SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013
YEAR 12 Mathematics Extension 2
HSC Task #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 120 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answers must be given in simplest exact form unless otherwise stated.

Total marks - 77

Multiple Choice Section (7 marks)
- Answer Questions 1-7 on the Multiple Choice answer sheet provided.

Sections A, B and C (70 marks)

- Start a new answer booklet for each section.

Examiner: *D.McQuillan*

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Multiple Choice Section [7 marks]

1 What is the value of $\int_1^3 x\left(x-2\right)^5 dx$? Use the substitution $u = x - 2$.

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$

(C) $\frac{1}{3}$ (D) $\frac{2}{3}$

2 What is the value of $\arg \overline{z}$ given the complex number $z = 1 - i\sqrt{3}$?

(A) $-\frac{\pi}{3}$

(B) $-\frac{2\pi}{3}$

(C) $-\frac{\pi}{3}$

(D) $\frac{\pi}{3}$

3 The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$.
What are the values of *a* and *b*?

(A) $a = -11$ and $b = -12$

(B) $a = -5$ and $b = -12$

(C) $a = -11$ and $b = 12$

(D) $a = -5$ and $b = 12$

4 Let α, β and γ be roots of the equation $x^3 + 3x^2 + 4 = 0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

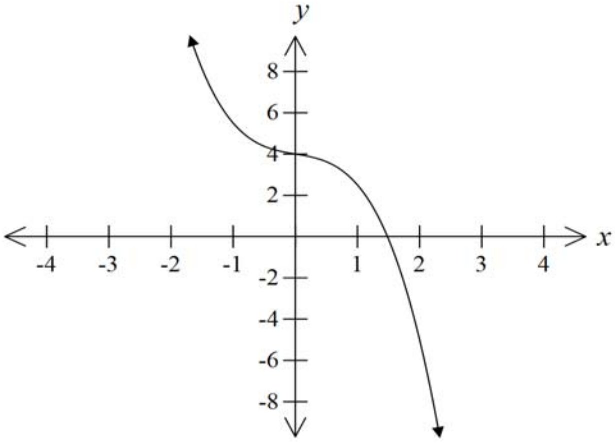
(A) $x^3 - 9x^2 - 24x - 4 = 0$

(B) $x^3 - 9x^2 - 12x - 4 = 0$

(C) $x^3 - 9x^2 - 24x - 16 = 0$

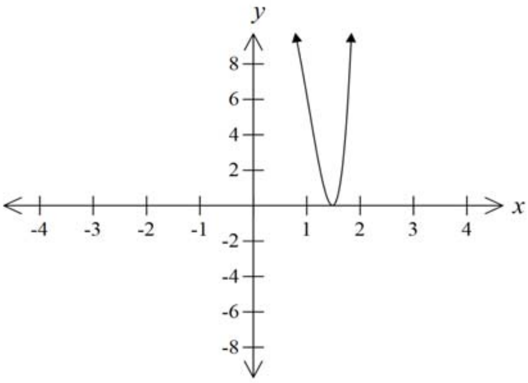
(D) $x^3 - 9x^2 - 12x - 16 = 0$

5 The diagram below shows the graph of the function $y = f(x)$.

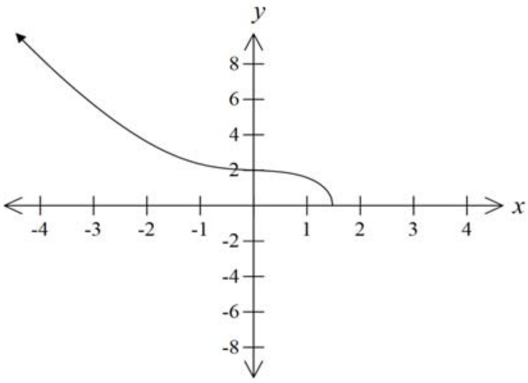


Which diagram represents the graph of $y^2 = f(x)$?

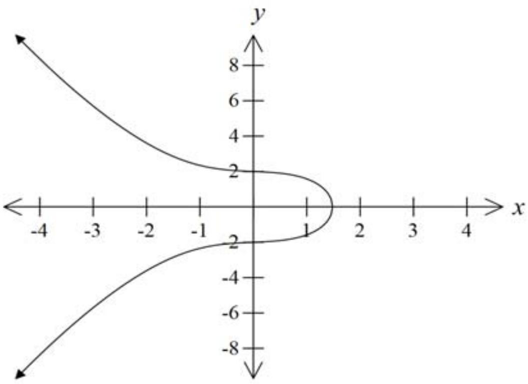
(A)



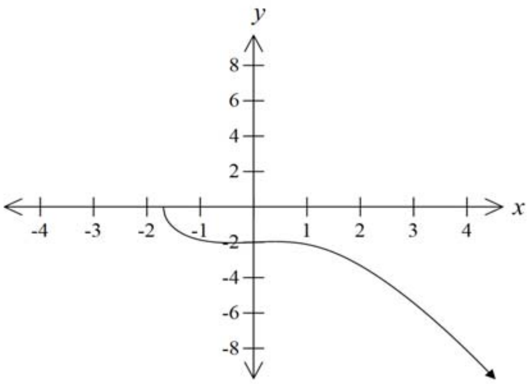
(B)



(C)



(D)



6 What is $-2+2\sqrt{3}i$ expressed in modulus-argument form?

(A) $2(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

(B) $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

(C) $2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

(D) $4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

7 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant. Which of the following is the correct expression for square of the velocity where x is the distance fallen?

(A) $v^2=\frac{g}{k}\left(1-e^{-2kx}\right)$

(B) $v^2=\frac{g}{k}\left(1+e^{-2kx}\right)$

(C) $v^2=\frac{g}{k}\left(1-e^{2kx}\right)$

(D) $v^2=\frac{g}{k}\left(1+e^{2kx}\right)$

End of Multiple Choice Section

Section A

Start each section in a new answer booklet.

Question 8 [13 marks]

(a) Find [8]

(i)

$$\int \frac{\cos x}{1+\sin x}dx$$

(ii)

$$\int \cos x \ln(\sin x) dx$$

(iii)

$$\int \frac{\sqrt{9-x^2}}{x^2}dx$$

(iv)

$$\int \frac{4}{x^3-x}dx$$

(b) Let $z = 2(\cos\theta - i\sin\theta)$ [5]

(i) Find $\bar{z}$ in mod-arg form.

(ii) Find $\overline{1+z}$.

(iii) Show that the real part of $\frac{1}{1+z}$ is $\frac{1+2\cos\theta}{5+4\cos\theta}$.

(iv) Express the imaginary part of $\frac{1}{1+z}$ in terms of θ.

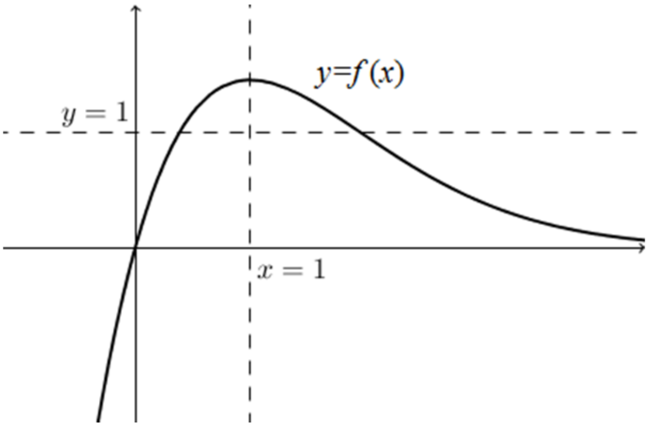
Question 9 [11 marks]

(a) [3]

(i) Show that $(2 - i)$ is a root of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$.

(ii) Hence factorise $P(x)$ into linear factors.

(b) [8]



Using four separate graphs sketch:

(i) $y = f'(x)$

(ii) $y = f(|\ x\ |)$

(iii) $y = \frac{1}{f(x)}$

(iv) $y = f\left(\frac{x}{3}\right)$

End of Section A

Section B

Start each section in a new answer booklet.

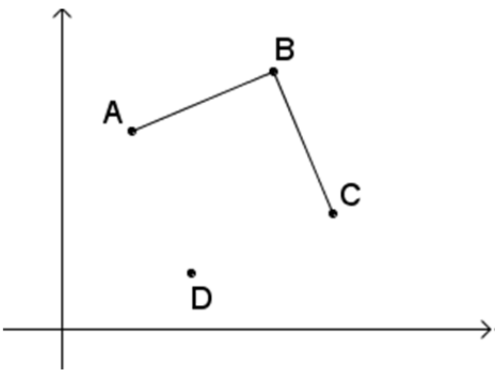
Question 10 [12 marks]

(a) A particle performs simple harmonic motion between $x = -3$ and $x = 7$. It takes 6 seconds to complete one oscillation returning to its starting position at $x = 2$. [4]

(i) If the initial movement was in the positive direction, write down the equation for its displacement in terms of time *t*.

(ii) Express its velocity in terms of its displacement.

(b) [3]



In the diagram the vertices of ΔABC are represented by the complex number z_1, z_2 and z_3 respectively. The triangle is isosceles and right-angled at *B*.

(i) Explain why $(z_3 - z_2)^2 = -(z_1 - z_2)^2$.

(ii) *D* is the point such that *ABCD* is a square. Find the complex number, in terms of z_1, z_2 and z_3, that represents *D*.

(c) Let $f(x) = \frac{x^3+1}{x}$.

[5]

(i) Show that

$$\lim_{x\to\pm\infty}[f(x) - x^2] = 0$$

(ii) Part (i) shows that the graph of $y = f(x)$ is asymptotic to the parabola $y = x^2$. Use this fact to help sketch the graph $y = f(x)$.

Question 11 [10 marks]

(a) [5]

(i) Evaluate

$$\int_0^1 e^x dx$$

(ii) Show that

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

(iii) Hence evaluate

$$\int_0^1 x^5 \, e^x dx$$

(b) An arrow is fired horizontally at 60 m/s from the top of a 20 m high wall. Taking g = 10 m/s^2, find the [5]

(i) time taken for the arrow to hit the ground.

(ii) distance the point of impact will be from the base of the wall.

(iii) angle with which the arrow will strike the ground.

End of Section B

Section C

Start each section in a new answer booklet.

Question 12 [13 marks]

(a) [5]

(i) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int \sec x \, dx = \log_e\left(\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right) + C$$

(ii) hence show that

$$\int \sec x \, dx = \log_e\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right) + C$$

(b) [4]

(i) Find the area bounded by $y = x^2 + 3$and $y = x + 9$.

(ii) Find the volume of the solid generated when the area from part (i) is revolved about the x-axis.

(c) Find the volume of the solid with a triangular base with vertices (0, 0), (2, 0) and (0, 1). Cross-sections perpendicular to the x-axis are isosceles triangles with equal height to the base. [4]

Question 13 [11 marks]

(a) [7]

(i) Show that

$$\int \frac{1}{a - by^2}\, dy = \frac{1}{2\sqrt{ab}} \log_e \left(\frac{\sqrt{a} + y\sqrt{b}}{\sqrt{a} - y\sqrt{b}} \right) + C$$

A boat of mass *m* is travelling at maximum power. At maximum power, its engines deliver a force *F* on the boat. The water and air exerts a resistive force proportional to the square of the boat's speed *v*.

(ii) Explain why

$$\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$$

(iii) Find the time *t* it takes the boat to reach the speed of *V* from rest.

(iv) Find the distance travelled during this period.

(b) A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at angle of 30^{o}. At what rate is the distance from the plane to the radar station 1 minute later? [4]

End of Section C

End of Exam

Student Number:______________________________

Mathematics Ext 2
Task ~~3~~ 2 2013

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Multiple Choice Section: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. A B C D
2. A B C D
3. A B C D
4. A B C D
5. A B C D
6. A B C D
7. A B C D

(EXT II)

Q8 a (i) $\int \frac{\cos x}{1+\sin x} \cdot dx$

let $u = 1+\sin x$.
$du = \cos x \, dx$

$= \int \frac{du}{u}$

$= \ln u + c$

$= \boxed{\ln(1+\sin x) + c}$

(ii). $\int \cos x \ln(\sin x) dx. = \int \frac{d}{dx}(\sin x) \cdot \ln(\sin x) . dx.$

$= \sin x . \ln(\sin x) - \int \sin x \cdot \frac{\cos x}{\sin x} . dx.$

$= \sin x . \ln(\sin x) - \int \cos x . dx.$

$= \boxed{\sin x . \ln(\sin x) - \sin x + c}$

(iii). $\int \frac{\sqrt{9-x^2}}{x^2} . dx.$

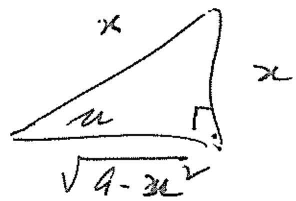
let $x = 3\sin u$
$dx = 3\cos u . du.$

$= \int \frac{\sqrt{9-9\sin^2 u}}{9\sin^2 u} . 3.\cos u . du$

$= \int \frac{3\sqrt{1-\sin^2 u} . 3\cos u . du}{9\sin^2 u .}$

$= \int \frac{9 \cos u . \cos u . du}{9\sin^2 u .}$

$= \int \cot^2 u . du$



$= \int (\operatorname{cosec}^2 u - 1) \, du$

$= -\cot u - u + c.$

$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + c}$

Q8 b $I = \int \frac{4}{x^3 - x} \cdot dx$

Let $\frac{4}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$.

$\therefore 4 \equiv A(x-1)(x+1) + B(x)(x+1) + Cx(x-1)$

let $x = 1$ $\quad 2B = 4$

$\boxed{B = 2}$

let $x = -1$ $\quad 2C = 4$

$\boxed{C = 2}$

let $x = 0$ $\quad -A = 4$

$A = -4$.

$\therefore I = \int \left(\frac{-4}{x} + \frac{2}{x-1} + \frac{2}{x+1} \right) dx$

$= -4\ln x + 2\ln(x-1) + 2\ln(x+1) + c$

$= \boxed{-4\ln x + 2\ln(x^2-1) + c}$

Q8b

$$z = 2(\cos\theta - i\sin\theta)$$

(i) $\boxed{\bar{z} = 2(\cos\theta + i\sin\theta)}$ (1)

(ii) $\boxed{\overline{1+z} = (2\cos\theta+1) + 2i\sin\theta.}$ (1)

(iii) now $(1+z)(\overline{1+z}) = |1+z|^2$. $[z\bar{z} = |z|^2]$

$$(1+z)(\overline{1+z}) = (2\cos\theta+1)^2 + (-2\sin\theta)^2$$

$$\therefore (1+z)((2\cos\theta+1)+2i\sin\theta) = 4\cos^2\theta + 4\cos\theta + 1 + 4\sin^2\theta$$

$$(1+z)((2\cos\theta+1)+2i\sin\theta) = 4(\sin^2\theta+\cos^2\theta) + 4\cos\theta + 1$$

$$= 5 + 4\cos\theta.$$

$$\therefore \frac{1}{1+z} = \frac{(2\cos\theta+1) + 2i\sin\theta}{5+4\cos\theta}$$

$$\therefore \boxed{\mathrm{Re}\left(\frac{1}{1+z}\right) = \frac{1+2\cos\theta}{5+4\cos\theta}.}$$ (2)

(iv) $$\boxed{\mathrm{Im}\left(\frac{1}{1+z}\right) = \frac{2\sin\theta}{5+4\cos\theta}.}$$ (1)

Q9. a (i) To show that $2-i$ is a zero of $P(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$.

We need to show that $P(2-i) = 0$

ie. $(2-i)^4 - 5(2-i)^3 + 7(2-i)^2 + 3(2-i) - 10 = 0$ (A)

now $(2-i)^2 = 3-4i$

$\therefore$ (A) becomes

$LHS = (3-4i)^2 - 5(3-4i)(2-i) + 7(3-4i) + 3(2-i) - 10$

$= -7-24i - 5(2-11i) + 21 - 28i + 6 - 3i - 10$

$= (-7-10+21+6-10) + i(-24+55-28-3)$

$= 0 + 0i$

$= 0$

$= RHS$. $\therefore$ $2-i$ is a zero.

(ii) By conjugate root theorem $2+i$ is also a zero

$\therefore$ $\left[x^2 - [2-i+2+i]x + (2-i)(2+i)\right]$ is a factor.

ie. $x^2 - 4x + 5$ is a factor

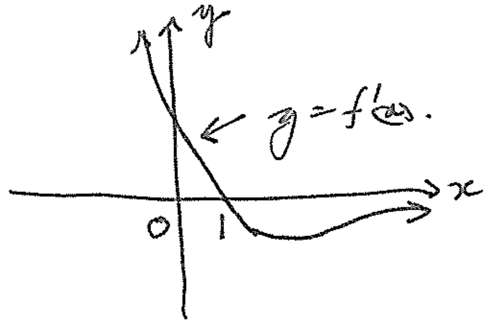
$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x^2-4x+5 & x^4 - 5x^3 + 7x^2 + 3x - 10 \\ & x^4 - 4x^3 + 5x^2 \\ \hline & -x^3 + 2x^2 + 3x \\ & -x^3 + 4x^2 - 5x \\ \hline & -2x^2 + 8x - 10 \\ & -2x^2 + 8x - 10 \\ \hline & 0 \end{array}$$

$\therefore$ other factor is $x^2 - x - 2$.

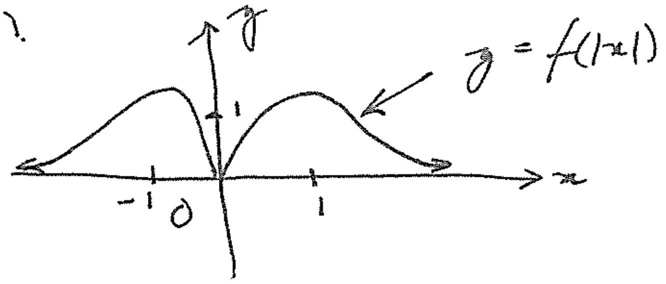
$\therefore$ $\boxed{P(x) = [x-(2-i)][x-(2+i)] \times (x-2)(x+1)}$

Q9 b

(i)



(ii)



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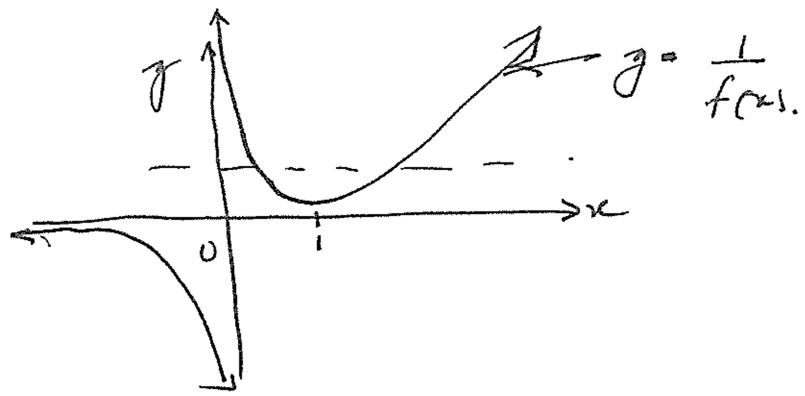

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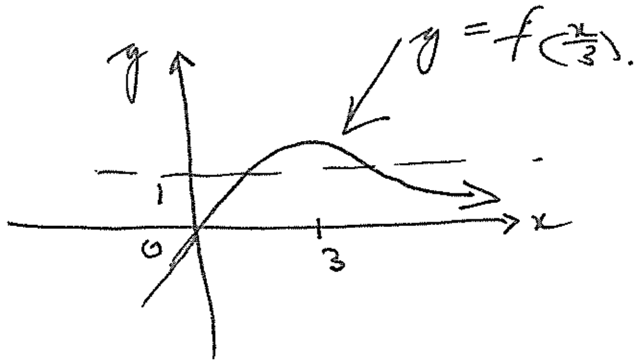


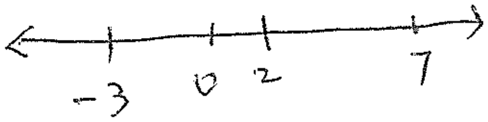
Ext 2 Task 2 2013

Section B

Question 10

(a)

(i)



Centre = 2 $\therefore$ Amplitude $a = 5$

$T = 6 \therefore n = \frac{2\pi}{6}$

$= \pi/3$

Motion initially positive.

$\therefore x = 5\sin\frac{\pi}{3}t + 2$

is a ~~solution~~. [2]

(ii) $\dot{x} = \frac{5\pi}{3}\cos\frac{\pi}{3}t$

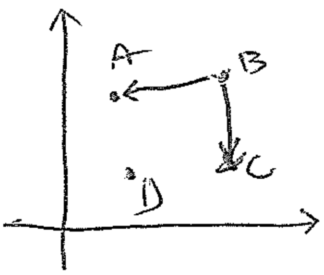
$= \pm\frac{5\pi}{3}\sqrt{1-\sin^2\frac{\pi}{3}t}$

$= \pm\frac{5\pi}{3}\sqrt{1-\left(\frac{x-2}{5}\right)^2}$

$\therefore \dot{x} = \pm\frac{\pi}{3}\sqrt{25-(x-2)^2}$

OR $\dot{x}^2 = \frac{\pi^2}{9}\left(25-(x-2)^2\right)$ [2]

(b)



(i) $\overrightarrow{BA} = z_1 - z_2$ $\overrightarrow{BC} = z_3 - z_2$

$\angle ABC = 90^\circ$

$\therefore z_3 - z_2 = i(z_1 - z_2)$

Square both sides

$(z_3 - z_2)^2 = -(z_1 - z_3)^2$

[1]

(ii) $\overrightarrow{BC} = \overrightarrow{AD}$

$\vec{C} - \vec{B} = \vec{D} - \vec{A}$

$\vec{D} = \vec{A} + \vec{C} - \vec{B}$ [2]

$= z_1 + z_3 - z_2$

OR $z_4 = z_1 + i(z_{\cdot}$

OR $z_4 = \frac{z_1 - iz_3}{1-i}$

(c) $f(x) = \frac{x^3+1}{x}$

(i) $\lim_{x\to\pm\infty}\left[f(x) - x^2\right]$

$= \lim_{x\to\pm\infty}\left[\frac{x^3+1}{x} - x^2\right]$

$= \lim_{x\to\pm\infty}\left[x^2 + \frac{1}{x} - x^2\right]$

$= \lim_{x\to\pm\infty}\left[\frac{1}{x}\right]$ [1]

$= 0$

(ii) $y = \frac{x^3+1}{x}$

$y' = \frac{2x^3-1}{x^2}$ $y'' = 2 + \frac{2}{x^3}$

$y' = 0$ for $2x^3 - 1 = 0$

$\therefore x = \frac{1}{\sqrt[3]{2}}$ $y''\left(\frac{1}{\sqrt[3]{2}}\right) = 3 > 0$

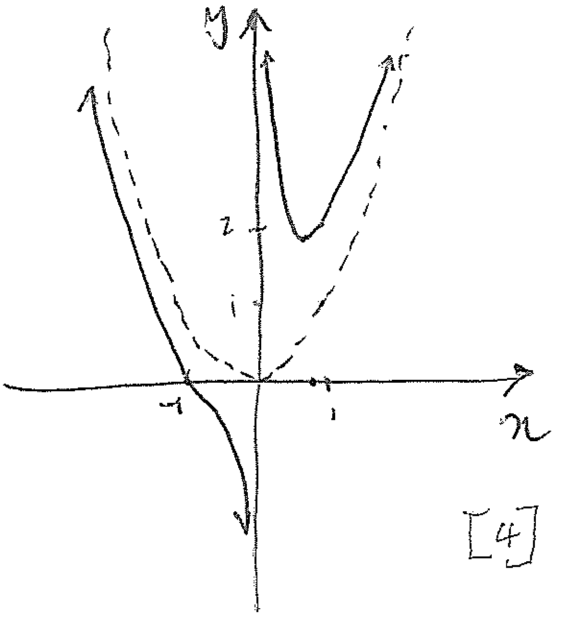
$\therefore$ Min T.P. at $x = \frac{1}{\sqrt[3]{2}}$

$y = \frac{3}{2}\sqrt[3]{2}$

When $y = 0$, $x = -1$

As $x \to 0$, $y \to \pm\infty$

As $x \to \pm\infty$ $y \to x^2$ (above)



[4]

Question 11

(a)(i) $\int_0^1 e^x\,dx = \left[e^x\right]_0^1$

$= e - 1 = I_0$ [1]

(ii) $\int x^n e^x\,dx = \int x^n \frac{d}{dx} e^x\,dx$

$= x^n e^x - \int \left(\frac{d}{dx}x^n\right) \cdot e^x\,dx$

$= x^n e^x - \int n x^{n-1} e^x\,dx$

$= x^n e^x - n\int x^{n-1} e^x\,dx$ [1]

(iii) Let $I_n = \int_0^1 x^n e^x\,dx$

$\therefore I_n = \left[x^n e^x\right]_0^1 - nI_{n-1}$ (from above

$\therefore I_n = e - nI_{n-1}$

$I_5 = e - 5I_4$

$= e - 5(e - 4I_3)$

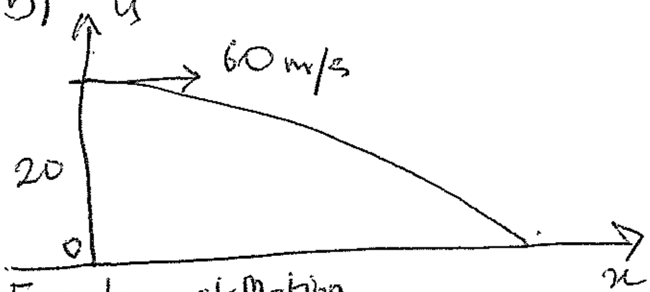
$= -4e + 20(e - 3I_2)$

$= 16e - 60(e - 2I_1)$

$= -44e + 120$

[3]

(b)



Equations of Motion

Horizontal | Vertical

$\ddot{x} = 0$ $\quad\quad \ddot{y} = -10$

$\dot{x} = C$ $\quad\quad \dot{y} = D - 10t$

When $t=0$, $\dot{x} = 60$, $\dot{y} = 0$

$\therefore C = 60 \quad D = 0$

$\therefore \dot{x} = 60 \quad \dot{y} = -10t$

$x = 60t + E \quad y = F - 5t^2$

When $t=0$, $x=0$, $y=20$

$\therefore E = 0 \quad F = 20$

Thus $x = 60t \quad y = 20 - 5t^2$

(i) When $y = 0$ $\quad 0 = 20 - 5t^2$

$t = \pm 2$

(-2 is extraneous)

$\therefore$ Time of flight $T = 2$ sec [2]

(ii) $x = 60T$

$= 60 \times 2 = 120$ m [1]

(iii) When $t = 2$

$\dot{x} = 60 \quad \dot{y} = -20$

$\theta = \tan^{-1} \frac{20}{60}$

$\doteqdot 18^\circ 26'' 6'$

$\doteqdot 0.32175^c$ [2]

Question 12

a) i) $\int \sec x \, dx$

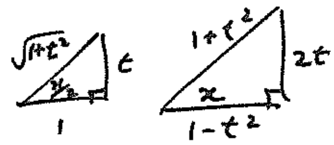
$t = \tan\left(\frac{x}{2}\right)$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = 2 \cdot \frac{1}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$



$$\int \sec x \, dx = \int \frac{1+t^2}{1-t^2} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int \frac{2\,dt}{1-t^2}$$

$$= \int \frac{2\,dt}{(1-t)(1+t)}$$

$$\frac{2}{(1-t)(1+t)} \equiv \frac{A}{1-t} + \frac{B}{1+t}$$

$$2 \equiv A(1+t) + B(1-t)$$

when $t=-1$

$2 = B(1-(-1))$

$B = 1$

when $t = 1$

$2 = A(1+(1))$

$A = 1$

$$= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt$$

$$= \int \left(-\frac{-1}{1-t} + \frac{1}{1+t}\right) dt$$

$$= -\ln(1-t) + \ln(1+t) + C$$

$$= \ln\left(\frac{1+t}{1-t}\right) + C$$

$$= \ln\left(\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right) + C$$

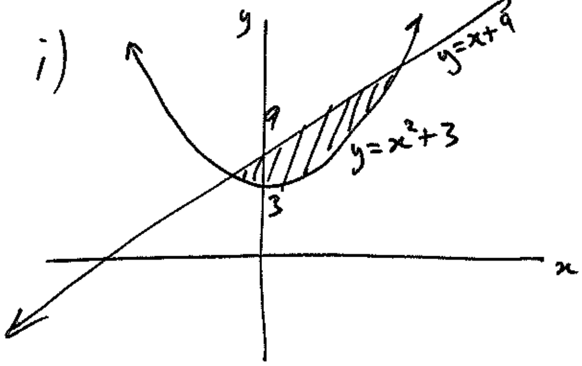
ii) $\int \sec x\,dx = \ln\left(\dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right) + C$

$= \ln\left(\dfrac{\tan\frac{\pi}{4}+\tan\left(\frac{x}{2}\right)}{1-\tan\frac{\pi}{4}\tan\left(\frac{x}{2}\right)}\right) + C$

since $\tan\frac{\pi}{4} = 1$

$= \ln\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right) + C$

b) i)



$y = x^2+3$ —— ①

$y = x+9$ —— ②

sub ① into ②

$x^2+3 = x+9$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = -2, 3$

$A = \int_{-2}^{3}\left(x+9-(x^2+3)\right)dx$

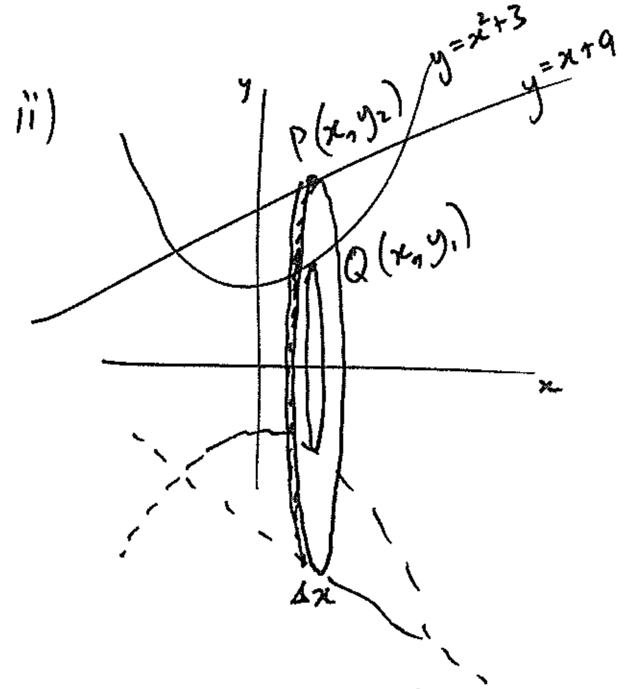
$= \int_{-2}^{3}(x - x^2 + 6)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 6x\right]_{-2}^{3}$

$= \frac{(3)^2}{2} - \frac{(3)^3}{3} + 6(3) - \left(\frac{(-2)^2}{2} - \frac{(-2)^3}{3} + 6(-2)\right)$

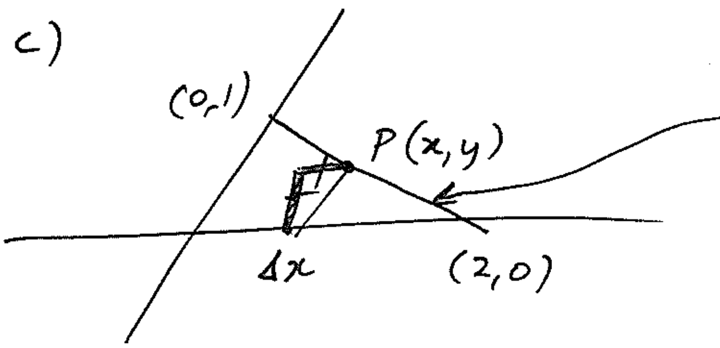
$= \frac{125}{6}$ square units

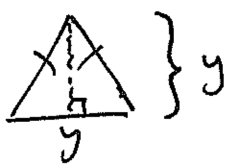
ii)



$$\Delta V = \pi\left(y_2^2 - y_1^2\right)\Delta x$$
$$= \pi\left((x+9)^2 - (x^2+3)^2\right)\Delta x$$
$$V = \lim_{\Delta x \to 0} \sum_{x=-2}^{3} \pi\left(x^2+18x+81-(x^4+6x^2+9)\right)\Delta x$$
$$V = \pi\int_{-2}^{3}\left(-x^4-5x^2+18x+72\right)dx$$
$$V = \pi\left[-\frac{x^5}{5} - \frac{5x^3}{3} + 9x^2 + 72x\right]_{-2}^{3}$$
$$V = \pi\left[-\frac{(3)^5}{5} - \frac{5(3)^3}{3} + 9(3)^2 + 72(3) - \left(-\frac{(-2)^5}{5} - \frac{5(-2)^3}{3} + 9(-2)^2 + 72(-2)\right)\right]$$
$$V = \frac{875\pi}{3} \text{ cubic units}$$

c)





$A = \frac{1}{2}y^2$

$$\Delta V = \frac{1}{2}y^2\Delta x$$
$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} \frac{1}{2}\left(-\frac{1}{2}x+1\right)^2\Delta x$$

$$V = \frac{1}{2}\int_0^2 \left(\frac{1}{4}x^2 - x + 1\right) dx$$

$$= \frac{1}{2}\left[\frac{1}{12}x^3 - \frac{x^2}{2} + x\right]_0^2$$

$$= \frac{1}{2}\left[\frac{1}{12}(2)^3 - \frac{(2)^2}{2} + (2) - (0)\right]$$

$$= \frac{1}{3} \text{ cubic units}$$

Question 13

a) i) $$\frac{1}{a-by^2} = \frac{1}{(\sqrt{a}-\sqrt{b}y)(\sqrt{a}+\sqrt{b}y)}$$

$$\frac{1}{a-by^2} \equiv \frac{A}{\sqrt{a}-\sqrt{b}y} + \frac{B}{\sqrt{a}+\sqrt{b}y}$$

$$1 \equiv A(\sqrt{a}+\sqrt{b}y) + B(\sqrt{a}-\sqrt{b}y)$$

when $y = \frac{\sqrt{a}}{\sqrt{b}}$

$$1 = A\left(\sqrt{a} + \sqrt{b}\left(\frac{\sqrt{a}}{\sqrt{b}}\right)\right) + 0$$

$$A = \frac{1}{2\sqrt{a}}$$

when $y = -\frac{\sqrt{a}}{\sqrt{b}}$

$$1 = 0 + B\left(\sqrt{a} - \sqrt{b}\left(-\frac{\sqrt{a}}{\sqrt{b}}\right)\right)$$

$$B = \frac{1}{2\sqrt{a}}$$

$$\int \frac{1}{a-by^2}\,dy = \frac{1}{2\sqrt{a}}\int\left(\frac{1}{\sqrt{a}-\sqrt{b}y} + \frac{1}{\sqrt{a}+\sqrt{b}y}\right)dy$$

$$= \frac{1}{2\sqrt{ab}}\int\left(-\frac{-\sqrt{b}}{\sqrt{a}-\sqrt{b}y} + \frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}y}\right)dy$$

$$= \frac{1}{2\sqrt{ab}}\left[-\ln(\sqrt{a}-\sqrt{b}y) + \ln(\sqrt{a}+\sqrt{b}y)\right] + C$$

$$= \frac{1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{a}+\sqrt{b}y}{\sqrt{a}-\sqrt{b}y}\right) + C$$

ii) 

Resultant Force

$$ma = F - kv^2$$

$$a = \frac{F}{m} - \frac{kv^2}{m}$$

$$\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$$

iii)
$$\frac{dt}{dv} = \frac{m}{F - kv^2}$$

$$\int dt = \int \frac{m}{F - kv^2}\, dv$$

$$\int_0^T dt = m \int_0^V \frac{dv}{F - kv^2}$$

using part(i)

$$T = m\left[\frac{1}{2\sqrt{Fk}} \ln\left(\frac{\sqrt{F} + v\sqrt{k}}{\sqrt{F} - v\sqrt{k}}\right)\right]_0^V$$

$$T = m\left[\frac{1}{2\sqrt{Fk}} \ln\left(\frac{\sqrt{F} + V\sqrt{k}}{\sqrt{F} - V\sqrt{k}}\right) - \cancel{\frac{1}{2\sqrt{Fk}} \ln\left(\frac{\sqrt{F}}{\sqrt{F}}\right)}\right]$$

$$T = \frac{m}{2\sqrt{Fk}} \ln\left(\frac{\sqrt{F} + V\sqrt{k}}{\sqrt{F} - V\sqrt{k}}\right)$$

iv)
$$v.\frac{dv}{dx} = \frac{1}{m}(F - kv^2)$$

$$\frac{dv}{dx} = \frac{F - kv^2}{mv}$$

$$\frac{dx}{dv} = \frac{mv}{F - kv^2}$$

$$\int dx = \int \frac{mv}{F-kv^2}\,dv$$

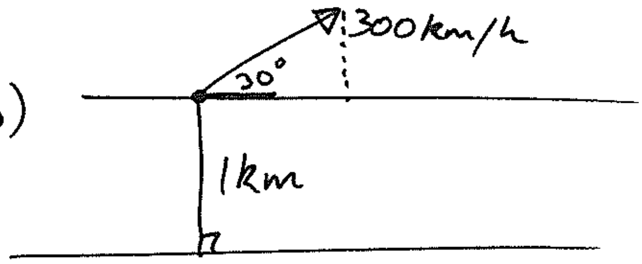
$$\int_0^D dx = \frac{-m}{2k}\int_0^V \frac{-2kv}{F-kv^2}\,dv$$

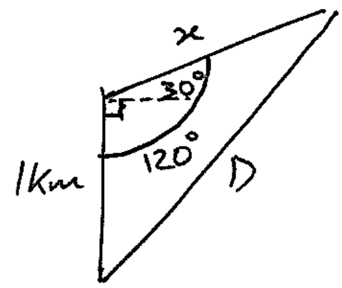
$$D = -\frac{m}{2k}\left[\ln\left(F-kv^2\right)\right]_0^V$$

$$D = -\frac{m}{2k}\left[\ln\left(F-kV^2\right) - \ln F\right]$$

$$D = -\frac{m}{2k}\ln\left(\frac{F-kV^2}{F}\right) \quad \text{or} \quad \frac{m}{2k}\ln\left(\frac{F}{F-kV^2}\right)$$

b)





$$\frac{dx}{dt} = 300$$

$$D^2 = 1^2 + x^2 - 2(1)(x).\cos 120^\circ$$

$$D^2 = 1 + x^2 + x$$

$$2D.\frac{dD}{dx} = 2x+1$$

$$\frac{dD}{dx} = \frac{2x+1}{2D}$$

$$\frac{dD}{dx} = \frac{2x+1}{2\sqrt{1+x^2+x}}$$

$$\frac{dD}{dt} = \frac{dD}{dx} \times \frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{2x+1}{2\sqrt{1+x^2+x}} \times 300$$

300 km/h
$= 5 \text{ km/min}$

when $x = 5$

$$\frac{dD}{dt} = \frac{2(5)+1}{2\sqrt{1+(5)^2+(5)}} \cdot 300$$

$$= \frac{1650}{\sqrt{31}} \quad \text{km/h}$$

$$\approx 296.35 \quad \text{km/h}$$