SYDNEY GRAMMAR SCHOOL

2012 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 2

Thursday 17th May 2012

General Instructions

- Writing time — 1 Hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection

- Write your candidate number clearly on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 86 boys

Examiner
BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which of the following expressions is a primitive of $\dfrac{1}{\sqrt{4-9x^2}}$? [1]

(A) $\frac{1}{3}\sin^{-1}\frac{3x}{2}$

(B) $\frac{1}{9}\sin^{-1}\frac{3x}{2}$

(C) $\sin^{-1}\frac{3x}{2}$

(D) $\frac{1}{2}\sin^{-1}\frac{3x}{2}$

QUESTION TWO

The remainder when $16x^4 - 8x + 3$ is divided by $x + i$ is: [1]

(A) $-13+8i$

(B) $19+8i$

(C) $-13-8i$

(D) $19-8i$

QUESTION THREE

The angle between the asymptotes of the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ is approximately: [1]

(A) 74°

(B) 37°

(C) 61°

(D) 29°

Exam continues next page ...

QUESTION FOUR

At the point $\left(2\cos\theta, \sin\theta\right)$ on the ellipse $x^2 + 4y^2 = 4$ the tangent has gradient: **1**

(A) $\frac{1}{2}\cot\theta$

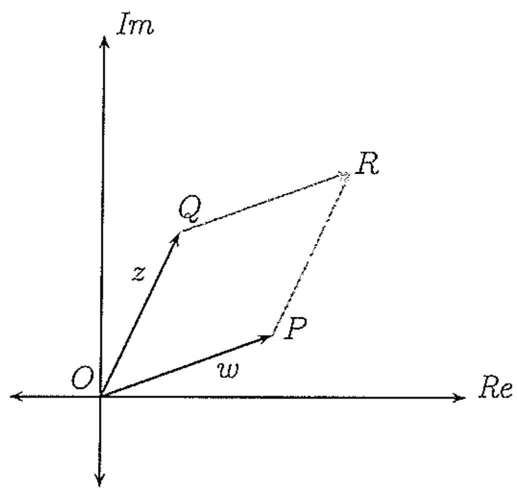
(B) $\frac{1}{2}\tan\theta$

(C) $-\frac{1}{2}\cot\theta$

(D) $-2\tan\theta$

QUESTION FIVE

1



Let P and Q represent complex numbers w and z respectively in the first quadrant of the Argand Plane, with $\arg z > \arg w$, as in the diagram above. The diagram also shows the addition parallelogram. Then $w - z$ is represented by the vector:

(A) $\overrightarrow{PQ}$

(B) $\overrightarrow{RP}$

(C) $\overrightarrow{OR}$

(D) $\overrightarrow{QP}$

QUESTION SIX

One root of the quadratic equation $iz^2 + 3z + 3 - 11i = 0$ is $3 + 2i$. The second root is: **1**

(A) $-2i$

(B) $-3 + i$

(C) $-6 - 2i$

(D) $-3 - 5i$

QUESTION SEVEN

A skydiver falls out of a plane which is flying horizontally. Initially his motion is determined by the acceleration due to gravity $10\,\text{m/s}^2$ and any resistance is negligible. After 5 seconds, he opens his parachute and his motion is determined by the equation $\ddot{x} = 10 - \frac{v}{4}$. His terminal velocity is $40\,\text{m/s}$. Downwards is taken as positive. **1**

Which statement best reflects the situation after he opens his parachute?

(A) He hits the ground with a vertical speed of $40\,\text{m/s}$.

(B) His vertical speed never exceeds $40\,\text{m/s}$.

(C) His vertical speed never drops below $40\,\text{m/s}$.

(D) We need to know his horizontal speed to complete this question.

——— End of Section I ———

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. Marks

(a) Using a suitable substitution, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \tan x \, dx$. **2**

(b) By a suitable use of a Pythagorean identity, find $\displaystyle\int \sin^3 x \cos^2 x \, dx$. **2**

(c) Consider the ellipse $\mathcal{E}$ with equation $9x^2 + 16y^2 = 144$.

(i) Find its eccentricity. **1**

(ii) Write down the coordinates of the foci. **1**

(iii) Write down the equations of the directrices. **1**

(iv) Sketch the ellipse, showing any intercepts with the axes and the information found above. **2**

(d) It is known that the equation $x^3 + 3x^2 + 7x + 5 = 0$ has roots α, β and γ.

(i) Find a simplified monic polynomial equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. **2**

(ii) Hence solve $x^3 + 3x^2 + 7x + 5 = 0$. **1**

QUESTION NINE (12 marks) Use a separate writing booklet. Marks

(a) Use a t-substitution to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x + 2\sin x} \, dx$ **4**

(b) Let $P(x) = 2x^3 + (1 - 4k)x^2 + (2k^2 - 2k)x + k^2$.

(i) Show that $x = k$ is a double root of $P(x) = 0$. **2**

(ii) Hence find the third root of $P(x) = 0$. **1**

(c) Consider the hyperbola $\mathcal{H}$ with equation $xy = c^2$ where c is a positive constant and let $P(cp, \frac{c}{p})$ be a variable point on $\mathcal{H}$. Suppose that the tangent at P intersects the x-axis at A and let M be the midpoint of AP.

(i) Show that the tangent to $\mathcal{H}$ at P has equation $x + p^2 y = 2cp$. **2**

(ii) Show that M has coordinates $\left(\frac{3}{2}cp, \frac{c}{2p}\right)$. **2**

(iii) Find the Cartesian equation of the locus of M, as P moves along the hyperbola $\mathcal{H}$. **1**

Exam continues overleaf ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$.

(i) Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$, for $n \geq 2$. **3**

(ii) Find I_4. **1**

(b) An object with unit mass is launched straight upwards with an initial speed of u m/s. It experiences the force of gravity and also a retarding force of $\dfrac{1}{40}v^2$ Newtons. Let x be its vertical displacement from its point of launch and take upwards as positive and the acceleration due to gravity as $g = 10\text{m/s}^2$.

(i) Show that the equation of motion is $\ddot{x} = \dfrac{-400 - v^2}{40}$. **1**

(ii) Show that the object rises to a maximum height of $20 \ln \left(\dfrac{400 + u^2}{400} \right)$. **2**

(iii) With an initial speed of 10 m/s the particle reaches a maximum height of $20 \ln \frac{5}{4}$ metres. What launch speed would double this maximum height? Working is required. **2**

(c) Suppose that $0 < x < \frac{1}{4}$.

(i) Prove that $1 - 2x > 2x > 0$. **1**

(ii) Find $\sin \cos^{-1} \dfrac{2x}{2x-1}$. **2**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Let $I = \int_0^{\frac{\pi}{2}} \ln \sin x \, dx$.

(i) Use the substitution $u = \frac{\pi}{2} - x$ to show that **1**

$$I = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx\,.$$

(ii) Use the substitution $u = \pi - x$ to show that **2**

$$I = \frac{1}{2} \times \int_0^{\pi} \ln \sin x \, dx\,.$$

(iii) Use part (i) to show that $2I = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx$. **1**

(iv) Hence use parts (ii) and (iii) to show that $I = -\dfrac{\pi}{2} \ln 2$. **2**

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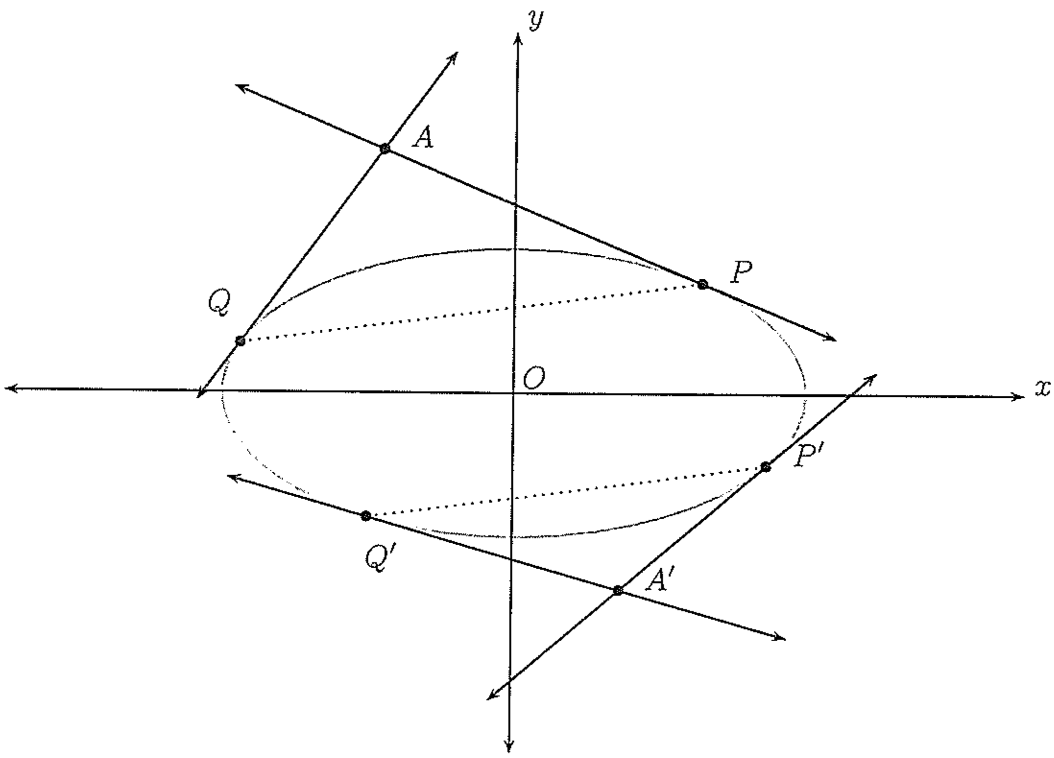
(b)



Consider the ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b < a$. Let $O(0, 0)$ be the centre of the ellipse.

Let PQ be the chord of contact defined by the point $A(x_0, y_0)$ external to the ellipse such that $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on $\mathcal{E}$.

Similarly let $P'Q'$ be the chord of contact defined by the point $A'(x_0', y_0')$ external to the ellipse such that $P'(x_1', y_1')$ and $Q'(x_2', y_2')$ lie on $\mathcal{E}$.

(i) Show that the tangent to $\mathcal{E}$ at $P(x_1, y_1)$ has equation $\dfrac{x_1 x}{a^2} + \dfrac{y_1 y}{b^2} = 1$. **[2]**

(ii) Assume the the chord of contact PQ has equation $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = 1$. Show that if A, A' and O are collinear, then the chords PQ and $P'Q'$ are parallel. **[2]**

(iii) If the chords PQ and $P'Q'$ are not parallel, then suppose they intersect at $R(x_3, y_3)$. Prove that the chord of contact from R lies on the line AA'. **[2]**

——— End of Section II ———

END OF EXAMINATION

SECTION I - Multiple Choice

QUESTION ONE

The required indefinite integral is:

$$\int \frac{1}{\sqrt{4-9x^2}} = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3x}{2}\right)^2}} \\ = \frac{1}{2} \times \frac{2}{3} \sin^{-1}\left(\tfrac{3x}{2}\right)$$

Hence A.

QUESTION TWO

Let $P(x) = 16x^4 - 8x + 3$. Then $P(-i) = 19 + 8i$. Hence B.

QUESTION THREE

The angle of inclination of the asymptote $y = \frac{b}{a}x$ is $\tan^{-1}(\frac{3}{4}) \doteqdot 37°$. The angle between the asymptotes is twice this, hence A.

QUESTION FOUR

By parametric differentiation, $\dfrac{dy}{dx} = \dfrac{\cos\theta}{-2\sin\theta}$, hence C.

QUESTION FIVE

The vector $w - z$ runs from Q to P, hence D.

QUESTION SIX

Let the second root be α. Then the sum of the roots is $\alpha + 3 + 2i = 3i$, thus $\alpha = -3 + i$. Hence B.

QUESTION SEVEN

After 5 seconds, the skydiver is travelling at $50\,\text{m/s}$, i.e. faster than the terminal velocity. Once he opens his parachute, his speed will slow to approach the terminal velocity, but never drop below this limiting velocity. Hence C.

SECTION II - Written Response

QUESTION EIGHT (12 marks)

Marks

(a) **2**

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x \tan x \, dx &= \left[\frac{1}{2}\tan^2 x\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{2}\left(\tan^2 \tfrac{\pi}{3} - \tan^2 0\right) \\ &= \frac{3}{2}\end{aligned}$$

(b) **2**

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin x(1-\cos^2 x)\cos^2 x \, dx \\ &= \left(\int \sin x \cos^2 x - \sin x \cos^4 x\right) dx \\ &= \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + C\end{aligned}$$

(c) (i) We have $b^2 = a^2(1-e^2)$ so $e^2 = 1 - \dfrac{b^2}{a^2}$. **1**

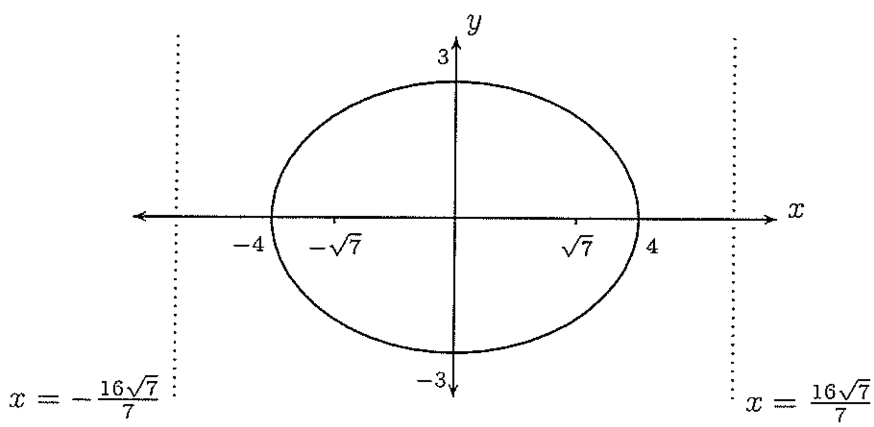
Since $a = 4$, $b = 3$ we have $e^2 = 1 - \dfrac{9}{16} = \dfrac{7}{16}$.

Hence $e = \frac{\sqrt{7}}{4}$.

(ii) The foci are $(\pm ae, 0) = (\pm\sqrt{7}, 0)$. **1**

(iii) The directrices are $x = \dfrac{a}{e}$ or $x = -\dfrac{a}{e}$. That is, $x = \dfrac{16\sqrt{7}}{7}$ or $x = -\dfrac{16\sqrt{7}}{7}$. **1**

(iv) **2**



(d) (i) Let $P(x) = x^3 + 3x^2 + 7x + 5 = 0$. The required polynomial is $P(x-1)$. The equation is: 2

$$\begin{aligned}(x-1)^3 + 3(x-1)^2 + 7(x-1) + 5 &= 0\\ x^3 - 3x^2 + 3x - 1 + 3(x^2 - 2x + 1) + 7x - 7 + 5 &= 0\\ x^3 + 4x &= 0\\ x(x^2+4) &= 0\end{aligned}$$

(ii) This new equation has roots $x = 0$, $x = \pm 2i$. Hence the original polynomial equation has roots -1, $-1 \pm 2i$. 1

QUESTION NINE (12 marks)

Marks

(a) 4

Let $t = \tan\frac{x}{2}$

Then $x = 2\tan^{-1} t$

Thus $dx = \dfrac{2dt}{1+t^2}$

$$\begin{aligned}\text{Hence } \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x + 2\sin x}\,dx &= \int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2} + \frac{4t}{1+t^2}}\,\frac{2}{1+t^2}\,dt\\ &= \int_0^1 \frac{2}{1+t^2+1-t^2+4t}\,dt\\ &= \int_0^1 \frac{2}{2+4t}\,dt\\ &= \int_0^1 \frac{1}{1+2t}\,dt\\ &= \left[\frac{1}{2}\ln(1+2t)\right]_0^1\\ &= \frac{1}{2}\ln 3\end{aligned}$$

(b) (i) 2

$$\begin{aligned}\text{We have } P(k) &= 2k^3 + (1-4k)k^2 + (2k^2-2k)k + k^2\\ &= 2k^3 + k^2 - 4k^3 + 2k^3 - 2k^2 + k^2\\ &= 0\\ \text{and } P'(k) &= 6k^2 + (1-4k)2k + (2k^2 - 2k)\\ &= 6k^2 + 2k - 8k^2 + 2k^2 - 2k\\ &= 0\end{aligned}$$

Hence k is a double root of $P(x)$.

(ii) Since the product of the roots is $-\dfrac{k^2}{2}$, the third root must be $-\dfrac{1}{2}$. 1

(c) (i) By parametric differentiation, [2]

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dp}{dx/dp}\\ &= \frac{-\frac{c}{p^2}}{-c}\\ &= -\frac{1}{p^2}\end{aligned}$$

Hence at the point $P(cp, \frac{c}{p})$ the tangent is

$$\begin{aligned}(y - \frac{c}{p}) &= -\frac{1}{p^2}(x - cp)\\ p^2y - cp &= -x + cp\\ x + p^2y &= 2cp\end{aligned}$$

(ii) The x-intercept has coordinates $A(2cp, 0)$. Point M is the midpoint of AP, hence it has coordinates [2]

$$\left(\frac{1}{2}(2cp+p), \frac{1}{2}(0+\tfrac{c}{p})\right) = \left(\frac{3}{2}cp, \frac{c}{2p}\right)$$

(iii) Eliminating the parameter p is easily done by noting [1]

$$\begin{aligned}xy &= \frac{3}{2}cp \times \frac{c}{2p}\\ xy &= \frac{3}{4}c^2\end{aligned}$$

This is a hyperbola— simply a stretched version of the original.

QUESTION TEN (12 marks)

Marks

(a) (i) Using integration by parts, [3]

$$\begin{aligned}I_n &= \int_0^{\frac{\pi}{2}} \cos^n x\,dx\\ &= \int_0^{\frac{\pi}{2}} \cos x \times \cos^{n-1} x\,dx\\ &= \Big[\sin x \cos^{n-1} x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \times -\sin x(n-1)\cos^{n-2} x\,dx\\ &= 0 + (n-1) \times \int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x\,dx\\ &= (n-1) \times \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2} x\,dx\\ &= (n-1) \times \left(\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx - \int_0^{\frac{\pi}{2}} \cos^n x\,dx\right)\\ &= (n-1) \times \Big(I_{n-2} - I_n\Big)\end{aligned}$$

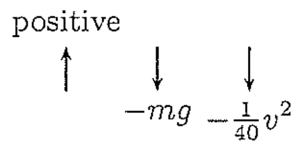
$$\begin{aligned} \text{Hence } I_n + (n-1)I_n &= (n-1)I_{n-2} \\ nI_n &= (n-2)I_{n-1} \\ I_n &= \frac{n-1}{n} I_{n-2} \end{aligned}$$

(ii) [1]

$$\begin{aligned} I_4 &= \frac{3}{4} I_2 \\ &= \frac{3}{4} \times \frac{1}{2} \times \int_0^{\frac{\pi}{2}} dx \\ &= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \\ &= \frac{3\pi}{16} \end{aligned}$$

(b) (i) [1]

The forces on the object are gravity $-mg$ and a retarding force opposing the motion $-\frac{1}{40}v^2$. By Newtons second law, the resulting force gives equation $m\ddot{x} = -mg - \frac{1}{40}v^2$. Taking $g = 10$ and $m = 1$ (since the object has unit mass) gives



$$\begin{aligned} \ddot{x} &= -10 - \tfrac{1}{40}v^2 \\ &= \frac{-400 - v^2}{40} \end{aligned}$$

(ii) Solving this equation of motion gives: [2]

$$\begin{aligned} \ddot{x} &= \frac{-400 - v^2}{40} \\ v\frac{dv}{dx} &= -\frac{400 + v^2}{40} \\ -\frac{40v\,dv}{400 + v^2} &= dx \end{aligned}$$

$$\begin{aligned} \text{Hence } \int dx &= -20 \times \int \frac{2v}{400 + v^2}\, dv \\ x &= -20\ln(400 + v^2) + C \end{aligned}$$

Since $x = 0$ when $v = u$, we have $C = 20\ln(400 + u^2)$. The maximum height $x = H$ is attained when $v = 0$, so

$$\begin{aligned} H &= -20\ln 400 + 20\ln(400 + u^2) \\ &= 20\ln\left(\frac{400 + u^2}{400}\right) \end{aligned}$$

(iii) We need to find what initial speed u gives a maximum height $40\ln\frac{5}{4}$. That is, we must solve **[2]**

$$\begin{aligned}40\ln\tfrac{5}{4} &= 2\ln\left(\frac{400+u^2}{400}\right)\\ 2\ln\tfrac{5}{4} &= \ln\left(\frac{400+u^2}{400}\right)\\ \ln\tfrac{25}{16} &= \ln\left(\frac{400+u^2}{400}\right)\\ \tfrac{25}{16} &= \frac{400+u^2}{400}\\ 625 &= 400+u^2\\ 225 &= u^2\\ u &= 15\end{aligned}$$

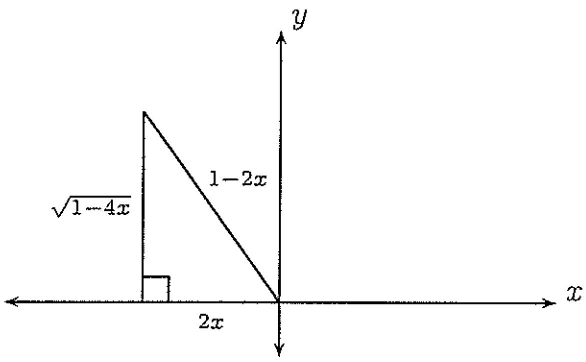
(c) (i) **[1]**

$$\begin{aligned}&0 < x < \tfrac{1}{4}\\ &0 < 2x < \tfrac{1}{2}\\ &0 > -2x > -\tfrac{1}{2}\\ &1 > 1-2x > \tfrac{1}{2}\end{aligned}$$

But $\frac{1}{2} > 2x > 0$, so $1-2x > 2x > 0$.

(ii) Let $\theta = \cos^{-1}\dfrac{2x}{2x-1}$. Since $\dfrac{2x}{2x-1} < 0$, the angle θ is in quadrant 2. The related triangle is drawn below: note that $|2x-1| = 1-2x > 0$. The missing side length ℓ in this triangle is given by **[2]**

$$\begin{aligned}\ell^2 &= (1-2x)^2-(2x)^2\\ &= 1-4x\end{aligned}$$



The angle is in quadrant two, hence $\sin\theta = +\dfrac{\sqrt{1-4x}}{1-2x}$.

QUESTION ELEVEN (12 marks) Marks

(a) (i) 1

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx \\
&= \int_{\frac{\pi}{2}}^{0} \ln \sin(\tfrac{\pi}{2} - u) \times -du \\
&= \int_0^{\frac{\pi}{2}} \ln \sin(\tfrac{\pi}{2} - u) \, du \\
&= \int_0^{\frac{\pi}{2}} \ln \cos u \, du \\
&= \int_0^{\frac{\pi}{2}} \ln \cos x \, dx
\end{aligned}
$$

relabelling u as x.

(ii) 2

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx \\
&= \int_{\pi}^{\frac{\pi}{2}} \ln \sin(\pi - u) \times -du \\
&= \int_{\frac{\pi}{2}}^{\pi} \ln \sin(\pi - u) \, du \\
&= \int_{\frac{\pi}{2}}^{\pi} \ln \sin u \, du \\
&= \int_{\frac{\pi}{2}}^{\pi} \ln \sin x \, dx
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^{\pi} \ln \sin x \, dx &= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx + \int_{\frac{\pi}{2}}^{\pi} \ln \sin x \, dx \\
&= 2I
\end{aligned}
$$

The result now follows on dividing by 2.

(iii) 1

$$
\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} \ln \sin x \, dx + \int_0^{\frac{\pi}{2}} \ln \cos x \, dx \\
&= \int_0^{\frac{\pi}{2}} \ln (\sin x \cos x) \, dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\tfrac{1}{2} \sin 2x\right) dx
\end{aligned}
$$

(iv) Use the substitution $u = 2x$ in part (iii). Then **2**

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln\left(\tfrac{1}{2}\sin 2x\right)\,dx \\ &= \int_0^{\pi} \ln\left(\tfrac{1}{2}\sin u\right)\frac{1}{2}du \\ &= \frac{1}{2}\times\int_0^{\pi}\left(\ln\tfrac{1}{2} + \ln\sin u\right)\,du \\ &= \frac{1}{2}\times\left(\int_0^{\pi}\ln\tfrac{1}{2}\,du + \int_0^{\pi}\ln\sin u\right)\,du \\ &= \frac{\pi}{2}\ln\tfrac{1}{2} + I \\ &= -\frac{\pi}{2}\ln 2 + I \end{aligned}$$

Hence $I = -\dfrac{\pi}{2}\ln 2$.

(v) By implicit differentiation, **2**

$$\begin{aligned} \frac{2x}{a^2} + \frac{2yy'}{b^2} &= 0 \\ y' &= \frac{-xb^2}{ya^2} \\ y' &= \frac{-x_1b^2}{y_1a^2} \qquad \text{at } (x_1, y_1) \end{aligned}$$

The tangent at P has equation

$$\begin{aligned} (y - y_1) &= \frac{-x_1b^2}{y_1a^2}(x - x_1) \\ \frac{yy_1 - y_1^2}{b^2} &= \frac{-xx_1 + x_1^2}{a^2} \\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} &= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} &= 1 \end{aligned}$$

since (x_1, y_1) is on $\mathcal{E}$.

(vi) We are given that gradient AO = gradient AO' and required to show that gradient PQ = gradient $P'Q'$. **2**

By rearranging the chord of contact formula, the gradient of PQ is $\dfrac{-x_0b^2}{y_0a^2}$.

Similarly the gradient of $P'Q'$ is $\dfrac{-x_0'b^2}{y_0'a^2}$.

Thus it is enough to show that $\dfrac{x_0}{y_0} = \dfrac{x_0'}{y_0'}$.

But gradient $AO = \dfrac{x_0 - 0}{y_0 - 0} = \dfrac{x_0}{y_0}$ and gradient $A'O = \dfrac{x_0' - 0}{y_0' - 0} = \dfrac{x_0'}{y_0'}$.
Since by assumption gradient AO = gradient AO', we are done.

(vii) The equations of the chords of contact PQ and $P'Q'$ from A and A' respectively are **2**

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1 \qquad \text{and} \qquad \frac{xx_0'}{a^2} + \frac{yy_0'}{b^2} = 1$$

Now since $R(x_3, y_3)$ is on the intersection of PQ and $P'Q'$, then it lies on both lines, that is its coordinates satisfy both equation. Hence

$$\frac{x_3x_0}{a^2} + \frac{y_3y_0}{b^2} = 1 \qquad \text{and} \qquad \frac{x_3x_0'}{a^2} + \frac{y_3y_0'}{b^2} = 1$$

But the chord of contact from $R(x_3, y_3)$ has equation $\dfrac{x_3x}{a^2} + \dfrac{y_3y}{b^2} = 1$.

By the previous two equations (x_0, y_0) and (x_0', y_0') lie on this line. Hence the line AA' is the line through the chord of contact from R.

BDD