SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION II

HSC ASSESSMENT TASK II

JUNE 2002

Time allowed: 70 minutes

Instructions:

- Show all necessary working in every question.
- Start each question on a new page.
- Attempt all questions.
- All questions are of equal value.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- This test forms part of your HSC assessment.
- These questions are to be handed in with your answers.

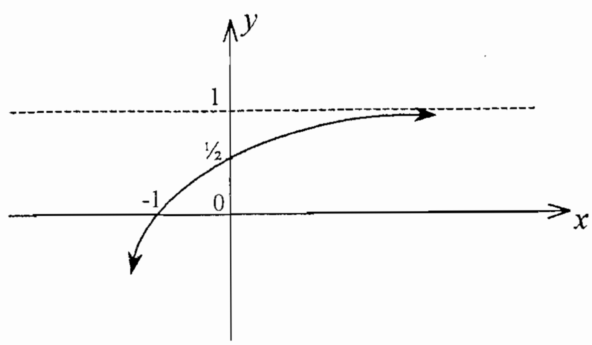
Name:____________________________

Question 1	Question 2	Question 3	Total

Question 1 **Marks**

a) Find $\int \frac{dx}{x^2 - 4x + 13}$ 2

b) The diagram shows the graph of the increasing function $y = f(x)$ 6



Draw separate one-third page sketches of the graphs of the following showing any intercepts

i) $y = \frac{1}{f(x)}$

ii) the inverse function of $y = f(x)$

iii) $y = \ln f(x)$

c) Find $\int \frac{dx}{\sqrt{e^{2x} - 1}}$ using the substitution $u = \sqrt{e^{2x} - 1}$ 2

d) A (1,4) is a fixed point while $P(2t, \frac{2}{t})$ is a variable point in the first quadrant on the hyperbola $xy = 4$. 7

i) what values can "t" take

ii) a) Show that the equation of the chord AP is $2x + ty - 4t - 2 = 0$

b) Which value/s of "t" must be excluded why?

iii) The chord AP cuts the x-*axis* at M. Find the coordinates of M

iv) Find the equation of the locus of the mid point of PM.

Question 2 **Marks**

a) The polynomial $x^3 + 5x - t = 0$ has 3 real roots α, β, γ 4

i) Find the cubic equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

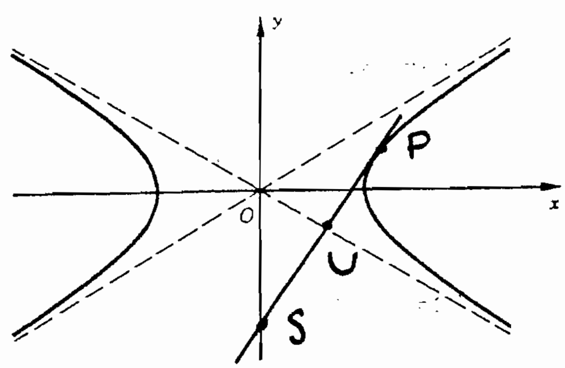
ii) Hence or otherwise find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

b) i) If $\frac{a}{x^2 - a^2} \equiv \frac{A}{x-a} + \frac{B}{x+a}$ find the values of A and B 5

ii) Use the substitution $x = u^2$ to find $\int \frac{\sqrt{x}}{x-1} dx$

(you may use the result in part i)

c) 8



Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

i) Write down the equation of each asymptote

ii) By differentiation find the gradient of the tangent to the hyperbola at $P(3\sec\theta, 4\tan\theta)$

iii) Show that the equation of the tangent at P is $4x = 3\sin\theta y + 12\cos\theta$

iv) Find the x-coordinate of U the point where the tangent meets the asymptote (as shown on the diagram)

v) Using the x-values only, find the value for θ such that U is the mid point of PS.

Question 3

a) Find $\int \tan^3 x \, dx$ 2

b) Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{2-2\cos 2x}\,dx$ 2

c) i) Write down the expansion for $\cos 3\theta$ *in terms of* $\cos\theta$ 5

ii) By solving an equation of the form $\cos 3\theta = K$ and using part i) find the exact roots of the equation $8x^3 - 6x - \sqrt{3} = 0$

d) i) Explain why a polynomial of degree 5 with real co-efficients has at least 1 real root. 7

The polynomial P (x) is given by $P(x) = x^5 - 5cx + 1$ where c is a real number

ii) Show by considering the turning points of P (x), that when $c < 0, P(x)$ has just on real root.

iii) Explain why this root is negative

iv) Prove that P (x) has three distinct real roots if $c > (\frac{1}{4})^{4/5}$.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

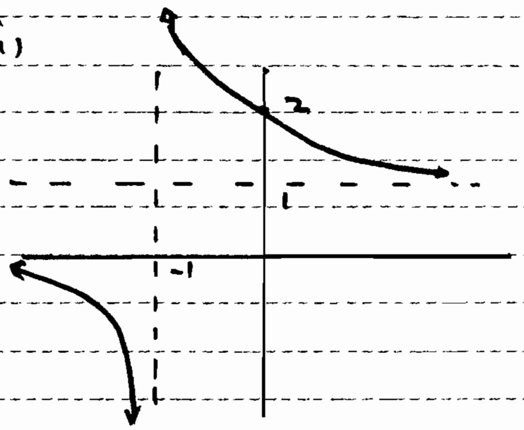
① a)

$$\int \frac{dx}{x^2-4x+13} = \int \frac{dx}{(x^2-4x+4)+9}$$

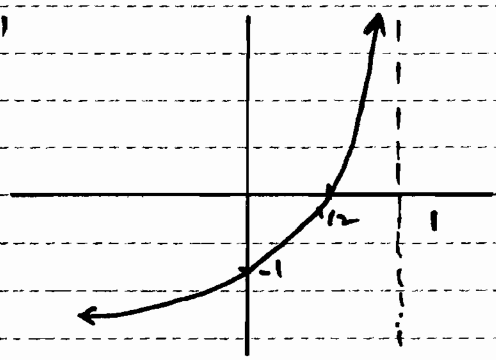
$$= \int \frac{dx}{9+(x-2)^2} \checkmark$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + c \checkmark$$

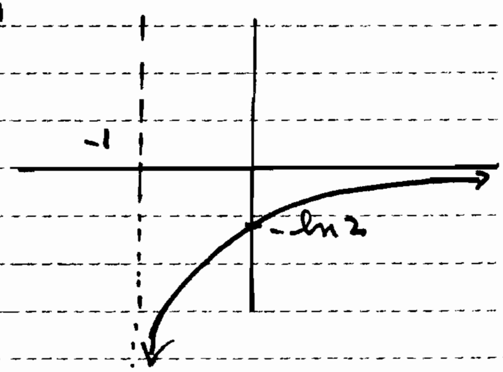
b) i)



ii)



iii)



each 1 shape.
1 intercepts / asymtotes

c) $u^2 = e^{2x} - 1$

$2u\,du = 2e^{2x}\,dx$

$dx = \frac{u\,du}{e^{2x}}$ ✓

$\therefore \int \frac{u\,du}{(u^2+1)\cdot u}$

$= \int \frac{du}{u^2+1}$

$= \tan^{-1}\sqrt{e^{2x}-1}$ ✓

d) i) $t > 0$ ✓

ii) a) $m_{AP} = \dfrac{\frac{2}{t} - 4}{2t-1}$

$= \frac{2-4t}{t(2t-1)}$

$= \frac{-2(2t-1)}{t(2t-1)}$

$= -\frac{2}{t}$ ✓

$y - 4 = -\frac{2}{t}(x-1)$

$ty - 4t = -2x + 2$ ✓

$2x + ty - 4t - 2 = 0$ ✓

b) $t \neq 1/2$ P coincides with A

iii) $y = 0$

$2x = 4t + 2$

$x = 2t + 1$

$\therefore M(2t+1, 0)$ ✓

iv) mid pt PM

$= \frac{2t+2t+1}{2}, \quad \frac{0+\frac{2}{t}}{2}$

$= \left(\frac{4t+1}{2}, \frac{1}{t}\right)$ ✓

$\therefore$ Locus

$t = 1/y$ sub in $x = \frac{4t+1}{2}$

a) i) Let $y = \frac{1}{\alpha^2}$

$\therefore \alpha = \frac{1}{\sqrt{y}}$. ✓

α is a root of $x^3 + 5x - t = 0$

$\therefore \left(\frac{1}{\sqrt{y}}\right)^3 + 5\left(\frac{1}{\sqrt{y}}\right) - t = 0$

$\frac{1}{y\sqrt{y}} + \frac{5}{\sqrt{y}} - t = 0$

$1 + 5y - ty\sqrt{y} = 0$

$1 + 5y = ty\sqrt{y}$ ✓

s.b.s

$1 + 10y + 25y^2 = t^2y^3$

$\therefore t^2y^3 - 25y^2 - 10y^2 - 1 = 0$ ✓

ii) $\sum \frac{1}{\alpha^2} = \frac{25}{t^2}$ ✓

b) i) $\frac{a}{x^2 - a^2} \equiv \frac{A}{x-a} + \frac{B}{x+a}$.

$A = 1/2$, $B = -1/2$ ✓

ii) $\int \frac{\sqrt{x}}{x-1}\,dx$

$x = u^2$

$dx = 2u\,du$

$\int \frac{u}{u^2-1} \cdot 2u\,du$ ✓

$\int \frac{2u^2\,du}{u^2-1}$

$2\int \frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}$ ✓

$= 2\int 1 + \frac{1/2}{u-1} - \frac{1/2}{u+1}$

$= 2\left[u + \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1)\right]$ ✓

$= 2\sqrt{x} + \ln(\sqrt{x}-1) - \ln(\sqrt{x}+1) + c$ ✓

c) i) $y = \frac{4}{3}x$, $y = -\frac{4}{3}x$ ✓

ii) $\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{16x}{9y}$

$= \frac{48\sec\theta}{36\tan\theta}$

$= \frac{4\sec\theta}{3\tan\theta}$ $\left(\frac{4}{3\sin\theta}\right)$ ✓

iii) $y - 4\tan\theta = \frac{4}{3\sin\theta}(x - 3\sec\theta)$ ✓

$3\sin\theta\, y - 12\sin\theta\tan\theta = 4x - 12\sec\theta$

$4x = 3\sin\theta\, y - 12[\sin\theta\tan\theta - \sec\theta]$

$4x = 3\sin\theta\, y - 12\left[\frac{\sin^2\theta - 1}{\cos\theta}\right]$ ✓

$4x = 3\sin\theta\, y + 12\cos\theta$

iv) Now $4x = -3y$.

$\therefore -3y = 3\sin\theta\, y + 12\cos\theta$

$(3\sin\theta + 3)y = -12\cos\theta$

$y = \frac{-12\cos\theta}{3\sin\theta + 3}$ ✓

$\therefore x = -\frac{3}{4}\left(\frac{-12\cos\theta}{3\sin\theta + 3}\right)$

$\therefore x = \frac{3\cos\theta}{\sin\theta + 3}$ ✓

v) $\frac{3\sec\theta}{2} = \frac{3\cos\theta}{1+\sin\theta}$ ✓

$\frac{3}{2\cos\theta} = \frac{3\cos\theta}{1+\sin\theta}$

$3+3\sin\theta = 6\cos^2\theta$

$3+3\sin\theta = 6-6\sin^2\theta$

$6\sin^2\theta + 3\sin\theta - 3 = 0$

$3(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = 1/2$ or -1

$\sin\theta \neq -1 \quad \left(\frac{3\cos\theta}{1+\sin\theta}\right)$

$\therefore \sin\theta = 1/2$ ✓

$\theta = \pi/6$

Question 3

a) $\int \tan^3 x\, dx = \int (\sec^2 x - 1)\tan x\, dx$ ✓

$= \int \sec^2 x \tan x\, dx - \int \tan x\, dx$ ✓

$= \frac{\tan^2 x}{2} + \ln(\cos x) + c$

b) $\int_0^{\pi/2} \sqrt{2-2\cos 2x}\, dx$

$= \sqrt{2}\int_0^{\pi/2} \sqrt{1-(1-2\sin^2 x)}\, dx$ ✓

$= \sqrt{2}\sqrt{2}\int_0^{\pi/2} \sin x\, dx$

$= -2[\cos x]_0^{\pi/2}$

$= -2[0-1]$

$= 2.$ ✓

c) i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ✓ ✓

ii) Let $x = \cos\theta$

$\therefore 4x^3 - 3x = \frac{\sqrt{3}}{2}$ ✓

if $8x^3 - 6x - \sqrt{3} = 0$

$\therefore \cos 3\theta = \frac{\sqrt{3}}{2}$ ✓

$3\theta = \pi/6, 11\pi/6, 13\pi/6 \ldots$

$\theta = \pi/18, 11\pi/18, 13\pi/18$ ✓

$\therefore$ roots $\cos \pi/18$, $\cos 11\pi/18$, $\cos 13\pi/18$

d) i) $x \to \infty \quad x^5 \to \infty$

$x \to -\infty \quad x^5 \to -\infty$ ✓

$\therefore$ at least 1 real root

ii) $P(x) = x^5 - 5cx + 1$

$P'(x) = 5x^4 - 5c$ ✓

if $c < 0$

$P'(x) = 5x^4 + 5k \quad k > 0$

For stationary pt.

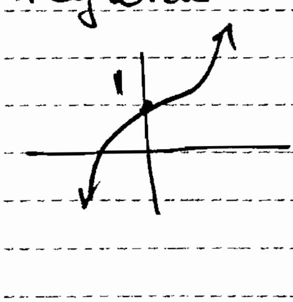
$P'(x) = 0$

$\therefore 5x^4 + 5k = 0$ ✓

which has no real roots

$\therefore P(x)$ has only one real root.

iii) $P(0) = 1 \quad \therefore$ root is negative ✓



iv) For 3 distinct roots

∴ 2 stationary points ✓

∴ $P'(x) = 5x^4 - 5c$

$= 5(x^4 - c)$

$= 5(x^2 - c^{1/2})(x^2 + c^{1/2})$

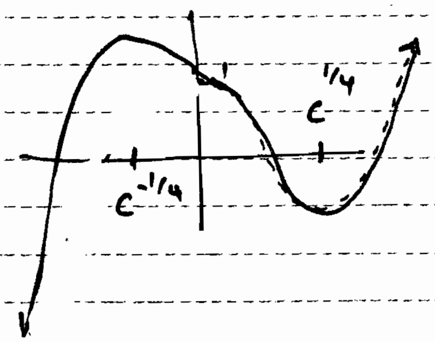
$= 5(x^2 + c^{1/2})(x - c^{1/4})(x + c^{1/4})$

$P'(x) = 0$ ∴

∴ 2 stationary pt when

$x = c^{1/4}$ or $-c^{1/4}$ ✓

Now when $x = c^{1/4}$ $P(x) < 0$



∴ $(c^{1/4})^5 - 5c \cdot c^{1/4} + 1 < 0$

$c^{5/4} - 5c^{5/4} + 1 < 0$

$-4c^{5/4} < -1$

$c^{5/4} > 1/4$ ✓

$c > (1/4)^{4/5}$