.
The point O is on the ground, vertically below A. The projectile strikes the ground at a point $64\,\text{m}$ distant from O. (In this question, take the acceleration due to gravity to be $10\,\text{ms}^{-1}$.)

(i) Derive the cartesian equation of the trajectory and hence find the speed of projection. $\boxed{4}$

(ii) Show that, at the highest point of its path, the projectile is at a height of $36\,\text{m}$ above the ground. $\boxed{2}$

(iii) Find the magnitude and the direction of the velocity of the projectile just before it strikes the ground. $\boxed{4}$

(iv) Find the time after projection at which the velocity of the projectile makes an angle β below the horizontal, where $\tan\beta = \frac{3}{4}$. $\boxed{3}$

(b) Find the number of ways of seating 6 men and 6 women around a table if:—

(i) there is no restriction on where the people sit, $\boxed{1}$

(ii) a particular woman wants to sit between two particular men, $\boxed{2}$

(iii) two particular women want to face each other across the table, $\boxed{2}$

(iv) two particular people do *not* wish to sit together. $\boxed{2}$

Section C

(Use a separate writing booklet.)

Marks

Question 5 (20 marks)

(a) Without using the calculus, sketch the following functions, showing any significant features.

(i) $f(x) = \dfrac{x^2+1}{x^2+x-2}$, $\boxed{4}$

(ii) $g(x) = \dfrac{-2x^3+6x}{2x^2-6x}$. $\boxed{4}$

(b) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. $\boxed{2}$

(ii) Hence or otherwise find all zeroes of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. $\boxed{2}$

(c) Find all integral solutions of the following simultaneous equations: $\boxed{4}$

$$\begin{aligned} x^2 + y &= 3 \ldots \boxed{1} \\ y^2 + x &= 5 \ldots \boxed{2} \end{aligned}$$

(d) The polynomial $P(x)$ leaves a remainder 3 when divided by $x+1$, and a remainder 1 when divided by $x-1$. Find the remainder when $P(x)$ is divided by x^2-1. $\boxed{4}$

Marks

Question 6 (20 marks)

(a) Nine people, of whom three are brothers and two are sisters, are divided into three groups of three.

(i) Show that the probability of the sisters being in the same group is $\frac{1}{4}$. **5**

(ii) Find the probability that one group contains at least two of the brothers. **4**

(b) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^n dx$, where n is an integer, $n \geqslant 0$.

(i) Using integration by parts, show that, for $n \geqslant 2$, **3**

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$$

(ii) Deduce that **3**

$$\begin{aligned} I_{2n} &= \frac{2n-1}{2n}.\frac{2n-3}{2n-2}\cdots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} \\ \text{and } I_{2n+1} &= \frac{2n}{2n+1}.\frac{2n-2}{2n-1}\cdots\frac{4}{5}.\frac{2}{3}.1. \end{aligned}$$

(iii) Explain why $I_k > I_{k+1}$. **1**

(iv) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that **4**

$$\begin{aligned} \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) &< \frac{2^2.4^2\ldots(2n)^2}{1.3^2.5^2\ldots(2n-1)^2(2n+1)} \\ &< \frac{\pi}{2}. \end{aligned}$$

End of Paper

STANDARD INTEGRALS

$$
\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \\
\int \frac{1}{x} dx &= \ln x, \quad x > 0 \\
\int e^{ax} dx &= \frac{1}{a}e^{ax}, \quad a \neq 0 \\
\int \cos ax dx &= \frac{1}{a}\sin ax, \quad a \neq 0 \\
\int \sin ax dx &= -\frac{1}{a}\cos ax, \quad a \neq 0 \\
\int \sec^2 ax dx &= \frac{1}{a}\tan ax, \quad a \neq 0 \\
\int \sec ax \tan ax dx &= \frac{1}{a}\sec ax, \quad a \neq 0 \\
\int \frac{1}{a^2+x^2} dx &= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0 \\
\int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a \\
\int \frac{1}{\sqrt{x^2-a^2}} dx &= \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0 \\
\int \frac{1}{\sqrt{x^2+a^2}} dx &= \ln\left(x+\sqrt{x^2+a^2}\right)
\end{aligned}
$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

Question 1 (20 marks)

(a) Find the exact value of $\displaystyle\int_0^{\ln 5} e^x\,dx$. $\boxed{2}$

> **Solution:** $$\begin{aligned}\int_0^{\ln 5} e^x\,dx &= e^x\Big]_0^{\ln 5},\\ &= 5-1,\\ &= 4.\end{aligned}$$

(b) Integrate

(i) $\cos^4 x \sin x$, $\boxed{2}$

> **Solution:** $\int \cos^4 x.(-d\cos x) = -\frac{1}{5}\cos^5 x + c.$

(ii) $\dfrac{x}{x^2+5}$, $\boxed{2}$

> **Solution:** $\frac{1}{2}\displaystyle\int \frac{2x\,dx}{x^2+5} = \ln(x^2+5)+c.$

(iii) $\dfrac{\sin\theta}{\sqrt{1-\cos\theta}}$. $\boxed{3}$

> **Solution:** $$\begin{aligned}\int \frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}} &= \int \frac{du}{u^{1/2}},\\ &= 2u^{1/2}+c,\\ &= 2\sqrt{1-\cos\theta}+c.\end{aligned}$$
> Put $\begin{aligned}1-\cos\theta &= u,\\ \sin\theta\,d\theta &= du.\end{aligned}$

(c) Evaluate $\displaystyle\int_1^e \ln(x^x)\,dx$. $\boxed{4}$

> **Solution:** $$\begin{aligned}\int_1^e x\ln x\,dx &= \left[\frac{x^2}{2}.\ln x\right]_1^e - \int_1^e \frac{x^2}{2}.\frac{1}{x}\,dx,\\ &= \frac{e^2}{2} - \frac{1}{2}\int_1^e x\,dx,\\ &= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e,\\ &= \frac{e^2}{2} - \left\{\frac{e^2}{4}-\frac{1}{4}\right\},\\ &= \frac{1}{4}\left(e^2+1\right).\end{aligned}$$
> $\begin{aligned}\text{Putting } u &= \ln x\\ \text{and } \tfrac{dv}{dx} &= x\\ \text{so that } \tfrac{du}{dx} &= \tfrac{1}{x}\\ \text{and } v &= \textstyle\int x\,dx\\ &= \tfrac{x^2}{2}\end{aligned}$

(d) (i) Express $\dfrac{x}{(x-2)(x-1)}$ in partial fractions and hence find $\displaystyle\int \frac{x\,dx}{(x-2)(x-1)}$. $\boxed{4}$

Solution:

$$\begin{aligned}
\frac{x}{(x-2)(x-1)} &\equiv \frac{\mathrm{A}}{x-2} + \frac{\mathrm{B}}{x-1}, \\
\textit{i.e. } x &\equiv \mathrm{A}(x-1) + \mathrm{B}(x-2). \\
\text{Put } x = 2, &\quad \mathrm{A} = 2, \\
x = 1, &\quad \mathrm{B} = -1. \\
\therefore \frac{x}{(x-2)(x-1)} &= \frac{2}{x-2} - \frac{1}{x-1}. \\
\int \frac{x\,dx}{(x-2)(x-1)} &= \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1}, \\
&= 2\ln(x-2) - \ln(x-1) + c.
\end{aligned}$$

(ii) Evaluate $\displaystyle\int_{-1}^{3} \frac{4\,dx}{x^2-2x+5}$ by completing the square on the denominator. $\boxed{3}$

Solution:

$$\begin{aligned}
\int_{-1}^{3} \frac{4\,dx}{x^2-2x+1+4} &= \int_{-1}^{3} \frac{4\,dx}{(x-1)^2+2^2}, \\
&= 4\left[\tfrac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)\right]_{-1}^{3}, \\
&= 2\left\{\frac{\pi}{4} - \frac{-\pi}{4}\right\}, \\
&= \pi.
\end{aligned}$$

Marks

Question 2 (20 marks)

(a) (i) Verify by substitution that $1 + 2i$ is a solution of the quadratic equation $x^2 + 5 = 2x$. $\boxed{3}$

> **Solution:** $\begin{aligned} \text{L.H.S.} &= (1+2i)^2 + 5, \\ &= 1 + 4i - 4 + 5, \\ &= 2 + 4i. \end{aligned}$ $\quad \begin{aligned} \text{R.H.S.} &= 2 + 4i, \\ &= \text{L.H.S.} \end{aligned}$
> Therefore true.

(ii) Write down the other solution. $\boxed{1}$

> **Solution:** $1 - 21$.

(b) Simplify, leaving in the form $a + ib$:

(i) $-i^{73}$, $\boxed{1}$

> **Solution:** $-i \times (i^2)^{36} = -i$ (or $0 - i$ if you like).

(ii) $(3 + i)(5 - i)$, $\boxed{2}$

> **Solution:** $15 - 3i + 5i + 1 = 16 + 2i$.

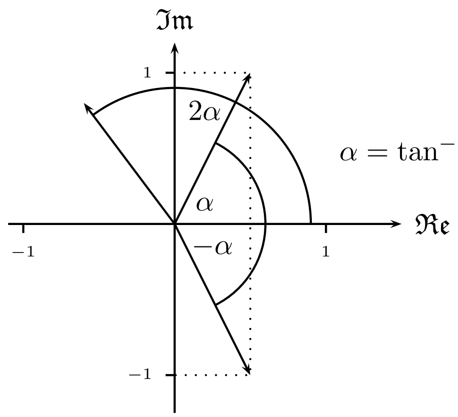
(iii) $\dfrac{3 - 4i}{4 + 3i}$, $\boxed{2}$

> **Solution:** $\begin{aligned} \frac{3-4i}{4+3i} \times \frac{4-3i}{4-3i} &= \frac{12 - 9i - 16i - 12}{16 + 9}, \\ &= -i. \end{aligned}$

(c) (i) Draw an Argand diagram showing the three vectors $\boxed{3}$

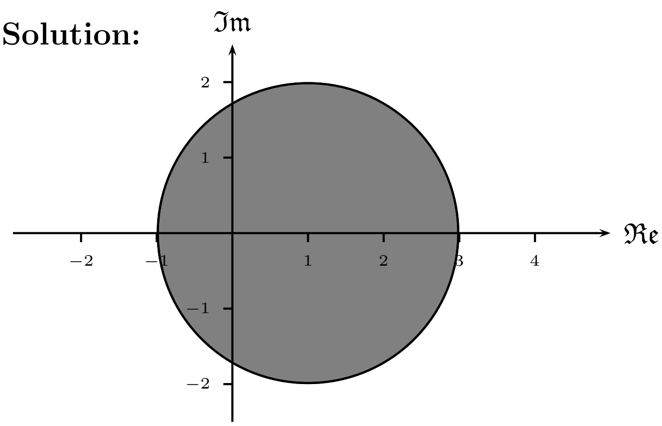
$$\frac{1+2i}{2}, \quad \frac{1-2i}{2}, \quad \frac{1+2i}{1-2i}.$$

Solution:



(ii) Sketch $|z-1| \leqslant 2$ on an Argand diagram. $\boxed{2}$

Solution:



(d) (i) Let $z = \cos\theta + i\sin\theta$, then show that $z^n + z^{-n} = 2\cos n\theta$. $\boxed{2}$

Solution:
$$\begin{aligned} z^n &= \cos n\theta + i\sin n\theta \text{ (from De Moivre's theorem)}, \\ \text{and } z^{-n} &= \cos n\theta - i\sin n\theta. \\ \text{Adding, } z^n + z^{-n} &= 2\cos n\theta. \end{aligned}$$

(ii) Hence or otherwise, find $\displaystyle\int \cos^6 x\,dx$. $\boxed{4}$

Solution:

$$\begin{aligned}
(2\cos\theta)^6 &= \left(z+\frac{1}{z}\right)^6,\\
&= \left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20,\\
64\cos^6\theta &= 2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20,\\
\cos^6\theta &= \tfrac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10).\\
\int\cos^6\theta &= \tfrac{1}{32}\int(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)d\theta,\\
&= \tfrac{1}{32}\left(\frac{\sin 6\theta}{6}+\frac{6\sin 4\theta}{4}+\frac{15\sin 2\theta}{2}+10\theta\right)+c,\\
&= \frac{1}{64}\left(\frac{\sin 6\theta}{3}+3\sin 4\theta+15\sin 2\theta+20\theta\right)+c.
\end{aligned}$$

Section B

(Use a separate writing booklet.)

Marks

Question 3 (20 marks)

(a) (i) If $z = x + iy$, the conjugate of z, $\overline{z} = x - iy$. Prove that

(α) $|z| = |\overline{z}|$, $\boxed{1}$

> **Solution:**
> $$\begin{aligned} |z| &= \sqrt{x^2 + y^2}, \\ |\overline{z}| &= \sqrt{x^2 + (-y)^2}, \\ &= \sqrt{x^2 + y^2}, \\ &= |z|. \end{aligned}$$

(β) $z\overline{z} = |z|^2$, $\boxed{1}$

> **Solution:**
> $$\begin{aligned} z\overline{z} &= (x + iy)(x - iy), \\ &= x^2 - (i^2y^2), \\ &= x^2 + y^2, \\ &= (\sqrt{x^2 + y^2})^2, \\ &= |z|^2. \end{aligned}$$

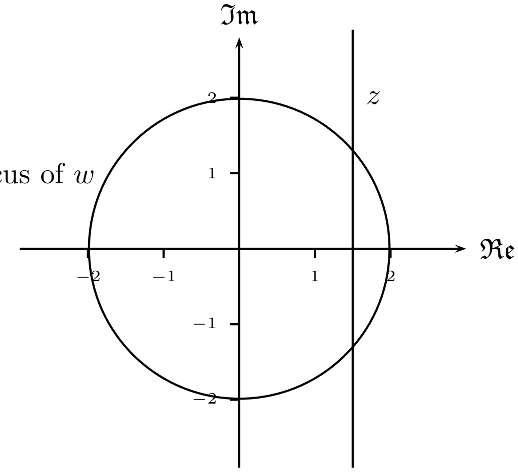
(γ) If $w = u + iv$, then $(\overline{z + w}) = \overline{z} + \overline{w}$. $\boxed{1}$

> **Solution:**
> $$\begin{aligned} z + w &= (x + iy) + (u + iv), \\ &= (x + u) + i(y + v). \\ \text{Now } (\overline{z + w}) &= (x + u) - i(y + v), \\ &= x - iy \ + \ u - iv, \\ &= \overline{z} + \overline{w}. \end{aligned}$$

(ii) Two variable complex numbers z and w are such that $z + \overline{z} = 3$ and $w\overline{w} = 4$.

(α) Show on the same Argand diagram the loci of the points which represent z and w. $\boxed{3}$

> **Solution:** If $z = x + iy$, $\overline{z} = x - iy$.
> Then $z + \overline{z} = 2x = 3 \Longrightarrow x = 1\frac{1}{2}$.
> Locus of z is the line $x = 1\frac{1}{2}$.
> Now $w\overline{w} = x^2 + y^2$, (from (i)($\beta$))
> *i.e.* $x^2 + y^2 = 4$,
> so the locus of w is a circle centre $(0,\, 0)$ of radius 2.



(β) Find in the form $a + ib$ the complex numbers represented by the points of intersection of the two loci. $\boxed{2}$

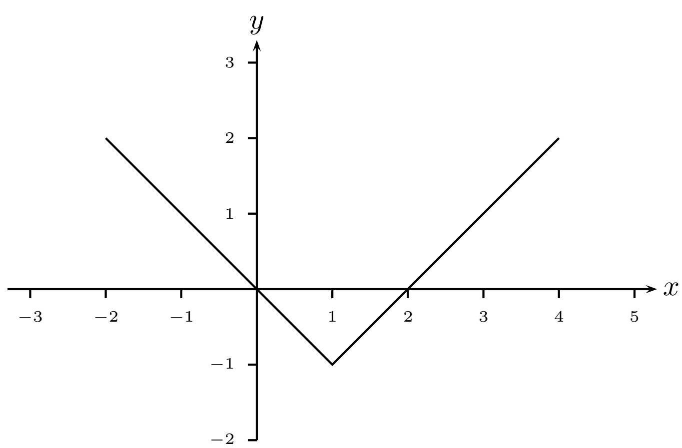
Solution: Substituting $x = \dfrac{3}{2}$ into $x^2 + y^2 = 4$,

$$\begin{aligned} \frac{9}{4} + y^2 &= 4, \\ y^2 &= \frac{7}{4}, \\ y &= \pm\frac{\sqrt{7}}{2}. \end{aligned}$$

$\therefore$ the points of intersection are $\left(\dfrac{3}{2} + i\dfrac{\sqrt{7}}{2}\right)$ and $\left(\dfrac{3}{2} - i\dfrac{\sqrt{7}}{2}\right)$.

(b) (i) Sketch $h(x) = |x - 1| - 1$, over $[-2, 4]$. $\boxed{2}$

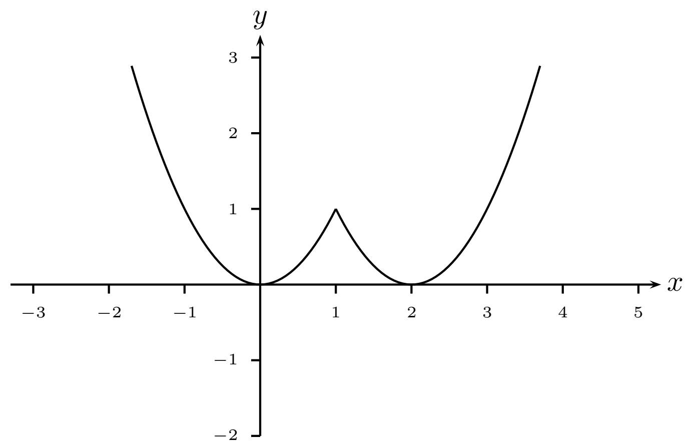
Solution:



Hence sketch the following:

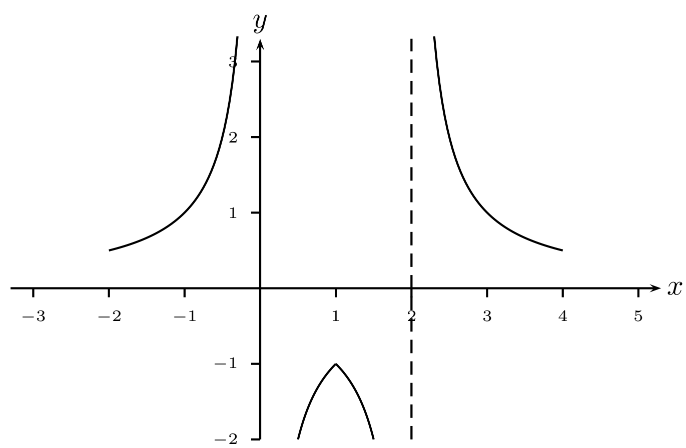
(ii) $\big(h(x)\big)^2$, $\boxed{3}$

Solution:



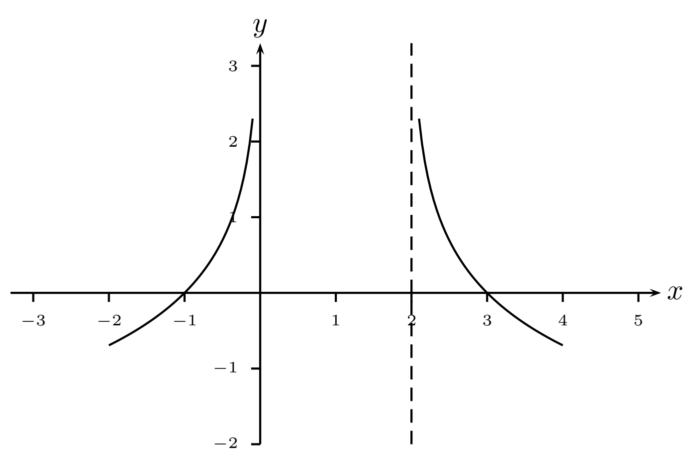
(iii) $\dfrac{1}{h(x)}$, $\boxed{3}$

Solution:



(iv) $\ln\left(\dfrac{1}{h(x)}\right)$. $\boxed{4}$

Solution:



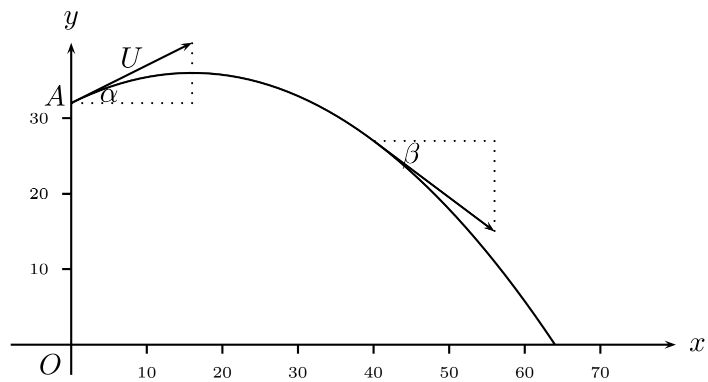
Marks

Question 4 (20 marks)

(a) A particle is fired from a point A which is 32 m above horizontal ground. The angle of projection above the horizontal is α, where $\tan\alpha = \frac{1}{2}$. The point O is on the ground, vertically below A. The projectile strikes the ground at a point 64 m distant from O. (In this question, take the acceleration due to gravity to be $10\,\text{ms}^{-1}$.)

(i) Derive the cartesian equation of the trajectory and hence find the speed of projection. $\boxed{4}$

Solution:



$$\begin{aligned} \ddot{x} &= 0, & \ddot{y} &= -10, \\ \dot{x} &= U\cos\alpha, & \dot{y} &= U\sin\alpha - 10t, \\ x &= Ut\cos\alpha, & y &= 32 + Ut\sin\alpha - 5t^2, \\ t &= \frac{x}{U\cos\alpha}, & y &= 32 + \frac{Ux\sin\alpha}{U\cos\alpha} - \frac{5x^2}{U^2\cos^2\alpha}, \\ & & &= 32 + \frac{x}{2} - \frac{5x^2}{U^2}\left(1 + \frac{1}{4}\right), \\ & & &= 32 + \frac{x}{2} - \frac{25x^2}{4U^2}. \end{aligned}$$

Now, when $x = 64$, $y = 0$, and thus
$$\begin{aligned} 0 &= 32 + 32 - \frac{25600}{U^2}, \\ 64U^2 &= 25600, \\ U^2 &= 400, \\ U &= 20. \end{aligned}$$

The initial speed is $20\,\text{ms}^{-1}$.

(ii) Show that, at the highest point of its path, the projectile is at a height of 36 m above the ground. $\boxed{2}$

Solution: At the highest point

$$\begin{aligned} \dot{y} &= 0, \\ \therefore\ t &= \frac{U \sin\alpha}{10}, \\ &= \frac{20}{\sqrt{5} \times 10}, \\ &= \frac{2\sqrt{5}}{5}. \\ \text{So } y &= 32 + \frac{20 \times 2\sqrt{5}}{5 \times \sqrt{5}} - \frac{5 \times 20}{25}, \\ &= 32 + 8 - 4, \\ &= 36 \text{ as required.} \end{aligned}$$

(iii) Find the magnitude and the direction of the velocity of the projectile just before it strikes the ground. $\boxed{4}$

Solution: Using $t = \dfrac{x}{U \cos\alpha}$, we have, at the moment of impact,

$$\begin{aligned} t &= \frac{64}{20} \times \frac{\sqrt{5}}{2}, \\ &= \frac{8}{\sqrt{5}}. \end{aligned}$$

Also, $\dot{x} = U \cos\alpha$, $\dot{y} = U \sin\alpha - 10t$, so evaluating the components of velocity at the moment of impact,

$$\begin{aligned} \dot{x} &= 20 \times \frac{2}{\sqrt{5}}, & \dot{y} &= \frac{20}{\sqrt{5}} - 10 \times \frac{8}{\sqrt{5}}, \\ &= \frac{40}{\sqrt{5}}, & &= -\frac{60}{\sqrt{5}}. \end{aligned}$$

The magnitude is $\sqrt{\dfrac{40^2}{5} + \dfrac{60^2}{5}} \approx 32.25\,\mathrm{ms}^{-1}$,

and the direction is $\tan^{-1} \frac{-60}{40}$, *i.e.* $\tan^{-1} \frac{3}{2}$ below horizontal.

(iv) Find the time after projection at which the velocity of the projectile makes an angle β below the horizontal, where $\tan\beta = \frac{3}{4}$. $\boxed{3}$

Solution:

$$\begin{aligned} -\frac{3}{4} &= \left(\frac{20}{\sqrt{5}} - 10t\right) \div \left(20 \times \frac{2}{\sqrt{5}}\right), \\ &= \frac{20}{\sqrt{5}} \times \frac{\sqrt{5}}{40} - \frac{10\sqrt{5}t}{40}, \\ &= \frac{1}{2} - \frac{\sqrt{5}t}{4}, \\ \frac{\sqrt{5}t}{4} &= \frac{5}{4}, \\ t &= \sqrt{5}, \end{aligned}$$

i.e. it will be $\sqrt{5}$ seconds after projection.

(b) Find the number of ways of seating 6 men and 6 women around a table if:—

(i) there is no restriction on where the people sit, $\boxed{1}$

> **Solution:** Sit any one of the 12 in any position. The remaining 11 may be seated in 11! ways.
> Thus the number of ways is $11! = 39\,916\,800$.

(ii) a particular woman wants to sit between two particular men, $\boxed{2}$

> **Solution:** Sit the particular woman anywhere then the two men can be seated left and right or right and left, *i.e.* 2 ways. The remaining 9 can be seated in 9! ways.
> Therefore the number of ways is $2 \times 9! = 725\,760$.

(iii) two particular women want to face each other across the table, $\boxed{2}$

> **Solution:** First seat one of the women in any position, there is then only one place facing for the other. the other ten can be seated in 10! ways.
> So the number of ways is $10! = 3\,628\,800$.

(iv) two particular people do *not* wish to sit together. $\boxed{2}$

> **Solution:** The ways they can be seated together is 2 and the others may be seated in 10! ways, *i.e.* the ways of seating them *together* is $2 \times 10!$ Now, from part (i), the total ways of seating all the people is 11! This implies that the ways of seating everyone so the two are *not* together is $11! - 2 \times 10! = 32\,659\,200$.

Section C

(Use a separate writing booklet.)

Marks

Question 5 (20 marks)

(a) Without using the calculus, sketch the following functions, showing any significant features.

(i) $f(x) = \dfrac{x^2+1}{x^2+x-2}$, $\boxed{4}$

Solution:
$$f(x) = \frac{x^2+1}{(x+2)(x-1)}, \quad = \frac{1+\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}.$$

Vertical asymptotes $x=-2$, and $x=1$.

$\lim\limits_{x\to\pm\infty} f(x) = 1$, hence horizontal asymptote $y = 1$.

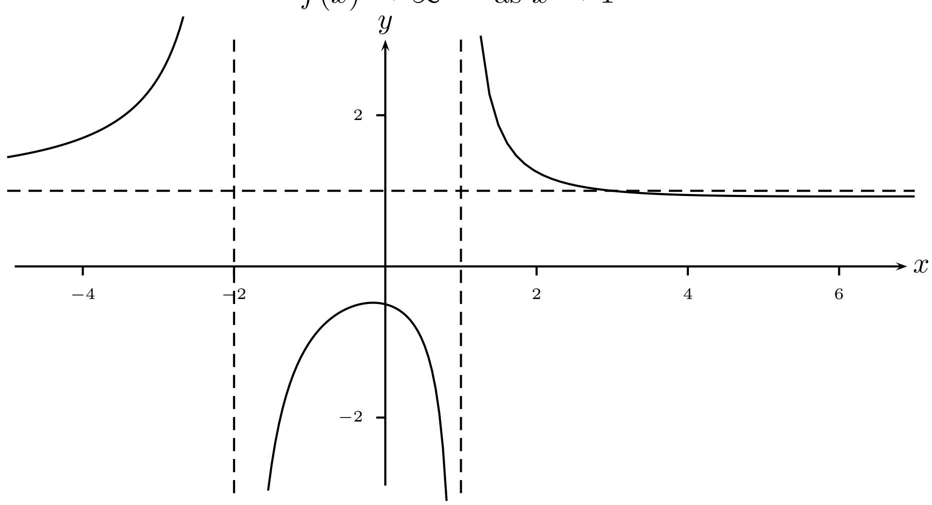
$$\begin{aligned}\text{Putting } f(x)=1,\ x^2+1 &= x^2+x-2,\\ 1 &= x-2,\\ x &= 3,\end{aligned}$$

so the curve cuts the asymptote at (3, 1).

$f(0) = -\frac{1}{2}$, (*i.e.* y-intercept),

but x^2+1 has no real zeroes, so there is no x-intercept.

Near the asymptotes,
$$\begin{aligned} f(x) &\to \infty && \text{as } x \to -2^- \\ f(x) &\to -\infty && \text{as } x \to -2^+ \\ f(x) &\to -\infty && \text{as } x \to 1^- \\ f(x) &\to \infty && \text{as } x \to 1^+ \end{aligned}$$



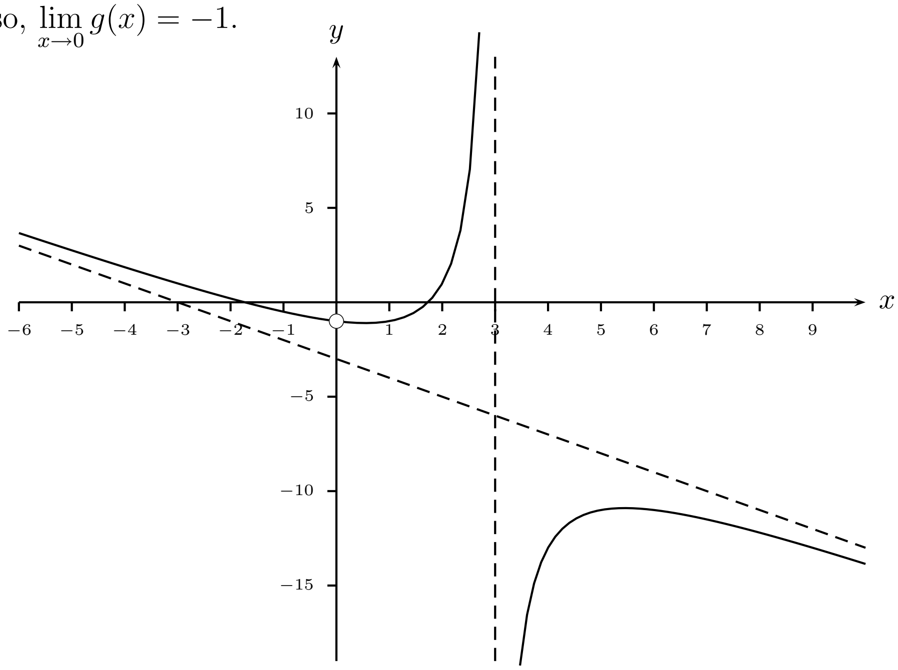
(ii) $g(x) = \dfrac{-2x^3+6x}{2x^2-6x}$. $\boxed{4}$

Solution:
$$g(x) = \frac{2x}{2x}.\frac{-x^2+3}{x-3}, \quad = \frac{-x^2+3}{x-3} \text{ if } x \neq 0$$

Now, using Horner's method,

$$\begin{array}{r|rrr} & -1 & 0 & 3 \\ 3 & & -3 & -9 \\ \hline & -1 & -3 & -6 \end{array} \quad \text{thus } g(x) = -x - 3 - \frac{6}{x-3}.$$

So the graph has a vertical asymptote at $x = 3$
and an oblique asymptote $y = -x - 3$.
Near the vertical asymptote, $g(x) \to \infty$ as $x \to 3^-$
$g(x) \to -\infty$ as $x \to 3^+$
x-intercepts when $g(x) = 0$, *i.e.* $-x^2 + 3 = 0$, $x = \pm\sqrt{3}$.
Also, $\lim_{x\to 0} g(x) = -1$.



(b) (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. $\boxed{2}$

Solution:
$$\begin{aligned} P(x) &= (x-\alpha)^3 Q(x), \text{ where } Q(\alpha) \neq 0, \\ P'(x) &= (x-\alpha)^3 Q'(x) + 3(x-\alpha)^2 Q(x), \\ &= (x-\alpha)^2[(x-\alpha)Q'(x) + 3Q(x)], \\ &= (x-\alpha)^2 R(x), \text{ where } R(\alpha) = 3Q(\alpha) \neq 0. \end{aligned}$$
So $P'(x)$ has a zero of multiplicity 2.

(ii) Hence or otherwise find all zeroes of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. $\boxed{2}$

Solution: Let $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$,
and let $x = \alpha$ be the zero of multiplicity 3.

$$\begin{aligned} P'(x) &= 32x^3 - 75x^2 + 54x - 11, \\ P''(x) &= 96x^2 - 150x + 54, \\ &= 6(16x^2 - 25x + 9), \\ &= 6(x-1)(16x-9). \end{aligned}$$

So the zeroes of $P''(x)$ are $x = 1$ and $x = \frac{9}{16}$.
Testing $x = 1$, $P(1) = 0$ and $P'(1) = 0$, so $P(x) = (x-1)^3 Q(x)$.
Let $x = \beta$ be the other zero,

$$\begin{aligned} \text{then} \quad \alpha + \alpha + \alpha + \beta &= \frac{25}{8}, \\ \beta &= \frac{25}{8} - 3, \\ &= \frac{1}{8}. \end{aligned}$$

So the zeroes of $8x^4 - 25x^3 + 27x^2 - 11x + 1$ are $x = 1,\ 1,\ 1,\ \frac{1}{8}$.

(c) Find all integral solutions of the following simultaneous equations: $\boxed{4}$

$$\begin{aligned} x^2 + y &= 3 \ldots \boxed{1} \\ y^2 + x &= 5 \ldots \boxed{2} \end{aligned}$$

Solution: From $\boxed{1}$: $y = 3 - x^2$.

$$\begin{aligned} \text{Sub. in } \boxed{2}: \ \left(3 - x^2\right)^2 + x &= 5, \\ x^4 - 6x^2 + x + 4 &= 0. \\ \text{Now, putting } P(x) &= x^4 - 6x^2 + x + 4, \\ P(1) &= 1 - 6 + 1 + 4 = 0. \end{aligned}$$

So $(x-1)$ is a factor of $P(x)$ and,
using long division,

$$\begin{array}{r} x^3 + x^2 - 5x - 4 \\ x - 1 \overline{\big) \ x^4 \qquad - 6x^2 + x + 4} \\ \underline{-x^4 + x^3 \qquad\qquad\quad} \\ x^3 - 6x^2 \qquad\quad \\ \underline{-x^3 + x^2 \qquad\quad} \\ -5x^2 + x \quad \\ \underline{5x^2 - 5x \quad} \\ -4x + 4 \\ \underline{4x - 4} \\ 0 \end{array}$$

or using Horner's method,

$$\begin{array}{c|ccccc} & 1 & 0 & -6 & 1 & 4 \\ 1 & & 1 & 1 & -5 & -4 \\ \hline & 1 & 1 & -5 & -4 & 0 \end{array}$$

$$\begin{aligned} \text{Now, putting } Q(x) &= x^3 + x^2 - 5x - 4, \\ Q(1) &= 1 + 1 - 5 - 4 \neq 0, \\ Q(-1) &= -1 + 1 + 5 - 4 \neq 0, \\ Q(2) &= 8 + 4 - 10 - 4 \neq 0, \\ Q(-2) &= -8 + 4 + 10 - 4 \neq 0. \end{aligned}$$

So there are no more integral solutions.
When $x = 1$, $y = 2$; so the only integral solution is $(1, 2)$.

(d) The polynomial $P(x)$ leaves a remainder 3 when divided by $x + 1$, and a remainder 1 when divided by $x - 1$. Find the remainder when $P(x)$ is divided by $x^2 - 1$. $\boxed{4}$

Solution:

$$\begin{aligned} P(x) &= (x+1)Q_1 + 3 \ldots\ldots\ldots\ldots\ldots\ldots \boxed{1} \\ P(x) &= (x-1)Q_2 + 1 \ldots\ldots\ldots\ldots\ldots\ldots \boxed{2} \\ P(x) &= (x+1)(x-1)Q_3 + (ax+b) \ldots \boxed{3} \\ P(-1) &= 3 = -a + b \ldots\ldots\ldots\ldots\ldots\ldots (\text{from } \boxed{1} \text{ and } \boxed{3}) \\ P(+1) &= 1 = \quad a + b \ldots\ldots\ldots\ldots\ldots\ldots (\text{from } \boxed{2} \text{ and } \boxed{3}) \end{aligned}$$

Solving simultaneously,

$$\begin{aligned} -a + b &= 3 \ldots \boxed{4} \\ a + b &= 1 \ldots \boxed{5} \\ 2b &= 4 \ldots \boxed{4} + \boxed{5} \\ b &= 2 \\ a + 2 &= 1 \ldots (\text{substitute in } \boxed{5}) \\ a &= -1 \end{aligned}$$

Thus the remainder after division by $x^2 - 1$ is $-x + 2$.

Question 6 (20 marks)

(a) Nine people, of whom three are brothers and two are sisters, are divided into three groups of three.

(i) Show that the probability of the sisters being in the same group is $\frac{1}{4}$. 5

Solution: Suppose the sisters are in the first group.
The 3rd person in this group can be chosen in $\binom{7}{1}$ ways, *i.e.* 7 ways.
The 2nd group of 3 can be formed from the remaining 6 in $\binom{6}{3}$ ways, *i.e.* 20 ways.
The third group is formed by the 3 people left — 1 way.
Now the sisters can be in the 1st, 2nd, or 3rd group.
$\therefore$ The number of ways of arranging the three groups so the sisters are in the same group is $7 \times 20 \times 1 \times 3 = 420$.
If there are no restrictions on the people forming a group,
the 1st group of 3 can be formed in $\binom{9}{3}$ ways, *i.e.* 84 ways,
the 2nd group of 3 can be formed in $\binom{6}{3}$ ways, *i.e.* 20 ways,
the 3rd group of 3 can be formed in $\binom{3}{3}$ ways, *i.e.* 1 way.
$\therefore$ The number of ways of dividing 9 people into 3 groups of 3 is $84 \times 20 \times 1 = 1680$.
So the probability of the 2 sisters being in the same group is $\dfrac{420}{1680} = \dfrac{1}{4}$.

(ii) Find the probability that one group contains at least two of the brothers. 4

Solution: The number of ways of arranging the brothers so there is one of them in each group is $3 \times 2 \times 1 = 6$ ways.
The 1st group can then be completed in $\binom{6}{2}$ ways, *i.e.* 15 ways.
The 2nd group can be completed in $\binom{4}{2}$ ways, *i.e.* 6 ways.
The last group can then be completed in only 1 way.
$\therefore$ The ways of selecting the groups so that each has a brother is $6 \times 15 \times 6 \times 1 = 540$ ways.
Probability of each group containing one brother is $\dfrac{540}{1680} = \dfrac{9}{28}$.
The probability of one group containing at least two brothers is 1 – probability of each group containing one brother.
That is, $1 - \dfrac{9}{28} = \dfrac{19}{28}$.

(b) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^n dx$, where n is an integer, $n \geqslant 0$.

(i) Using integration by parts, show that, for $n \geqslant 2$, $\boxed{3}$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$$

Solution:

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^{n-1} x . \sin x \, dx, \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-1} x . \frac{d(-\cos x)}{dx} \, dx, \\
&= \Big[-\cos x . \sin^{n-1} x \Big]_0^{\frac{\pi}{2}} \\
&\qquad - (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x . \cos x . -\cos x \, dx, \\
&= 0 + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x . (1 - \sin^2 x) \, dx. \\
i.e.\ I_n &= (n-1) I_{n-2} - (n-1) I_n, \\
I_n + (n-1) I_n &= (n-1) I_{n-2}, \\
n I_n &= (n-1) I_{n-2}, \\
I_n &= \left(\frac{n-1}{n}\right) I_{n-2}, \text{ as required.}
\end{aligned}$$

(ii) Deduce that $I_{2n} = \dfrac{2n-1}{2n}.\dfrac{2n-3}{2n-2}\cdots\dfrac{3}{4}.\dfrac{1}{2}.\dfrac{\pi}{2}$ $\boxed{3}$

and $I_{2n+1} = \dfrac{2n}{2n+1}.\dfrac{2n-2}{2n-1}\cdots\dfrac{4}{5}.\dfrac{2}{3}.1.$

Solution: From (i),
$$\begin{aligned}
I_{2n} &= \frac{2n-1}{2n}I_{2n-2},\\
&= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}I_{2n-4},\ \text{etc.}\\
\text{Note that } I_0 &= \int_0^{\frac{\pi}{2}}(\sin x)^0\,dx,\\
&= \int_0^{\frac{\pi}{2}} 1\,dx,\\
&= \frac{\pi}{2},\\
\text{and } I_2 &= \frac{1}{2}\times\frac{\pi}{2},\\
I_4 &= \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2},\\
\therefore\ I_{2n} &= \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\cdots\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}.\\
\text{Similarly, } I_{2n+1} &= \frac{2n}{2n+1}I_{2n-1},\\
I_{2n+1} &= \frac{2n-2}{2n-1}I_{2n-3},\ \text{etc.}\\
\text{Also } I_1 &= \int_0^{\frac{\pi}{2}}\sin x\,dx,\\
&= -\Big[\cos x\Big]_0^{\frac{\pi}{2}}\\
&= 1.\\
\text{and } I_3 &= \frac{2}{3}\times I_1 = \frac{2}{3}\times 1,\\
\text{and } I_5 &= \frac{4}{5}\times I_3 = \frac{4}{5}\times\frac{2}{3}\times 1,\\
\therefore\ I_{2n+1} &= \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\cdots\times\frac{4}{5}\times\frac{2}{3}\times 1.
\end{aligned}$$

(iii) Explain why $I_k > I_{k+1}$. $\boxed{1}$

Solution: For $0 < x < \frac{\pi}{2}$, $0 < \sin x < 1$.
$\therefore\ 0 < (\sin x)^{k+1} < (\sin x)^k < 1$, for
$0 < x < \frac{\pi}{2}$ and $k \geqslant 0$ and an integer.
$\therefore\ \displaystyle\int_0^{\frac{\pi}{2}}(\sin x)^{k+1}\,dx < \int_0^{\frac{\pi}{2}}(\sin x)^k\,dx$
that is $I_k > I_{k+1}$.

(iv) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that $\boxed{4}$

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2.4^2\ldots(2n)^2}{1.3^2.5^2\ldots(2n-1)^2(2n+1)} < \frac{\pi}{2}.$$

Solution: Given that $I_{2n} > I_{2n+1}$:

$$\frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\cdots\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} > \frac{2n}{2n+1}\times\frac{2n-2}{2n-1}\times\cdots\times\frac{4}{5}\times\frac{2}{3}\times 1$$

$$\therefore\ \frac{\pi}{2} > \frac{(2n)^2(2n-2)^2\times\cdots\times 4^2\times 2^2}{(2n+1)(2n-1)^2(2n-3)^2\times\cdots\times 5^2\times 3^2\times 1^2}\ \ldots\ldots\ldots\ldots\ \boxed{1}$$

Given that $I_{2n-1} > I_{2n}$:

$$\frac{2n-2}{2n-1}\times\frac{2n-4}{2n-3}\times\cdots\times\frac{4}{5}\times\frac{2}{3}\times 1 > \frac{2n-1}{2n}\times\frac{2n-3}{2n-2}\times\cdots\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$\therefore\ \frac{(2n)(2n-2)^2(2n-4)^2\times\cdots\times 4^2\times 2^2}{(2n-1)^2(2n-3)^2\times\cdots\times 5^2\times 3^2\times 1^2} > \frac{\pi}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \boxed{2}$$

$\boxed{2}\times\dfrac{2n}{2n+1}$:

$$\frac{(2n)^2(2n-2)^2(2n-4)^2\times\cdots\times 4^2\times 2^2}{(2n+1)(2n-1)^2(2n-3)^2\times\cdots\times 5^2\times 3^2\times 1^2} > \frac{\pi}{2}\left(\frac{2n}{2n+1}\right)\ldots\ldots\ \boxed{3}$$

From $\boxed{1}$ and $\boxed{3}$, we get:

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2\times 4^2\times\cdots\times(2n)^2}{(2n+1)(2n-1)^2(2n-3)^2\times\cdots\times 5^2\times 3^2\times 1^2} < \frac{\pi}{2},\ \text{as required.}$$

End of Paper