SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2011

YEAR 12 Mathematics Extension 2

HSC Task #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answer must be given in simplest exact form.

Total Marks – 83

- Attempt questions 1-6
- Start each new section of a separate answer booklet

Examiner: *D.McQuillan*

SECTION A

Question 1

(a) Let $w_1 = -8 + 3i$ and $w_2 = 5 - 2i$. Find $w_1 - \overline{w}_2$. 1

(b) Find 5

(i)

$$\int x \tan^{-1} x \, dx$$

(ii)

$$\int \frac{\tan\theta}{1 + \cos\theta} \, d\theta$$

(c) Evaluate 6

(i)

$$\int_{-2}^{-1} \frac{dx}{x^2 + 4x + 5}$$

(ii)

$$\int_{0}^{\frac{\pi}{4}} \frac{-\sin x}{\cos^3 x} \, dx$$

(d) $27x^3 - 36x + k = 0$ has a double root. Find the possible values of k. 2

Question 2

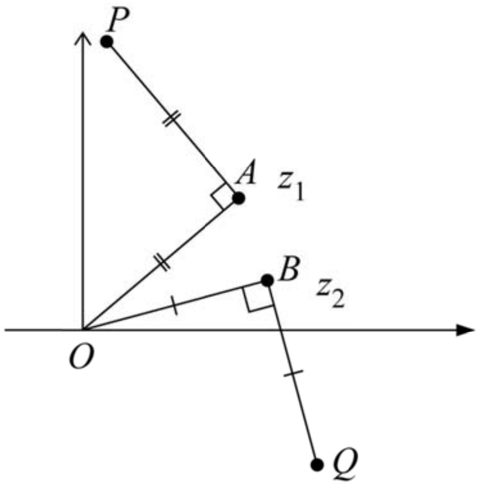
(a) In how many ways can 5 mathematics books and 3 science books be arranged on a shelf so that the books of each subject come together? 2

(b) In the expansion of $\left(2x^2 - \frac{3}{x}\right)^9$ what is the term independent of x? 2

(c) On an Argand diagram, shade the region specified by both the conditions 4

$$Re(z) \leq 4 \text{ and } |z - 4 + 5i| \leq 3$$

(d) The points A and B in the complex plane correspond to complex numbers z_1 and z_2 respectively. Both triangle OAP and OBQ are right-angled isosceles triangles. 4



(i) Explain why P corresponds to the complex number $(1 + i)z_1$.

(ii) Let M be the midpoint of PQ. What complex number corresponds to M?

END OF SECTION

Start each SECTION in a NEW writing BOOKLET

SECTION B

Question 3

(a) A golf ball is hit with a velocity of 40 m/s at an angle of 38° to the horizontal. If it just clears a tree 20 metres away, find the height of the tree to two decimal places. **3**

(b) Sketch the graphs of the following functions for $-2\pi \leq x \leq 2\pi$. **6**

(i) $y = \sin x + \frac{1}{x}$

(ii) $y = x \sin x$

(iii) $y = \frac{\sin x}{x}$

(c) Consider the polynomial equation $x^4 + ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are all integers. Suppose the equation has a root of the form ki, where k is real, and $k \neq 0$. **6**

(i) State why the conjugate $-ki$ is also a root.

(ii) Show that $c = k^2 a$.

(iii) Show that $c^2 + a^2 d = abc$.

(iv) If 2 is also a root of the equation, and $b = 0$, show that c is even.

Question 4

(a) The probability that a missile will hit a target is $\frac{2}{5}$. What is the probability that the target will be hit at least twice if 4 missiles are fired in quick succession? 2

(b) 3

(i) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k = 1$.

(ii) Show that if the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are arbitrary integers.

(c) A body of mass one kilogram is projected vertically upwards from the ground with an initial speed of 20 metres per second. The body is subject to both gravity of 10 m/s^2 and air resistance of $\frac{v^2}{40}$ where v is the body's velocity at that time. 7

(i) While the body is travelling upwards, the equation of motion is $\ddot{x} = -\left(10 + \frac{v^2}{40}\right)$.

(1) Using $\ddot{x} = v\frac{dv}{dx}$, calculate the greatest height reached by the body.

(2) Using $\ddot{x} = \frac{dv}{dt}$, calculate the time taken to reach the greatest height.

(ii) After reaching its greatest height, the body falls back to its starting point. The body is still affected by gravity and air resistance.

(1) Write the equation of motion of the body as is falls.

(2) Find the speed of the body when it returns to its starting point.

(d) 3

(i) Find the remainder when $x^2 + 6$ is divided by $x^2 + x - 6$.

(ii) Hence, find $\int \frac{x^2+6}{x^2+x-6}\,dx$.

END OF SECTION

Start each SECTION in a NEW writing BOOKLET

SECTION C

Question 5

(a) There are 3 pairs of socks in a drawer. Each pair is a different colour. If two socks are selected at random, what is the probability that they are a matching pair? 2

(b) By considering $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. 5

(i) Show that

$$\binom{2m}{0} + \binom{2m}{2} + \cdots + \binom{2m}{2m} = \binom{2m}{1} + \binom{2m}{3} + \cdots + \binom{2m}{2m-1}$$

(ii) Show that

$$\binom{n}{0} + \frac{\binom{n}{1}}{2} + \frac{\binom{n}{2}}{3} + \cdots + \frac{\binom{n}{n}}{n+1} = \frac{2^{n+1}-1}{n+1}$$

(c) 6

(i) Show that $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

(ii) Hence show that $\cos n\theta - \cos(n+1)\theta = 2\sin\left(n+\frac{1}{2}\right)\theta \sin\frac{\theta}{2}$.

(iii) Show that $1 + z + z^2 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, $z \neq 1$.

(iv) Let $z = \cos\theta + i\sin\theta$, $0 < \theta < 2\pi$. By consider the real parts of the expression in (iii), show that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

Question 6

(a) Let $I_n = \int_1^e (\log_e x)^n dx$. 4

(i) Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$

(ii) Hence evaluate I_4.

(b) Show that the sum of the *x*- and *y*-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to *c*. 4

(c) Let α, β and γ be the roots of $x^3 + qx + r = 0$. Define $s_n = \alpha^n + \beta^n + \gamma^n$ for $n = 1, 2, 3, \ldots$

6

(i) Explain why $s_1 = 0$ and show that $s_2 = -2q$.

(ii) By considering that $\alpha^3 + q\alpha + r = 0$ show that $s_3 = -3r$.

(iii) Show that $s_5 = 5qr$.

END OF EXAM

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

X2

Q1

(a) $w_1 - \overline{w_2} = -8+3i-(5+2i)$
$= \boxed{-13+i}$ [1]

(b) (i) $\int x \tan^{-1}x \, dx = \int \frac{d}{dx}\left(\frac{x^2}{2}\right) \tan^{-1}x \, dx$

$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1} \, dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$

$= \boxed{\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2} + c}$ [3]

(ii) $\int \frac{\tan\theta \, d\theta}{1+\cos\theta} = \int \frac{\tan\theta\sec\theta}{\sec\theta+1} \, d\theta$

$= \boxed{\ln(\sec\theta+1) + c}$ [2]

[NB there were many approaches to this question including 't-results' & substitution]

(c) (i) $\int_{-2}^{-1} \frac{dx}{x^2+4x+5} = \int_{-2}^{-1} \frac{dx}{(x+2)^2+1}$

$= \left[\tan^{-1}(x+2)\right]_{-2}^{-1}$

$= \tan^{-1}1 - \tan^{-1}0$

$\boxed{= \frac{\pi}{4}}$ [3]

P.B.

Q1 (CONTD)

(ii) $\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos^3 x}\,dx$ let $u = \cos x$

$du = -\sin x\,dx$.

$= \int_1^{\frac{1}{\sqrt{2}}} \frac{du}{u^3}$

$= -\frac{1}{2}\left[\frac{1}{u^2}\right]_1^{\frac{1}{\sqrt{2}}}$

$= -\frac{1}{2}\left(\frac{1}{1/2} - 1\right)$

$= -\frac{1}{2}(2-1)$

$= \boxed{-\frac{1}{2}}$ [3]

(d) $P(x) = 27x^3 - 36x + k.$

$P'(x) = 81x^2 - 36$

Possible double roots where $P'(x) = 0$

ie. $81x^2 - 36 = 0.$

$x^2 = \frac{36}{81}$

$= \frac{4}{9}$

$\therefore x = \pm\frac{2}{3}$

Let $P(\frac{2}{3}) = 0$	Let $P(-\frac{2}{3}) = 0$
$8 - 24 + k = 0.$	$-8 + 24 + k = 0$
$k = 16.$	$k = -16$

[2]

$\therefore \boxed{k = \pm 16}$

P.B.

X2

Q2. (a) $2 \times 5! \times 3! = \boxed{1440}$ [2]

(b) $T_{r+1} = \binom{9}{r}(2x^2)^{9-r}\left(\frac{-3}{x}\right)^r$

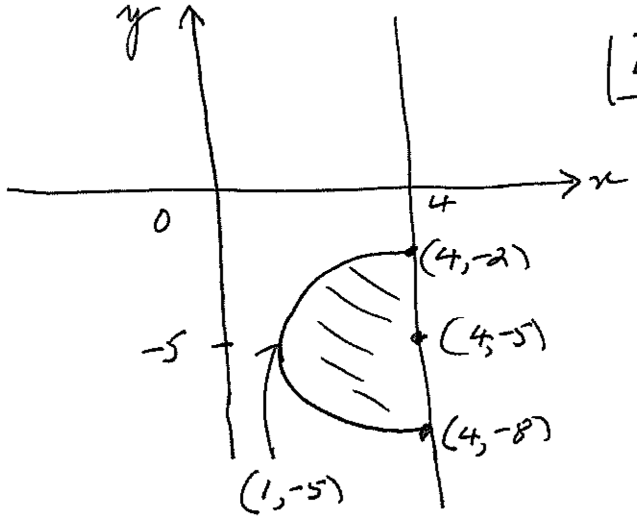
$= \binom{9}{r} 2^{9-r}(-3)^r \cdot x^{18-3r}$

For constant term $18-3r=0$
$r=6$.

$\therefore T_7 = \binom{9}{6} 2^3 \times (-3)^6$

$= \boxed{489\,888}$ [2]

(c)



[4] [NB $|z-4+5i| \leq 3$
$\Rightarrow |z-(4-5i)| \leq 3$
ie circle centre (4,-5) radius 3 is the boundary]

(d) (i) $\vec{OP} = \vec{OA} + i\,\vec{OA}$ [multiplication by i is a rotation through $\frac{\pi}{2}$, anti-clockwise

$\therefore$ P represents $z_1 + i z_1 = (1+i)z_1$ [2]

(ii) Similarly Q represents $z_2 + -i z_2$
$= (1-i)z_2$ [2]

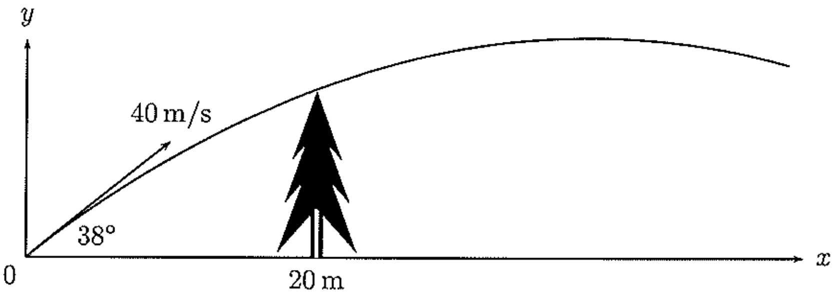
$\therefore$ M represents $\boxed{\dfrac{(1+i)z_1 + (1-i)z_2}{2}}$

P.B.

2011 Extension 2 Mathematics Task 2:
Solutions— Section B

3. (a) A golf ball is hit with a velocity of 40 m/s at an angle of 38° to the horizontal. If it just clears a tree 20 metres away, find the height of the tree to two decimal places. [3]

Solution:

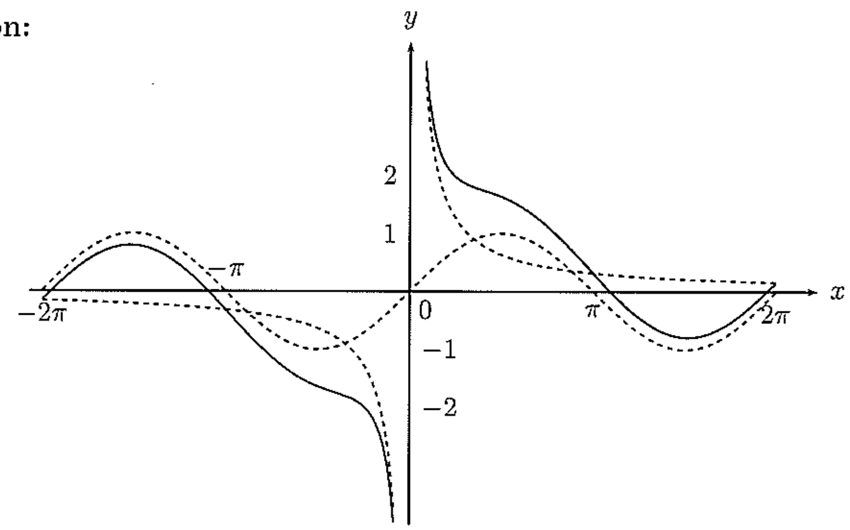


$$\ddot{x} = 0, \qquad \ddot{y} = -10, \text{ (taking } g = 10\,\text{m/s}^2\text{)}$$
$$\dot{x} = 40\cos 38°, \qquad \dot{y} = 40\sin 38° - 10t,$$
$$x = 40\cos 38° t, \qquad y = 40\sin 38° t - 5t^2.$$
When $x = 20,\ t = \dfrac{1}{2\cos 38°}$, so $y = \dfrac{40\sin 38°}{2\cos 38°} - \dfrac{5}{4\cos^2 38°}$,
$$\approx 13.61\text{ m}.$$
(Or $y \approx 13.65$ m, if using $g = 9.8\,\text{m/s}^2$.)

(b) Sketch the graphs of the following functions for $-2\pi \leqslant x \leqslant 2\pi$: [6]

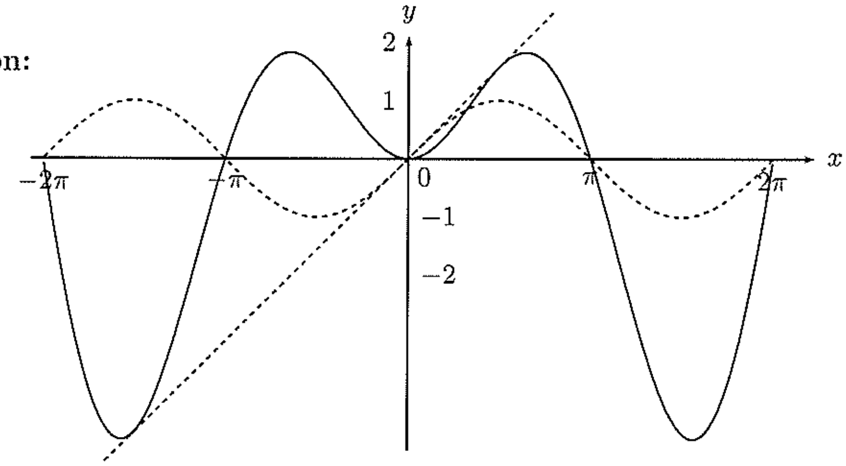
(i) $y = \sin x + \dfrac{1}{x}$,

Solution:



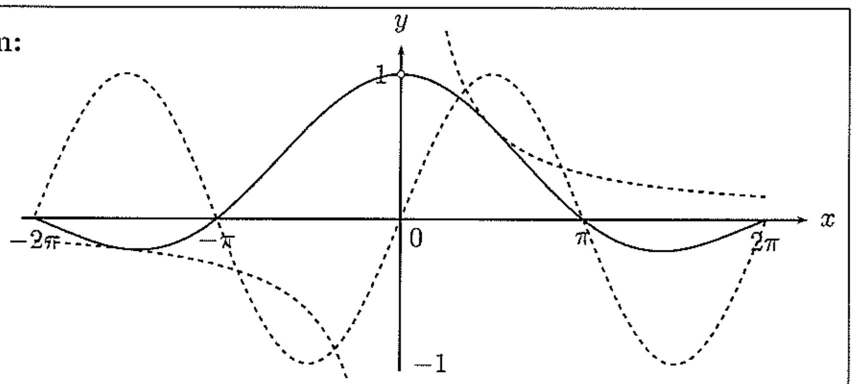
(ii) $y = x \sin x$,

Solution:



(iii) $y = \dfrac{\sin x}{x}$.

Solution:



(c) Consider the polynomial equation $x^4 + ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are all integers. Suppose the equation has a root of the form ki, where k is real and $k \neq 0$. [6]

(i) State why the conjugate, $-ki$, is also a root.

Solution: If a polynomial has real coefficients, any complex roots occur in conjugate pairs.

(ii) Show that $c = k^2 a$.

Solution: Method 1—

$$\begin{aligned} P(x) &= x^4 + ax^3 + bx^2 + cx + d, \\ P(ki) &= k^4 - iak^3 - bk^2 + ick + d = 0. \\ -ak^3 + ck &= 0, \text{ (equating imaginary coefficients)} \\ \textit{i.e.,}\ c &= ak^2. \end{aligned}$$

Solution: Method 2—

$$\begin{aligned} P(x) &= x^4 + ax^3 + bx^2 + cx + d, \\ P(ki) &= k^4 - iak^3 - bk^2 + ick + d = 0, \ldots. \boxed{1} \\ P(ki) &= k^4 + iak^3 - bk^2 - ick + d = 0, \ldots. \boxed{2} \\ \boxed{1} - \boxed{2}: \quad -2iak^3 + 2ick &= 0, \\ i.e., \ c &= ak^2. \end{aligned}$$

(iii) Show that $c^2 + a^2d = abc$.

Solution: Method 1—
Equating real coefficients of $P(ki)$,

$$\begin{aligned} k^4 - bk^2 + d &= 0, \\ \frac{c^2}{a^2} - \frac{bc}{a} + d &= 0, \ \left(\text{substituting } k^2 = \tfrac{c}{a}\right) \\ c^2 - abc + a^2d &= 0, \\ \therefore abc &= c^2 + a^2d. \end{aligned}$$

Solution: Method 2—

$$\begin{aligned} 2k^4 - 2bk^2 + 2d &= 0, \ \boxed{1} + \boxed{2} \text{ (from part (ii) above)} \\ \frac{c^2}{a^2} - \frac{bc}{a} + d &= 0, \ \left(\text{substituting } k^2 = \tfrac{c}{a}\right) \\ c^2 - abc + a^2d &= 0, \\ \therefore abc &= c^2 + a^2d. \end{aligned}$$

(iv) If 2 is also a root of the equation and $b = 0$, show that c is even.

Solution: $P(2) = 16 + 8a + 2c + d = 0$, so d is even.
From part (iii), if $b = 0$ then $c^2 = -a^2d$.
Hence c^2 is even, and thus c is even.

4. (a) The probability that a missile will hit a target is $\frac{2}{5}$. What is the probability that the target will be hit at least twice if 4 missiles are fired in quick succession? [2]

Solution:
$$\begin{aligned} P(\text{hit} \geqslant 2) &= 1 - \left\{ \left(\frac{3}{5}\right)^4 + \binom{4}{1} \times \frac{2}{5} \times \left(\frac{3}{5}\right)^3 \right\} \\ &= 1 - \frac{81 + 216}{625}, \\ &= \frac{328}{625} \quad (= 0.5248). \end{aligned}$$

(b) (i) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k = 1$. [3]

Solution:
$$\begin{aligned} \left(\cos\left(\tfrac{4\pi}{7}\right) + i\sin\left(\tfrac{4\pi}{7}\right)\right)^k &= 1 = \cos(2n\pi) + i\sin(2n\pi), \ n \in \mathbb{J}, \\ \frac{4k\pi}{7} &= 2n\pi, \\ k &= \frac{7n}{2}, \\ &= 7 \text{ when } n = 2. \end{aligned}$$

(ii) Show that if the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are arbitrary integers.

Solution:
$$\begin{aligned} w^n &= 1, \ (\text{as } w \text{ is a solution of } z^n = 1) \\ \text{now } (w^n)^m &= 1^m = 1, \\ w^{nm} &= 1 \quad = w^{mn}, \\ (w^m)^n &= 1^n, \\ \text{so } w^m &= 1, \text{ which establishes the result.} \end{aligned}$$

(c) A body of mass one kilogram is projected vertically upwards from the ground with an initial speed of 20 metres per second. The body is subjected to both gravity of $10\,\text{m/s}^2$ and air resistance of $\frac{v^2}{40}$ where v is the body's velocity at that time. [7]

(i) While the body is travelling upwards, the equation of motion is
$$\ddot{x} = -\left(10 + \frac{v^2}{40}\right).$$

(α) Using $\ddot{x} = v\dfrac{dv}{dx}$, calculate the greatest height reached by the body.

Solution:

$$\begin{aligned}
v\frac{dv}{dx} &= -\frac{400+v^2}{40}, \\
\int_0^h dx &= -\int_{20}^0 \frac{20\times 2v}{400+v^2}\,dv, \\
x\Big]_0^h &= -20\Big[\ln(400+v^2)\Big]_{20}^0, \\
h &= -20\ln\frac{400}{800}, \\
&= 20\ln 2, \\
&\approx 13.9\,\text{m (3 sig. fig.)}
\end{aligned}$$

(β) Using $\ddot{x} = \dfrac{dv}{dt}$, calculate the time taken to reach the greatest height.

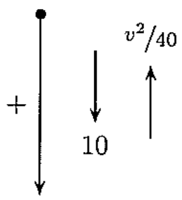
Solution:

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{400+v^2}{40}, \\
-\int_0^t dt &= 40\int_{20}^0 \frac{dv}{20^2+v^2}, \\
-t\Big]_0^t &= 40\times\frac{1}{20}\Big[\tan^{-1}\frac{v}{20}\Big]_{20}^0, \\
-t &= 2\tan^{-1}0 - 2\tan^{-1}1, \\
t &= \frac{\pi}{2}, \\
&\approx 1.57\,\text{s (3 sig. fig.)}
\end{aligned}$$

(ii) After reaching its greatest height, the body falls back to its starting point. The body is still affected by gravity and air resistance.

(α) Write the equation of motion of the body as it falls.

Solution:



$$\ddot{x} = 10 - \frac{v^2}{40}.$$

(β) Find the speed of the body when it returns to its starting point.

Solution:

$$\begin{aligned} v\frac{dv}{dx} &= \frac{400-v^2}{40}, \\ -\int_0^v \frac{-40v\,dv}{400-v^2} &= \int_0^{20\ln 2} dx, \\ -20\Big[\ln(400-v^2)\Big]_0^v &= x\Big]_0^{20\ln 2}, \\ -20\ln\left(\frac{400-v^2}{400}\right) &= 20\ln 2 - 0, \\ \frac{400-v^2}{400} &= \frac{1}{2}, \\ v^2 &= 400-200, \\ v &= \sqrt{200} \text{ (taking downwards +ve)}, \\ &= 10\sqrt{2}, \\ &\approx 14.1\,\text{m/s (3 sig. fig.)}. \end{aligned}$$

(d) (i) Find the remainder when x^2+6 is divided by x^2+x-6. [3]

Solution:

$$\begin{array}{r|l} & \qquad\qquad 1 \\ x^2+x-6 & \; x^2 \qquad\quad +6 \\ & -x^2 - x \;\; +6 \\ \hline & \qquad -x + 12 \end{array}$$

So the remainder is $12-x$.

Alternatively:

$$\begin{aligned} \frac{x^2+6}{x^2+x-6} &= \frac{x^2+x-6-x+12}{x^2+x-6}, \\ &= 1 + \frac{-x+12}{x^2+x-6}. \end{aligned}$$

So again the remainder is $12-x$.

(ii) Hence find $\displaystyle\int \frac{x^2+6}{x^2+x-6}dx$.

Solution:

$$\begin{aligned} \int \frac{x^2+6}{x^2+x-6}dx &= \int \left\{1 + \frac{12-x}{x^2+x-6}\right\} dx. \\ \frac{12-x}{x^2+x-6} &\equiv \frac{A}{x+3} + \frac{B}{x-2}, \\ 12-x &\equiv A(x-2)+B(x+3), \\ \text{put } x=2, \quad 10 &= 5B \implies B = 2, \\ x=-3, \quad 15 &= -5A \implies A = -3. \\ \int \frac{x^2+6}{x^2+x-6}dx &= \int \left\{1 - \frac{3}{x+3} + \frac{2}{x-2}\right\} dx, \\ &= x - 3\ln(x+3) + 2\ln(x-2) + c. \end{aligned}$$

SECTION C

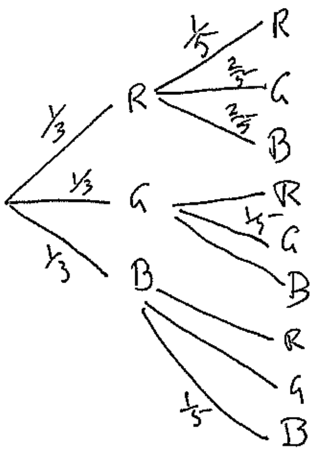
Question 5

(a) $p(\text{2 matching}) = p(\text{1st matching}) \times p(\text{2nd matching})$

$= 1 \times \frac{1}{5}$

$= \frac{1}{5}$

OR



$p(RR) = \frac{1}{15}$

$p(GG) = \frac{1}{15}$

$P(BB) = \frac{1}{15}$

$\therefore p(\text{matching}) = \frac{3}{15}$

$= \frac{1}{5}$

[2]

(b)

(b)(i) $(x+1)^{2m} = {}^{2m}C_0 x^{2m} + {}^{2m}C_1 x^{2m-1}\cdot 1 + {}^{2m}C_2 x^{2m-2} \dots {}^{2m}C_{2m}$

Let $x = -1$

$0 = {}^{2m}C_0 - {}^{2m}C_1 + {}^{2m}C_2 - \dots - {}^{2m}C_{2m-1} + {}^{2m}C_{2m}$

$\therefore {}^{2m}C_0 + {}^{2m}C_2 + \dots + {}^{2m}C_{2m} = {}^{2m}C_1 + {}^{2m}C_3 + \dots + {}^{2m}C_{2m-1}$ [2]

(ii) $(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_{n-2}x^2 + {}^nC_{n-1}x + {}^nC_n$

Integrate wrt x:

$\frac{(x+1)^{n+1}}{n+1} = \frac{{}^nC_0 x^{n+1}}{n+1} + \frac{{}^nC_1 x^n}{n} + \frac{{}^nC_2 x^{n-1}}{n-1} + \dots + \frac{{}^nC_{n-2} x^3}{3} + \frac{{}^nC_{n-1} x^2}{2} + {}^nC_n x + c$

Let $x = 0$ $\frac{1}{n+1} = C$. Let $x = 1$

$\frac{2^{n+1}}{n+1} = \frac{{}^nC_n}{n+1} + \frac{{}^nC_{n-1}}{n} + \dots + \frac{{}^nC_{n-2}}{3} + \frac{{}^nC_{n-1}}{2} + {}^nC_0 + \frac{1}{n+1}$

$\frac{2^{n+1}-1}{n+1} = - - - -$ as req'd.

[3]

3

(c) (i) $\cos(A-B) - \cos(A+B)$

$= (\cos A\cos B + \sin A \sin B) - (\cos A\cos B - \sin A \sin B)$

$= 2\sin A \sin B$. — ① [1]

(ii) In ① let $A-B = n\theta$

$A+B = (n+1)\theta$

$\therefore 2A = 2n\theta + \theta \quad 2B = \theta$

$A = n\theta + \frac{1}{2}\theta \quad B = \frac{\theta}{2}$

$= (n+\frac{1}{2})\theta$

$\therefore \cos n\theta - \cos(n+1)\theta = 2\sin(n+\frac{1}{2})\theta \sin\frac{\theta}{2}$ [1]

(iii) RTP $1 + z + z^2 + \dots + z^n = \frac{1-z^{n+1}}{1-z}$

Consider $(1 + z + z^2 + \dots z^n)$ — ①

and $z(1+z+z^2+\dots+z^n)$ — ②

$= z + z^2 + z^3 + \dots + z^n + z^{n+1}$ — ③ [2]

Now ① − ③ $= (1-z)(1+z+z^2+\dots+z^n)$

$= 1 - z^{n+1}$

$\therefore$ — as req'd.

OR GS $a=1$, $r=z$

no. of terms $= n+1$

$\therefore S_{n+1} = \frac{1(1-z^{n+1})}{1-z}$

QED

4

(iv) RTP: $1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\theta/2}$

In (iii) putting $z = \text{cis}\,\theta$

$1 + \text{cis}\,\theta + \text{cis}\,2\theta + \dots + \text{cis}\,n\theta = \frac{1-\text{cis}(n+1)\theta}{1-\text{cis}\,\theta}$

Equating real parts:

$1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta = \text{Re}[LHS]$

$RHS = \frac{1-\text{cis}(n+1)\theta}{1-\text{cis}\,\theta} = \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{1-\cos\theta - i\sin\theta}$

$= \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{(1-\cos\theta) - i\sin\theta} \times \frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}$

$\text{Re}[RHS] = \frac{(1-\cos\theta) - \cos(n+1)\theta + \cos\theta\cos(n+1)\theta + \sin\theta\sin(n+1)\theta}{(1-\cos\theta)^2 + \sin^2\theta}$

$= \frac{1-\cos\theta - \cos(n+1)\theta + \cos\theta\cos(n+1)\theta + \sin\theta\sin(n+1)\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \frac{1-\cos\theta - \cos(n+1)\theta + \cos(n+1-1)\theta}{2-2\cos\theta}$

$= \frac{1}{2} + \frac{-\cos(n+1)\theta + \cos n\theta}{2-2\cos\theta}$

$= \frac{1}{2} + \frac{2\sin(n+\frac{1}{2})\theta\sin\frac{\theta}{2}}{2-2\cos\theta}$ from (ii)

(iv) Contd 5

$= \frac{1}{2} + \frac{2\sin(n+\frac{1}{2})\theta \sin\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$2-2\cos\theta = 4(\frac{1}{2}-\frac{1}{2}\cos\theta)$
$= 4\sin^2\theta/2$

$= \frac{1}{2} + \frac{2\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$

[2]

But Re[LHS] = Re[RHS]

∴ QED

Question 6 6

(a)(i) $I_n = \int_1^e (\ln x)^n dx$

$= \int_1^e \frac{d}{dx}(x).(\ln x)^n dx$

$= x.(\ln x)^n \Big|_1^e - \int_1^e x.\frac{d}{dx}(\ln x)^n dx$

$= e - \int_1^e x.n(\ln x)^{n-1}.\frac{1}{x}.dx$

$= e - n\int_1^e (\ln x)^{n-1} dx$ [2]

$\therefore I_n = e - nI_{n-1}$

(ii) $I_0 = \int_1^e 1.dx$

$= [x]_1^e$

$= e-1$

$I_1 = e-(e-1)$
$= e-e+1$
$= 1$

$I_2 = e - 2I_1$
$= e-2$

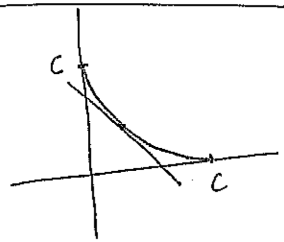
$I_3 = e - 3(e-2)$
$= -2e+6$

$I_4 = e-4(-2e+6)$
$= 9e-24$

[2]

Q6

(b) $\sqrt{x}+\sqrt{y}=\sqrt{c}$



Differentiating Implicitly:

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot y' = 0$$

$$\frac{1}{2\sqrt{y}}\, y' = -\frac{1}{2\sqrt{x}}$$

$$y' = -\frac{\sqrt{y}}{\sqrt{x}}$$

$y'=0$ when $y=0$

$y'\to\infty$ when $x=0$

$$y-y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x-x_1)$$

x-int: ($y=0$) $\quad -y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x-x_1)$

$$y_1 = \frac{\sqrt{y_1}}{\sqrt{x_1}}(x-x_1)$$

$$y_1\sqrt{x_1} = \sqrt{y_1}(x-x_1)$$

$$\sqrt{y_1}\sqrt{x_1} = x-x_1$$

$$x = x_1+\sqrt{x_1}\sqrt{y_1}$$

y-int: ($x=0$)

$$y-y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(0-x_1)$$

$$y = y_1+\sqrt{y_1}\sqrt{x_1}$$

Sum of Intercepts $x+y = x_1+\sqrt{x_1}\sqrt{y_1}+y_1+\sqrt{x_1}\sqrt{y_1}$

$$= x_1+2\sqrt{x_1}\sqrt{y_1}+y_1$$

$$= (\sqrt{x_1}+\sqrt{y_1})^2$$

$$= c$$

[4]

Q6

(c) $x^3+qx+r=0$ Roots α,β,γ $\quad S_n=\alpha^n+\beta^n+\gamma^n$

(i) $S_1 = \alpha+\beta+\gamma$

$= 0$ Sum of roots is coeff of x^2 (=0).

$S_2 = \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 0-2(\Sigma\alpha\beta)$

$= 0-2q$

$= -2q$

[2]

(ii) $\alpha^3+q\alpha+r=0$ — (1)

$\beta^3+q\beta+r=0$ — (2)

$\gamma^3+q\gamma+r=0$ — (3)

(1)+(2)+(3): $\alpha^3+\beta^3+\gamma^3+q(\alpha+\beta+\gamma)+3r=0$

$\therefore \alpha^3+\beta^3+\gamma^3 = -3r$ $\quad (\Sigma\alpha=0)$

$\therefore S_3 = -3r$

[2]

(iii) Consider $S_3\times S_2 = (-3r)\times(-2q)$

$= 6qr$

Expand LHS $= (\alpha^3+\beta^3+\gamma^3)(\alpha^2+\beta^2+\gamma^2)$

$= \alpha^5+\beta^5+\gamma^5+\alpha^3\beta^2+\alpha^3\gamma^2+\beta^3\alpha^2+\beta^3\gamma^2+\gamma^3\alpha^2+\gamma^3\beta^2$

$= S_5+\alpha^2\beta^2(\alpha+\beta)+\alpha^2\gamma^2(\alpha+\gamma)+\beta^2\gamma^2(\beta+\gamma)$

Q6 9

(iii) Cont'd $\alpha+\beta+\gamma=0$

$\therefore \alpha+\beta=-\gamma, \quad \alpha+\gamma=-\beta, \quad \beta+\gamma=-\alpha$

$LHS = S_5 - \alpha^2\beta^2\gamma - \alpha^2\gamma^2\beta - \beta^2\gamma^2\alpha$

$= S_5 - \alpha\beta\gamma(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= S_5 - (-r)(q)$ $\quad \alpha\beta\gamma = -r$

$= S_5 + qr$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma = q$

$\therefore S_5 + qr = 6qr$ [2]

$\therefore S_5 = 5qr$ QED