SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 2 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Each of the three Sections (A, B, and C) is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 91

- Attempt questions 1 – 6
- Board approved calculators maybe used.
- Full marks may not be awarded for careless or badly arranged work.
- Unless otherwise stated, give answers in simplest exact form.

Examiner: *P.R.Bigelow*

Section A
(Start a new answer sheet.)

Question 1. (17 marks)

Marks

(a) Find $\int \frac{2x+1}{x^2+2x+2}\,dx$. **2**

(b) Evaluate $\arg\left((2+i)\overline{z}\right)$ where $z=-1-3i$ **2**

(c) (i) Sketch the curve: **2**

$$(x+1)(y-2)=1$$

(ii) Express the function in (i) in the form $y=f(x)$.
Hence or otherwise sketch:

(α) $y=f(x-2)$ **1**

(β) $y^2=f(x)$ **2**

(γ) $y=\frac{1}{f(x)}$ **2**

(d) Find:

$$\int \frac{d\theta}{8\cos^2\theta+1}$$ **2**

(e) If z is a complex number such that $z=k(\cos\theta+i\sin\theta)$ where k is real, show that **2**

$$\arg(z+k)=\frac{\theta}{2}.$$

(f) Use the substitution $u=1-\sqrt{x}$ to find: **2**

$$\int \frac{dx}{1-\sqrt{x}}$$

Question 2. (15 marks)

Marks

(a) (i) Show that: **2**

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a}\left[f(x)+f(-x)\right]dx$$

(ii) Hence evaluate $\displaystyle\int_{-2}^{2}\frac{dx}{1+e^{-x}}$ **2**

(b) (i) Express $-\sqrt{27}-3i$ in mod-arg form. **2**

(ii) Hence find $\left(-\sqrt{27}-3i\right)^{6}$, giving your answer in the form $a+ib$ where *a* and *b* are real. **2**

(c) The complex number *z* lies on the locus $\arg(z+i)=\frac{\pi}{4}$:

(i) Sketch the locus, showing any intercepts with the axes. **1**

(ii) Find the least value of $|z|$. **2**

(d) (i) Use the graph of $y=\sin x$ to sketch on separate diagrams, the graphs of the functions $y=|\sin x|$ and $y=\sin|x|$ for $-2\pi \le x \le 2\pi$. **2**

(ii) Evaluate: $\displaystyle\int_{-2\pi}^{2\pi}\left(\sin|x|+|\sin x|\right)dx$ **2**

Section B
(Start a new answer sheet.)

Question 3. (16 marks)

Marks

(a) The nine letters of the word *REDIVIDER* are arranged at random in a line.

(i) How many letter sequences are possible? **2**

(ii) What is the probability that the sequence is the same from left to right as from right to left. **2**

(b) An object of mass 10 kg falls from rest from a stationary balloon, and experiences air resistance of magnitude $4v^2$. Take the value of g to be 10 m/s^2, and take downwards to be positive. **2**

(i) Show that the equation of motion is: **1**

$$\ddot{x} = \frac{50 - 2v^2}{5}$$

(ii) Find the terminal velocity of the object. **1**

(iii) Show that the velocity v at time t is given by: **3**

$$t = \frac{1}{4}\ln\left(\frac{5+v}{5-v}\right)$$

(iv) Show $\quad v = \dfrac{5\left(e^{4t} - 1\right)}{e^{4t} + 1}$ **2**

(v) Find the vertical distance fallen as a function of t. **3**

Question 4 (14 marks)

(a) Let $p = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

(i) Show that $1 + p + p^2 + p^3 + p^4 = 0$ **2**

(ii) Simplify $(p + p^4)(p^2 + p^3)$ **1**

(iii) Form a quadratic equation with roots $(p + p^4)$ and $(p^2 + p^3)$ **1**

(iv) Hence or otherwise show that $\cos\frac{2\pi}{5}.\cos\frac{4\pi}{5} = -\frac{1}{4}$ **2**

(b) There are eight people, 5 men and 3 women, to be seated at a round table. How many arrangements are there if:

(i) there are no restrictions? **1**

(ii) no two women are next to one another? **2**

(Give your answers in the form of powers and factorials.)

(c) The constants *m* and *n* are such that the equation $x^3 - mx + n = 0$has three no-zero roots α, β, γ.

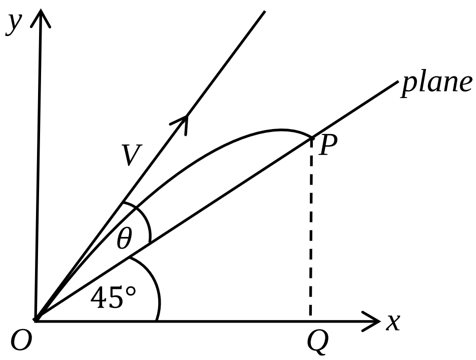
(i) Find in terms of *m* and *n* the cubic equation (in expanded form) whose roots are α^3, β^3 and γ^3. **2**

(ii) Hence or otherwise find $\alpha^7\beta\gamma + \alpha\beta^7\gamma + \alpha\beta\gamma^7$ in terms of *m* and *n*. **3**

Section C
(Start a new answer booklet)

Question 5 (14 marks)

(a)



A particle is projected from a point *O* on a plane inclined at 45° to the horizontal. The velocity of projection is *V* at an angle of θ to the inclined plane, where $\theta = \tan^{-1}\frac{1}{2}$.

[You are given that $y = x\tan\alpha - \dfrac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$ where $\alpha = 45^o + \theta$]

(i) Show that $\tan\alpha = 3$. **1**

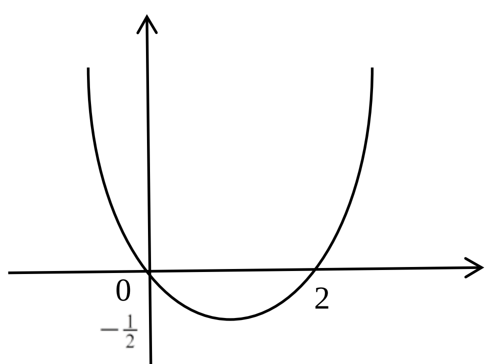
(ii) Hence show that $y = 3x - \dfrac{5gx^2}{V^2}$. **1**

(iii) Show that the range *OP* is $\dfrac{2\sqrt{2}V^2}{5g}$. **2**

(iv) Show that it meets the plane at right-angles. **3**

(b) The sketch is of the parabola **7**

$$f(x) = \frac{x^2 - 2x}{2}.$$



Without calculus, draw the following on separate diagrams, showing essential features:

(i) $y = |f(x)|$

(ii) $y = \dfrac{x}{2}|x-2|$

(iii) $y = \tan^{-1}\left(f(x)\right)$

(iv) $|y| = f(x)$

Question 6 (15 marks)

(a) Sketch the following, showing essential features:

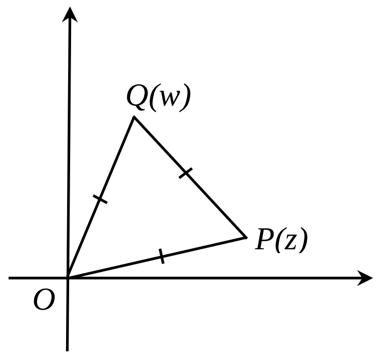
(i) $y = \ln(\ln x)$ **2**

(ii) $f(x) = \cos^{-1}(\sin x)$ **2**

(b) The triangle OPQ is equilateral. **2**

P represents the complex number z, and Q represents the complex number w.

Show that $z^3 + w^3 = 0$.



(c) Let $I_n = \int_0^1 (1-x^2)^n \, dx$ and $J_n = \int_0^1 x^2 (1-x^2)^n \, dx$.

(i) Using integration by parts, on I_n, show that $I_n = 2nJ_{n-1}$. **2**

(ii) Hence show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$. **2**

(iii) Show that $J_n = I_n - I_{n+1}$. **2**

(iv) Hence deduce that $J_n = \dfrac{1}{2n+3} I_n$. **2**

(v) Hence find a reduction formula for J_n in terms of J_{n-1}. **1**

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

SBHS 2014 Extension 2 Assessment #2 Solutions

1) a) $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$

$= \displaystyle\int \left(\frac{2x+2}{x^2+2x+2} - \frac{1}{x^2+2x+1+1}\right)dx$

$= \displaystyle\int \left(\frac{2x+2}{x^2+2x+2} - \frac{1}{(x+1)^2+1}\right)dx$

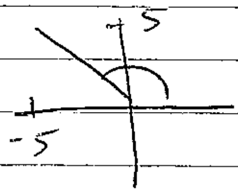
$= \ln(x^2+2x+2) - \tan^{-1}(x+1) + C$

b) $\arg\left((2+i)(\overline{-1-3i})\right)$

$= \arg\left((2+i)(-1+3i)\right)$

$= \arg(-2+6i-i-3)$

$= \arg(-5+5i)$



$= \dfrac{3\pi}{4}$

c) i) $(x+1)(y-2)=1$

$x+1 \neq 0$

$x \neq -1$

$y-2 \neq 0$

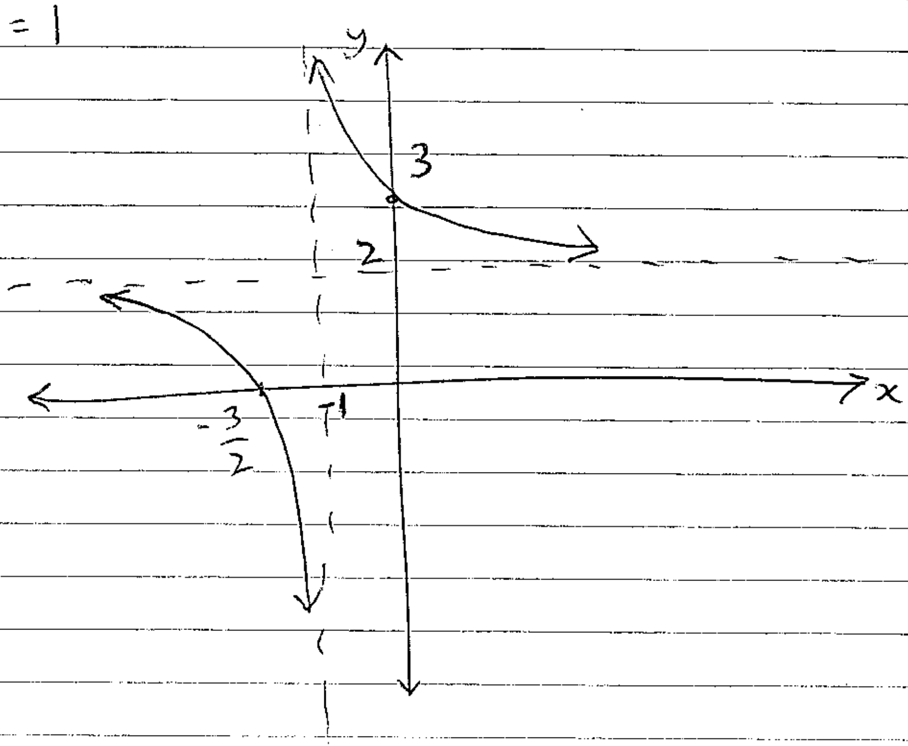
$y \neq 2$

when $x=0$

$y=3$

when $y=0$

$x=-\frac{3}{2}$

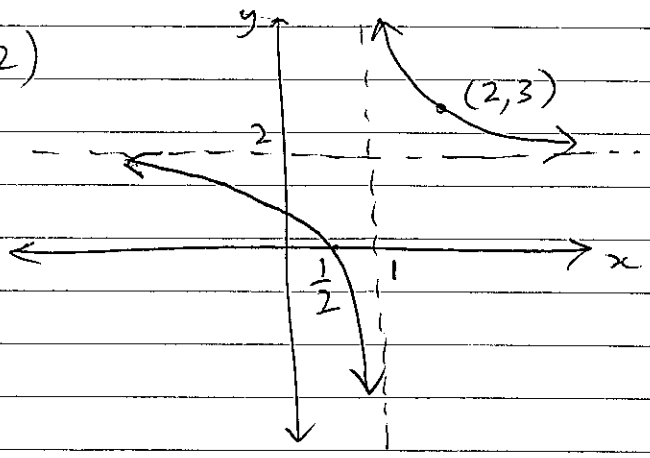


ii) $(x+1)(y-2)=1$

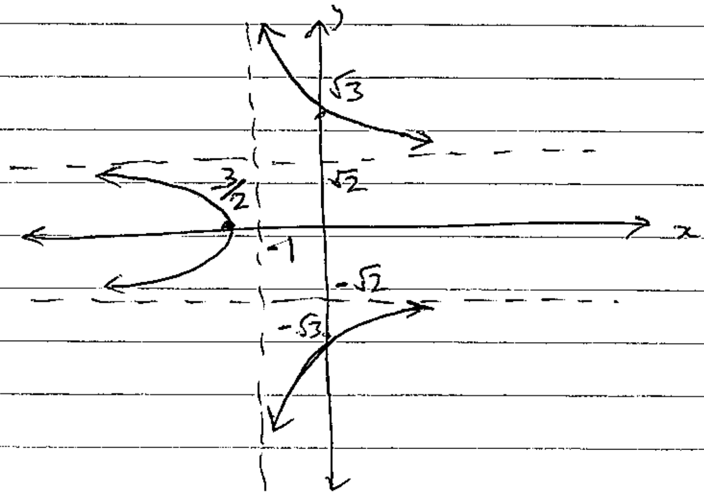
$$y-2=\frac{1}{x+1}$$

$$y=\frac{1}{x+1}+2$$

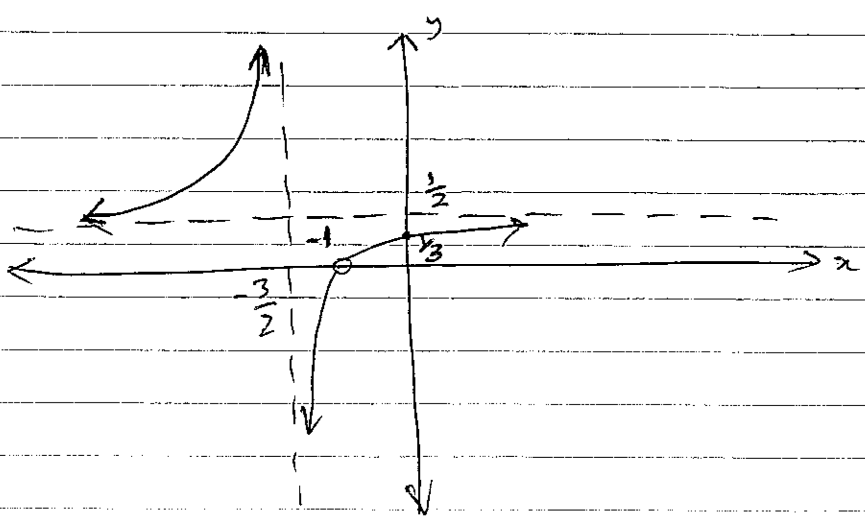
α) $y=f(x-2)$



β)



γ)



d) $\displaystyle\int \frac{d\theta}{8\cos^2\theta + 1}$

$$= \int \frac{d\theta}{8\cos^2\theta + \sin^2\theta + \cos^2\theta}$$

$$= \int \frac{d\theta}{9\cos^2\theta + \sin^2\theta}$$

$$= \int \frac{\frac{1}{\cos^2\theta}}{\frac{9\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}} \cdot d\theta$$

$$= \int \frac{\sec^2\theta \, d\theta}{9 + \tan^2\theta}$$

let $u = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta$

$d\theta = \frac{du}{\sec^2\theta}$

$$= \int \frac{\cancel{\sec^2\theta}}{9 + u^2} \cdot \frac{du}{\cancel{\sec^2\theta}} \qquad a = 3$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{\tan\theta}{3}\right) + C$$

e) $\arg(z+k) = \arg\left(k(\cos\theta + i\sin\theta) + k\right)$

$= \arg\left(k(\cos\theta + 1) + ki\sin\theta\right)$

$= \arg\left(k\left(2\cos^2\frac{\theta}{2} \cancel{- 1 + 1}\right) + ki\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$

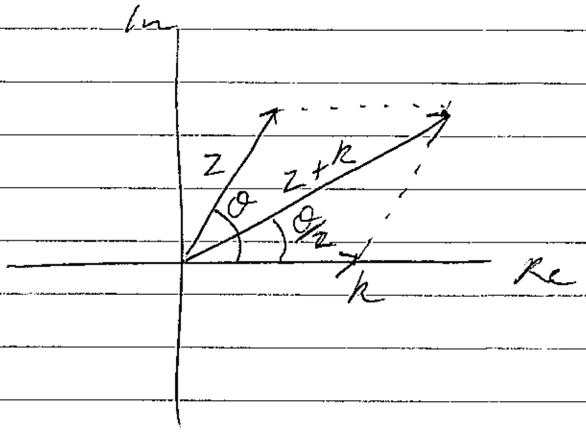
$= \arg\left(2k\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right)$

$= \frac{\theta}{2}$

Graphically

Note: $|z| = k$

$|k| = k$ $\therefore$ $z+k$ is the diagonal of a rhombus



since the diagonals of a rhombus bisect the angles. $\arg(z+k) = \dfrac{\theta}{2}$.

f) $\int \frac{dx}{1-\sqrt{x}}$

let $u = 1-\sqrt{x}$

$\frac{du}{dx} = -\frac{1}{2\sqrt{x}}$

$dx = -2\sqrt{x}\,du$

$= \int \frac{-2\sqrt{x}\,du}{u}$

$= \int \frac{-2(1-u)}{u}\,du$

$= 2\int \frac{u-1}{u}\,du$

$= 2\int \left(1-\frac{1}{u}\right)du$

$= 2\left[u - \ln u\right] + C$

$= 2\left[1-\sqrt{x} - \ln\left(1-\sqrt{x}\right)\right] + C$

2) a) i) $\displaystyle\int_{-a}^{a} f(x)\,dx$

$$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

consider $\displaystyle\int_{-a}^{0} f(x)\,dx$

let $u = -x$ when $x=-a, u=$

$\dfrac{du}{dx} = -1$ $x=0, u=$

$dx = -du$

$$= \int_{a}^{0} f(-u)\cdot -du$$

$$= -\int_{a}^{0} f(-u)\,du$$

$$= \int_{0}^{a} f(-u)\,du$$

for definite integrals the variable can be changed

$$= \int_{0}^{a} f(-x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} \left[f(x) + f(-x)\right] dx$$

ii) $\displaystyle\int_{-2}^{2} \frac{dx}{1+e^{-x}}$

$$= \int_{0}^{2} \left[\frac{1}{1+e^{-x}} + \frac{1}{1+e^{x}}\right] dx$$

$$= \int_{0}^{2} \left[\frac{e^{x}}{e^{x}+1} + \frac{1}{1+e^{x}}\right] dx$$

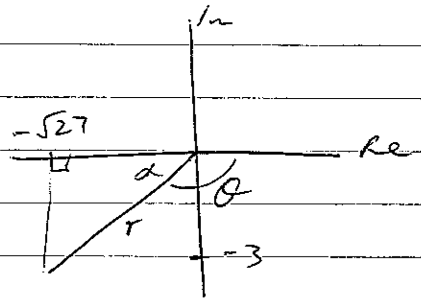
$$= \int_0^2 \left[\overset{1}{\cancel{\frac{1+e^{x}}{1+e^{x}}}} \right] dx$$

$$= \Big[x \Big]_0^2$$

$$= (2) - (0)$$

$$= 2$$

b) i) $-\sqrt{27} - 3i$



$$r^2 = (\sqrt{27})^2 + (3)^2$$
$$= 36$$
$$\underline{r = 6}$$

$$\tan\alpha = \frac{3}{\sqrt{27}}$$
$$= \frac{3}{3\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6}$$
$$\therefore \theta = -\frac{5\pi}{6}$$

$$-\sqrt{27} - 3i = 6\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

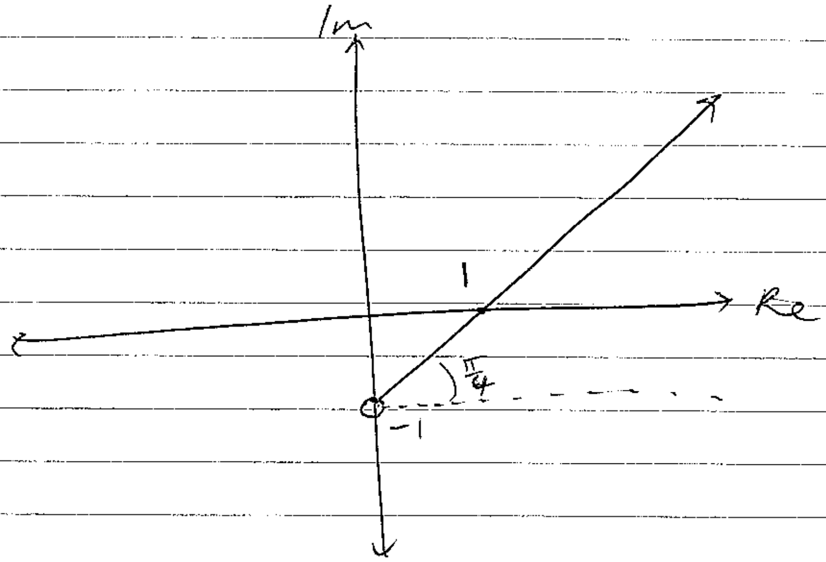
ii) $$(-\sqrt{27} - 3i)^6 = \left[6\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^6$$

$$= 6^6\left(\cos(-5\pi) + i\sin(-5\pi)\right)$$ using de moivres theorem

$$= 46656\left((-1) + i(0)\right)$$

$$= -46656$$

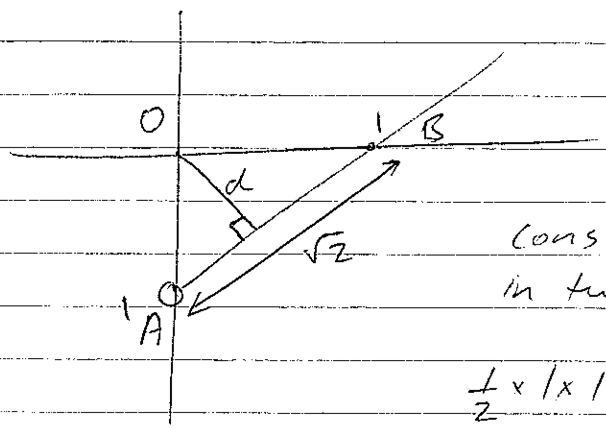
c) i)



$\arg(z+i) = \frac{\pi}{4}$

$= \arg(z-(-i)) = \frac{\pi}{4}$

ii)



consider the area of $\triangle OAB$ in two ways

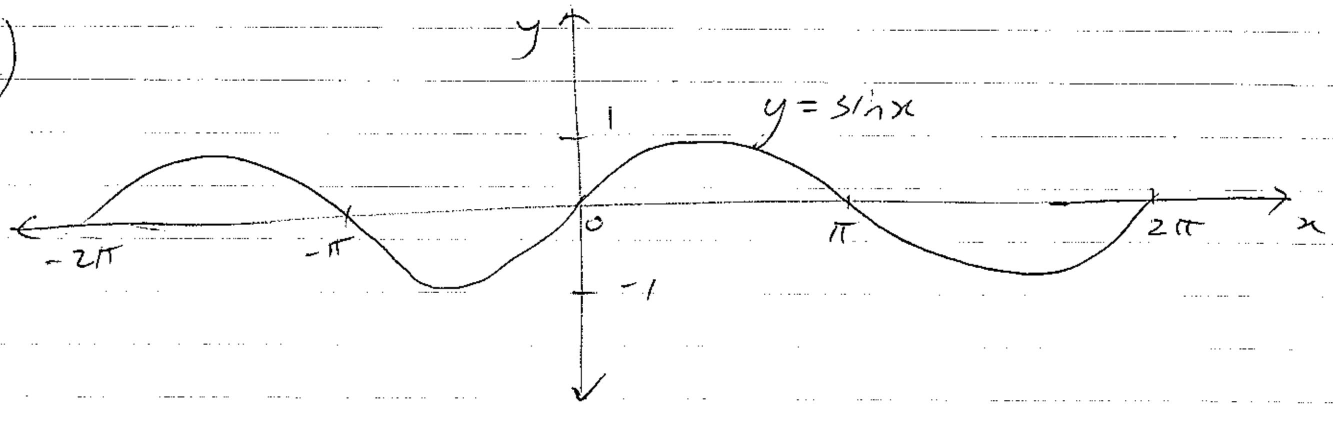
$$\frac{1}{2} \times 1 \times 1 = \frac{1}{2} \times \sqrt{2} \times d$$

$$d = \frac{1}{\sqrt{2}}$$

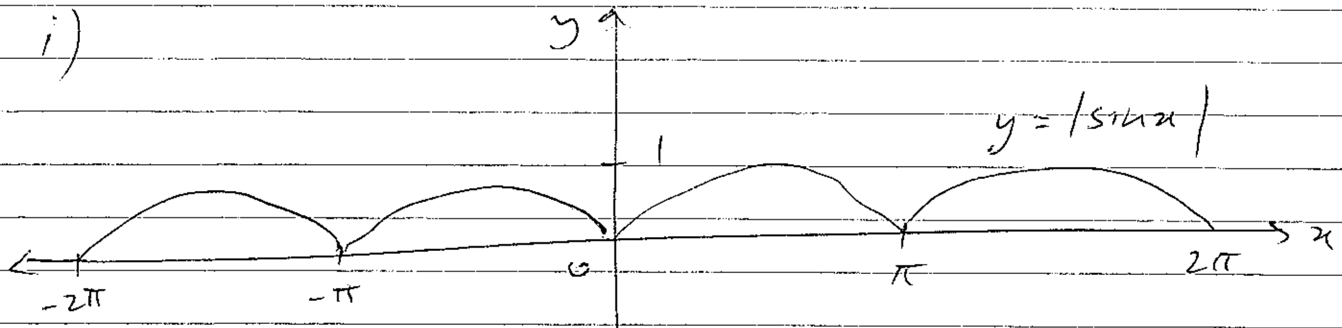
the least value of $|z| = \frac{1}{\sqrt{2}}$

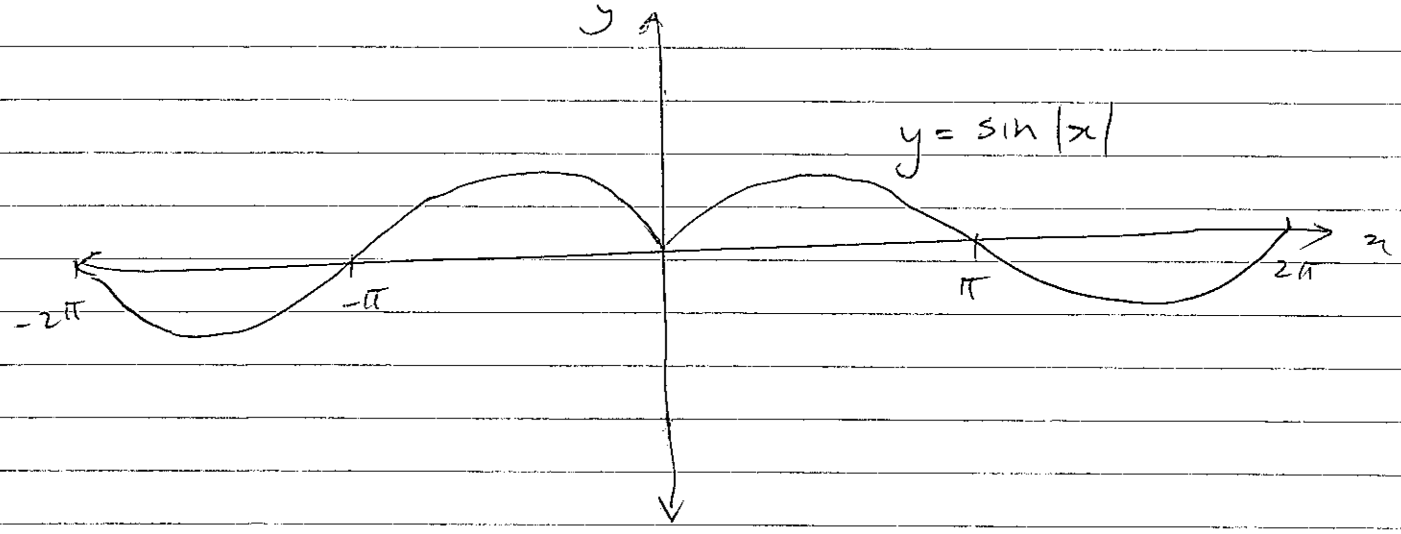
$$= \frac{\sqrt{2}}{2}$$

d)



i)





ii) $\displaystyle\int_{-2\pi}^{2\pi} \left(\sin|x| + |\sin x|\right) dx$

$= \displaystyle\int_{-2\pi}^{2\pi} \sin|x|\, dx + \int_{-2\pi}^{2\pi} |\sin x|\, dx$

$= 0 + 4\displaystyle\int_{0}^{\pi} \sin x\, dx$

$= 4\left[-\cos x\right]_0^{\pi}$

$= 4\left[-\cos\pi - (-\cos 0)\right]$

$= 4\left[-(-1) + 1\right]$

$= 8.$

| Solutions |

Question (3)

$V \to 1 \quad D, E, R, I \to 2$

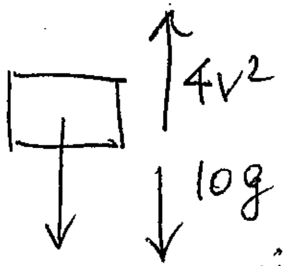
(i) $\therefore$ The possible sequences are: $\boxed{\dfrac{9!}{2^4} = \dfrac{362880}{16} = 22680}$

(ii) One way to put 'V' in the middle. The rest of D, E, R, I can be arranged $4!$ ways.
Now, once the letters on either the left/ right of V are determined there is only 1 way to arrange right/left side $= 1 \times 4!$

$\therefore$ Prob. $= \boxed{\dfrac{4!}{(9!/2^4)} = \dfrac{1}{945}}$

$= \boxed{0.010582}$

(b) (i)



$10\ddot{x} = 10g - 4v^2$

$\ddot{x} = g - \dfrac{2v^2}{5}$ $\quad g = 10$

$\therefore \boxed{\ddot{x} = \dfrac{50 - 2v^2}{5}}$

(ii) Terminal velocity $\boxed{\text{when } \ddot{x} = 0.}$

$\Rightarrow \boxed{V = 5 \text{ m/sec}}$

(iii) $\dfrac{dv}{dt} = \dfrac{50 - 2v^2}{5} \Rightarrow \dfrac{dt}{dv} = \dfrac{5}{50 - 2v}.$

$\therefore \quad t = \dfrac{5}{2} \displaystyle\int \dfrac{dv}{25 - v^2}$

Now Let $\dfrac{1}{25 - v^2} = \dfrac{a}{5+v} + \dfrac{b}{5-v}$

$\therefore \quad 1 = a(5-v) + b(5+v)$

equate coeff. of v $\quad b - a = 0 \Rightarrow a = b.$

equate constant term $\quad 5a + 5b = 1$

$\therefore \quad a + b = \dfrac{1}{5}, \ 2a = \dfrac{1}{5}$

$\therefore \quad a = b = \dfrac{1}{10}.$

$$\therefore \quad t = \frac{5}{20}\int\left(\frac{1}{5+v}+\frac{1}{5-v}\right)dv$$

$$\Rightarrow \quad t = \frac{1}{4}\left[\ln(5+v) - \ln(5-v)\right]$$

$$\therefore \boxed{t = \frac{1}{4}\ln\left(\frac{5+v}{5-v}\right)}$$

(iv) Now, $4t = \ln\left(\frac{5+v}{5-v}\right)$.

$$\therefore \frac{5+v}{5-v} = e^{4t}$$

$$5e^{4t} - (e^{4t})v = 5+v$$

$$5(e^{4t}-1) = v(1+e^{4t})$$

$$\therefore \boxed{v = \frac{5(e^{4t}-1)}{e^{4t}+1}}$$

(v)

Now, $v = \frac{dx}{dt}$

$$\therefore \frac{dx}{dt} = \frac{5(e^{4t}-1)}{e^{4t}+1}$$

separating the variables:

$$\int dx = 5\int\left[\frac{(e^{4t}+1)-2}{e^{4t}+1}\right]dt$$

$$= 5\int\left(1 - \frac{2}{e^{4t}+1}\right)dt$$

$$= 5t - 10\int\frac{dt}{e^{4t}+1}$$

Now, $10\int\frac{dt}{e^{4t}+1}\left(\times\frac{e^{-4t}}{e^{-4t}}\right)$

$$= 10\int\frac{e^{-4t}}{1+e^{-4t}}\,dt$$

$\boxed{\text{let } u = 1+e^{-4t} \text{ then } du = -4e^{-4t}dt}$

$$= -\frac{5}{2}\int\frac{(-4e^{-4t})dt}{1+e^{-4t}}$$

$$= -\frac{5}{2}\int\frac{du}{u} = -\frac{5}{2}\ln u$$

$$= -\frac{5}{2}\ln(1+e^{-4t})$$

$$\therefore x = 5t - \frac{5}{2}\ln(1+e^{-4t}) + c.$$

When $t=0$, $x=0$, $\Rightarrow c = \frac{5}{2}\ln 2$.

$$\therefore x = 5t + \frac{5}{2}\ln\left(\frac{2}{1+e^{-4t}}\right)$$

$\doteqdot 5.17$

- **Method (1).** Q(3) (b) (v)

$$v = \frac{5(e^{4t}-1)}{e^{4t}+1}$$

$$\frac{dx}{dt} = \frac{5(e^{4t}-1)}{e^{4t}+1}$$

$$x = 5\int\left(\frac{(e^{4t}+1)-2}{e^{4t}+1}\right)dt$$

$$= 5\int\left(1-\frac{2}{e^{4t}+1}\right)dt$$

$$= 5\int\left(1+\frac{-2e^{-4t}}{1+e^{-4t}}\right)dt$$

$$= 5\int\left(1+\frac{1}{2}\left(\frac{-4e^{-4t}}{1+e^{-4t}}\right)\right)dt$$

$$= 5t + \frac{5}{2}\ln\left(1+e^{-4t}\right)+c$$

When $t=0$, $x=0$, $0 = \frac{5}{2}\ln 2 + c$

$$\therefore \boxed{x = 5t + \frac{5}{2}\ln\left(\frac{1+e^{-4t}}{2}\right)}$$

1

- **Method (2)** $v = 5\int\left(\frac{e^{4t}-1}{e^{4t}+1}\right)dt$

but

$$\therefore\ x = 5\int\left(\frac{e^{2t}-e^{-2t}}{e^{2t}+e^{-2t}}\right)dt.$$

Let $u = e^{2t}+e^{-2t}$, $du = 2(e^{2t}-e^{-2t})dt$

$$\therefore\ x = \frac{5}{2}\int\frac{du}{u}$$

$$\therefore\ x = \frac{5}{2}\ln\left(e^{2t}+e^{-2t}\right)+c$$

When $t=0$, $x=0$

$$\Longrightarrow\ c = -\frac{5}{2}\ln 2.$$

$$\therefore \boxed{x = \frac{5}{2}\ln\left(\frac{e^{2t}+e^{-2t}}{2}\right)}$$

2

Method (3) Q(3) (b) (V)

$$x = 5\int \frac{e^{4t}}{e^{4t}+1}\,dt - 5\int \frac{dt}{e^{4t}+1}$$

$$= \frac{5}{4}\ln\left(e^{4t}+1\right) - 5\int \frac{4e^{4t}\,dt}{4e^{4t}\left(e^{4t}+1\right)}$$

$$\boxed{-\frac{5}{4}\int \frac{4e^{4t}\,dt}{e^{4t}\left(e^{4t}+1\right)}}$$

Now, to integrate this,

Let $u = e^{4t}$, $du = 4e^{4t}dt$.

by partial fractions

$$\frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1}$$

$$\therefore -\frac{5}{4}\int\left(\frac{4e^{4t}\,dt}{e^{4t}+1}\right)$$

$$= -\frac{5}{4}\left(\ln u - \ln(u+1)\right)$$

$$= -\frac{5}{4}\ln\left(\frac{e^{4t}}{e^{4t}+1}\right)$$

$$\therefore x = \frac{5}{4}\ln\left(e^{4t}+1\right) - \frac{5}{4}\ln\left(\frac{e^{4t}}{e^{4t}+1}\right) + c$$

$$x = \frac{5}{4}\left[\ln\left(e^{4t}+1\right) - \ln\left(e^{4t}\right) + \ln\left(e^{4t}+1\right)\right] + c.$$

$$x = \frac{5}{4}\left(2\ln\left(e^{4t}+1\right) - 4t\right) + c$$

When $t = 0$, $x = 0$

$$0 = \frac{5}{4}(2\ln 2) + c$$

$$\Longrightarrow c = -\frac{5}{2}\ln 2.$$

$$\therefore \boxed{x = \frac{5}{2}\ln\left(\frac{e^{4t}+1}{2}\right) - 5t}$$

3

Conclusion: (All 3 are SAME because of the log expansions)

Now, expanding

(1) method (1) we have

$$x = 5t + \frac{5}{2}\ln\left(\frac{1 + \frac{1}{e^{4t}}}{2}\right)$$

$$\therefore \boxed{x = 5t + \frac{5}{2}\ln\left(\frac{e^{4t}+1}{2e^{4t}}\right)}$$

---

(2) Expand method (2)

$$x = \frac{5}{2}\ln\left(\frac{e^{2t} + \frac{1}{e^{2t}}}{2}\right)$$

$$\boxed{x = \frac{5}{2}\ln\left(\frac{e^{4t}+1}{2e^{2t}}\right)}$$

(3) Method (3) left as it is

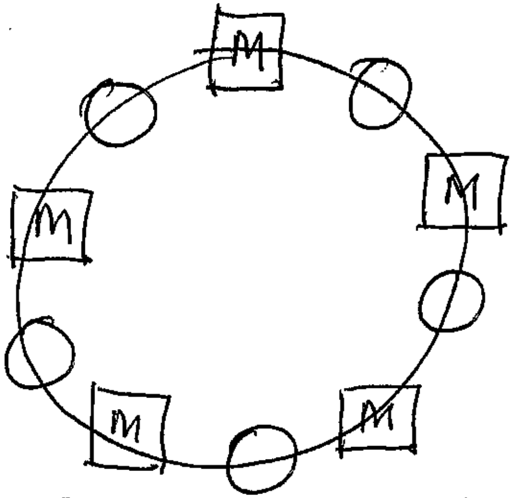
$$\therefore \boxed{x = \frac{5}{2}\ln\left(\frac{e^{4t}+1}{2}\right) - 5t}$$

depending on whether you take out e^{2t}, v e^{-2t}

as a factor!

Question (4)

(b)



(i) For a circular arrangement $\boxed{\frac{8!}{8} = 7!}$

(ii) Fix one man as a ref. (one way) The rest of the men can arrange themselves in $\boxed{4!}$ way.

The women can arrange themselves between the men in $\boxed{(5 \times 4 \times 3)}$ ways

∴ The number of ways to arrange 5M, 3M so that no two women are next to one another

$$= (4!) \times (5 \times 4 \times 3)$$

$$= \underbrace{}_{}5! \times 12 = \boxed{1440}$$

(c) $x^3 - mx + n = 0$ —— (i)

(i) Let $\boxed{y = x^3 \Rightarrow x = y^{1/3}}$.

∴ (1) becomes $y - my^{1/3} + n = 0$

$$\Longrightarrow y^{1/3} = \left(\frac{y+n}{m}\right).$$

$$\therefore \boxed{m^3 y = (y+n)^3}$$

$$m^3 y = y^3 + 3ny^2 + 3n^2 y + n^3$$

Collecting like terms

~~we have~~

$$\boxed{y^3 + 3ny^2 + (3n^2 - m^3)y + n^3 = 0}$$

(ii)

$$\alpha^6 + \beta^6 + \gamma^6 = \left(\sum \alpha_i\right)^2 - 2\sum_{i \neq j} \alpha_i \alpha_j$$

$$\left(\sum \alpha_i\right)^2 = (-3n)^2 - 2(3n^2 - m^3)$$

$$= 9n^2 - 6n^2 + 2m^3.$$

$\alpha\beta\gamma = -n$, (from $x^3 - mx + n = 0$)

$$\therefore \alpha\beta\gamma(\alpha^6 + \beta^6 + \gamma^6) = -n(3n^2 + 2m^3).$$

Question (4)

(a) $1+p+p^2+p^3+p^4 = \frac{p^5-1}{p-1} = 0$ —— (1)

$p^5 = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

(i) $= \cos 2\pi + i\sin 2\pi$ (DeMoivre's Thm)

$= 1$

$\therefore \frac{p^5-1}{p-1} = \frac{1-1}{p-1} = 0$

(ii) $(p+p^4)(p^2+p^3)$

$= p^3+p^4+p^6+p^7 \quad [\because p^5=1]$

$= p^3+p^4+p+p^2$

from (1) $\boxed{1+p+p^2+p^3+p^4=0}$

$= -1$ —— (2)

(iii) The quadratic equation is:

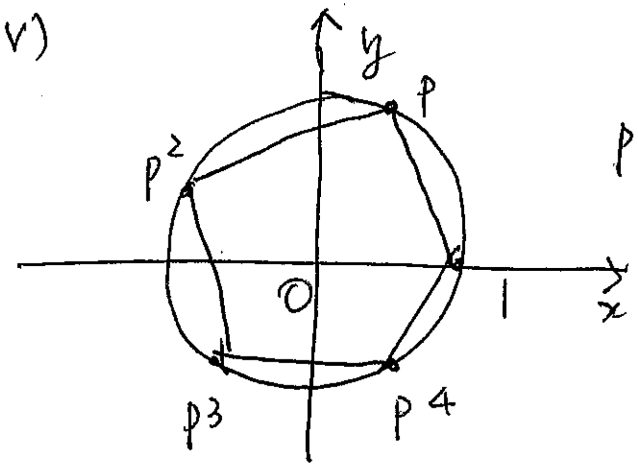
$z^2 - [(p+p^4)+(p^2+p^3)]z + (p+p^4)(p^2+p^3) = 0$

From (1) & (2)

$z^2 - (-1)z + (-1) = 0$

i.e $\boxed{z^2 + z - 1 = 0}$

(iv)



$p^4 = \left(\operatorname{cis}\frac{2\pi}{5}\right)^4$

$= \operatorname{cis}\frac{8\pi}{5}$

$= \operatorname{cis}\left(-\frac{2\pi}{5}\right)$

$= \bar{p}$

Also, $p^3 = \left(\operatorname{cis}\frac{2\pi}{5}\right)^3$

$= \left(\operatorname{cis}\frac{6\pi}{5}\right) = \left(\operatorname{cis} -\frac{4\pi}{5}\right) = \overline{p^2}$

Now, $(p+p^4)(p^2+p^3) = -1$ (proven from (ii))

$\therefore (p+\bar{p})(p^2+\overline{p^2}) = -1.$

$-1 = 2\operatorname{Re}(p) \times 2\operatorname{Re}(p^2)$

$-1 = 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}.$

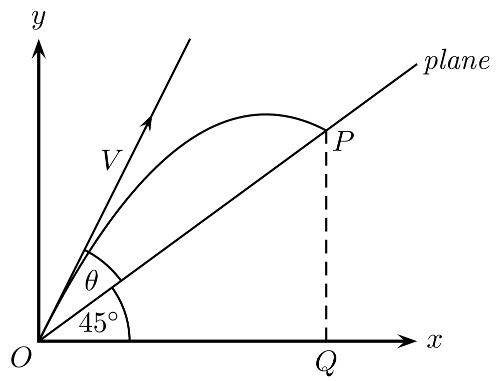
i.e $\boxed{\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -1}$

2014 Extension 2 Mathematics Task 2:
Solutions— Section C

Marks

Question 5 (14 marks)

(a)



A particle is projected from a point O on a plane inclined at $45°$ to the horizontal. The velocity of projection is V at an angle of θ to the inclined plane, where $\theta = \tan^{-1}\frac{1}{2}$.

[You are given that $y = x\tan\alpha - \dfrac{gx^2}{2V^2}(1+\tan^2\alpha)$ where $\alpha = 45° + \theta$.]

(i) Show that $\tan\alpha = 3$. **1**

Solution:
$$\begin{aligned}\tan\alpha &= \tan(\theta + 45°),\\ &= \frac{\frac{1}{2}+1}{1-\frac{1}{2}\times 1},\\ &= 3.\end{aligned}$$

(ii) Hence show that $y = 3x - \dfrac{5gx^2}{V^2}$. **1**

Solution:
$$\begin{aligned}y &= 3x - \frac{gx^2}{2V^2}(1+9),\\ &= 3x - \frac{10gx^2}{2V^2},\\ &= 3x - \frac{5gx^2}{V^2}.\end{aligned}$$

(iii) Show that the range OP is $\dfrac{2\sqrt{2}V^2}{5g}$. **2**

Solution:
$$\begin{aligned}\text{At } P,\ y &= x,\\ \text{so } 2x &= \frac{5gx^2}{V^2},\\ x &= \frac{2V^2}{5g} \text{ as } x \neq 0.\\ \text{Now } OP &= \sqrt{x^2+x^2},\\ &= \sqrt{2}x,\\ &= \frac{2\sqrt{2}V^2}{5g}.\end{aligned}$$

(iv) Show that it meets the plane at right-angles. $\boxed{3}$

Solution: Method 1—

$$\begin{aligned}\frac{dy}{dx} &= 3 - \frac{10gx}{V^2}.\\ \text{At } P,\ x &= \frac{2V^2}{5g},\\ \text{so } \frac{dy}{dx} &= 3 - \frac{10g}{V^2} \times \frac{2V^2}{5g},\\ &= -1 \text{ at } P.\end{aligned}$$

Now the slope of the plane is 1, so the particle meets it at right-angles.

Solution: Method 2—

$$\begin{aligned}\ddot{y} &= -g & \ddot{x} &= 0\\ \dot{y} &= -gt + V\sin\alpha & \dot{x} &= V\cos\alpha\\ y &= -\frac{gt^2}{2} + Vt\sin\alpha & x &= Vt\cos\alpha\end{aligned}$$

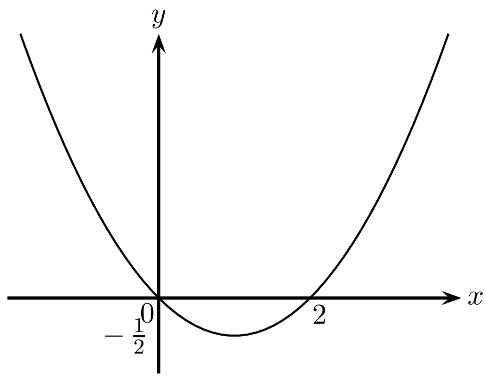
$$\begin{aligned}\text{At } P,\quad Vt\cos\alpha &= Vt\sin\alpha - \frac{gt^2}{2},\ t \neq 0\\ \frac{gt}{2} &= V\sin\alpha - V\cos\alpha\\ t &= \frac{2V}{g} \times \left(\frac{3-1}{\sqrt{10}}\right)\\ &= \frac{4V}{g\sqrt{10}}\\ \dot{y} &= \frac{3V}{\sqrt{10}} - \frac{4V}{\sqrt{10}}\\ &= -\frac{V}{\sqrt{10}}\\ \therefore \frac{\dot{y}}{\dot{x}} &= -\frac{V}{\sqrt{10}} \times \frac{\sqrt{10}}{V}\\ &= -1.\end{aligned}$$

Now the slope of the plane is 1, so the particle meets it at right-angles.

(b) The sketch is of the parabola $\boxed{7}$

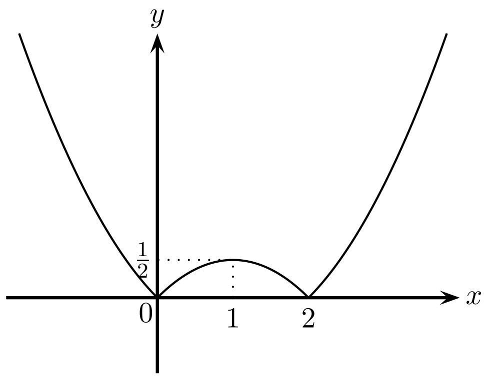
$$f(x) = \frac{x^2 - 2x}{2}.$$

Without calculus, draw the following on separate diagrams, showing essential features:



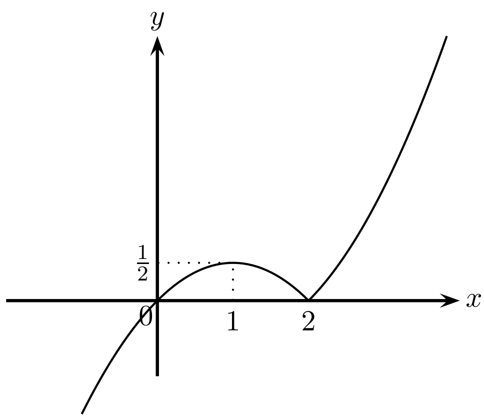
(i) $y = |f(x)|$

Solution:



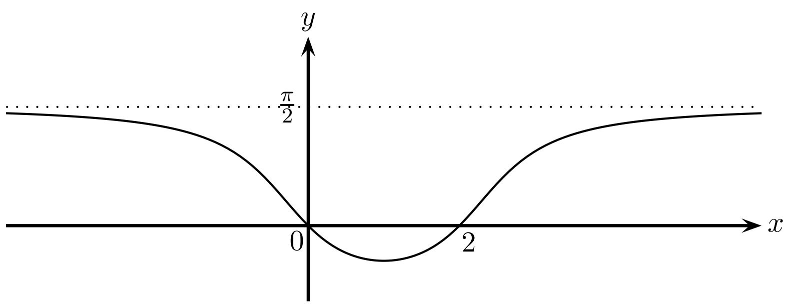
(ii) $y = \dfrac{x}{2}|x-2|$

Solution:



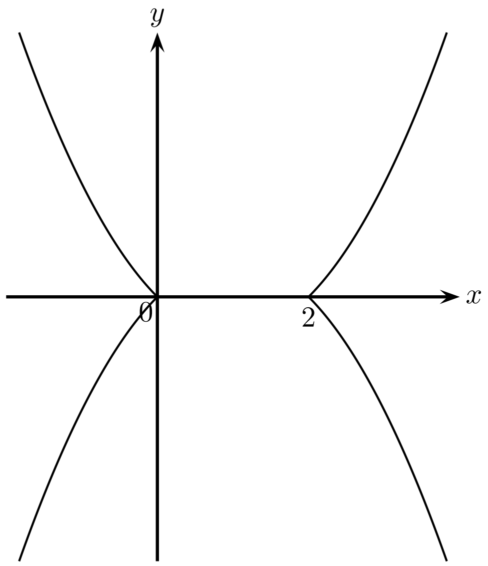
(iii) $y = \tan^{-1}\big(f(x)\big)$

Solution:



(iv) $|y| = f(x)$

Solution:



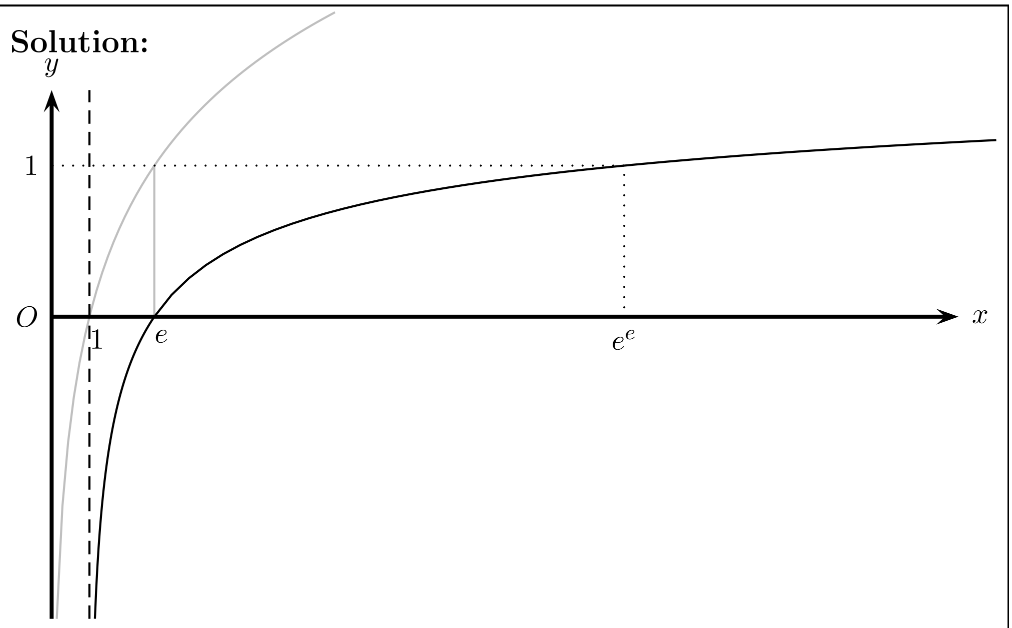
Marks

Question 6 (15 marks)

(a) Sketch the following, showing essential features:

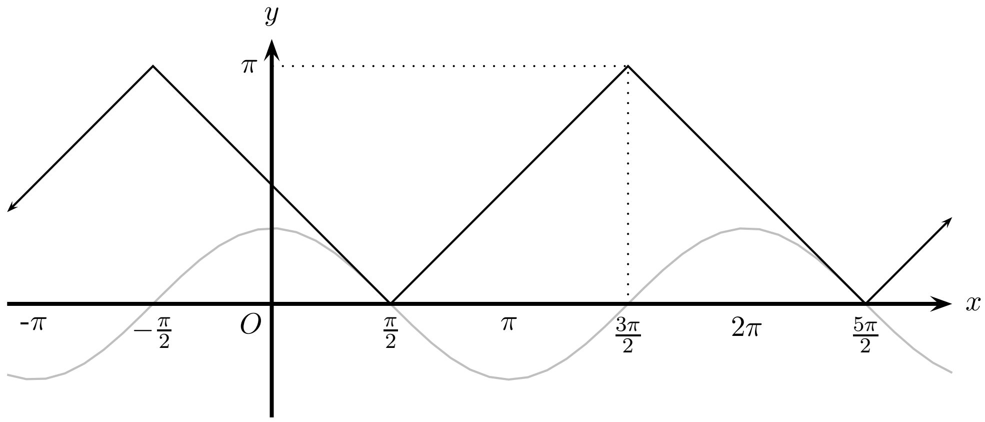
(i) $y = \ln(\ln x)$ $\boxed{2}$

Solution:



(ii) $f(x) = \cos^{-1}(\sin x)$ $\boxed{2}$

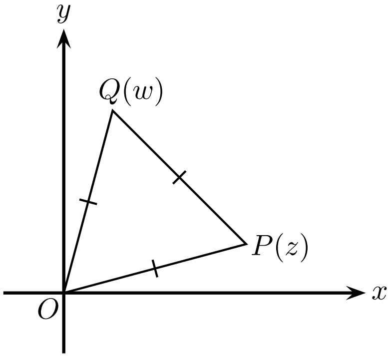
Solution:
$$\begin{aligned} f'(x) &= \frac{-\cos x}{\sqrt{1-\sin^2 x}}, \\ &= \frac{-\cos x}{|\cos x|}, \\ &= \begin{cases} -1 & \text{when } \cos x > 0, \\ \ \ 1 & \text{when } \cos x < 0, \end{cases} \end{aligned}$$



(b) The triangle OPQ is equilateral. $\boxed{2}$

P represents the complex number z, and
Q represents the complex number w.

Show that $z^3 + w^3 = 0$.



Solution:

$$\begin{aligned} w &= z\,\text{cis}\,\tfrac{\pi}{3}, \\ w^3 &= z^3\,\text{cis}\,\pi, \\ &= -z^3. \\ \therefore z^3 + w^3 &= z^3 - z^3, \\ &= 0. \end{aligned}$$

(c) Let $I_n = \displaystyle\int_0^1 \left(1 - x^2\right)^n dx$ and $J_n = \displaystyle\int_0^1 x^2\left(1 - x^2\right)^n dx$.

(i) Using integration by parts on I_n, show that $I_n = 2nJ_{n-1}$. $\boxed{2}$

Solution:

$$\begin{aligned} u &= \left(1 - x^2\right)^n & v' &= dx \\ u' &= -2xn\left(1 - x^2\right)^{n-1} dx & v &= x \end{aligned}$$

$$\begin{aligned} I_n &= \Big[x\left(1 - x^2\right)^n\Big]_0^1 + \int_0^1 2nx^2\left(1 - x^2\right)^{n-1} dx, \\ &= 0 + 2n\int_0^1 x^2\left(1 - x^2\right)^{n-1} dx, \\ &= 2nJ_{n-1}. \end{aligned}$$

(ii) Hence show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$. $\boxed{2}$

Solution:

$$\begin{aligned} I_n &= -2n\int_0^1 -x^2\left(1 - x^2\right)^{n-1} dx, \\ &= -2n\int_0^1 \Big(\left(1 - x^2\right)\left(1 - x^2\right)^{n-1} - \left(1 - x^2\right)^{n-1}\Big)\, dx, \\ &= -2n\int_0^1 \left(1 - x^2\right)^n dx + 2n\int_0^1 \left(1 - x^2\right)^{n-1} dx, \\ &= 2nI_{n-1} - 2nI_n, \\ (1 + 2n)I_n &= 2nI_{n-1}, \\ \therefore I_n &= \frac{2n}{2n+1} I_{n-1}. \end{aligned}$$

(iii) Show that $J_n = I_n - I_{n+1}$. $\boxed{2}$

Solution: Method 1—

$$\begin{aligned}
I_n - I_{n+1} &= \int_0^1 \left(1-x^2\right)^n dx - \int_0^1 \left(1-x^2\right)^{n+1} dx, \\
&= \int_0^1 \left(\left(1-x^2\right)^n - \left(1-x^2\right)\left(1-x^2\right)^n\right) dx, \\
&= \int_0^1 x^2\left(1-x^2\right)^n dx, \\
&= J_n.
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
\cancel{2n} J_{n-1} &= \frac{\cancel{2n}}{2n+1} I_{n-1} \big(\text{from (i) and (ii)}\big), \\
\therefore 2nJ_{n-1} + J_{n-1} &= I_{n-1}, \\
I_n + J_{n-1} &= I_{n-1}, \\
J_{n-1} &= I_{n-1} - I_n, \\
\text{so } J_n &= I_n - I_{n+1}.
\end{aligned}$$

(iv) Hence deduce that $J_n = \dfrac{1}{2n+3} I_n$. $\boxed{2}$

Solution: Method 1—

$$\begin{aligned}
I_{n+1} &= \big(2(n+1)\big) J_n \ \big(\text{from (i)}\big), \\
\therefore J_n &= I_n - (2n+2)J_n \ \big(\text{from (iii)}\big), \\
J_n(2n+2+1) &= I_n, \\
J_n &= \frac{1}{2n+3} I_n
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
I_{n+1} &= \frac{2(n+1)}{2(n+1)+1} I_{(n+1)-1} \ \big(\text{from (ii)}\big), \\
&= I_n \left(\frac{2n+2}{2n+3}\right). \\
\text{So } J_n &= I_n \left(\frac{2n+3-(2n+2)}{2n+3}\right) \ \big(\text{from (iii)}\big), \\
&= \frac{1}{2n+3} I_n.
\end{aligned}$$

(v) Hence find a reduction formula for J_n in terms of J_{n-1}. $\boxed{1}$

Solution:

$$\begin{aligned}
J_n &= \frac{1}{2n+3} \times 2nJ_{n-1}, \\
&= \frac{2n}{2n+3} J_{n-1}.
\end{aligned}$$