SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION II

HSC Assessment TASK II

JUNE 2004

Time allowed: 70 minutes

Instructions:

- Show all necessary working in every question.
- Start each question on a new page.
- Attempt all questions.
- All questions are of equal value.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- This test forms part of your HSC assessment.
- These questions are to be handed in with your answers.
- Standard integrals are attached at the back of this paper.

Name: ______________________________

Question 1	Question 2	Question 3	Total

Question 1

a) Find $\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$ **2**

b) $(x+1)^2$ is a factor of $P(x) = x^5 + 2x^3 + ax + b$. **3**

Find a and b.

(c) The real roots of $P(x) = x^3 + 4x - m$ are $\alpha,\ \beta,\ \gamma$. **5**

i) Find the value of $\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\beta^2\alpha\gamma} + \frac{1}{\gamma^2\alpha\beta}$

ii) Explain why $\frac{1}{\alpha^2\beta\gamma} = \frac{1}{m\alpha}$

iii) Hence or otherwise form a cubic polynomial whose roots are

$\frac{1}{\alpha^2\beta\gamma},\ \frac{1}{\beta^2\alpha\gamma}$ and $\frac{1}{\gamma^2\alpha\beta}$.

(d) Evaluate $\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sin^2 x}\, dx$ **3**

(e) i) The point $P(4\sec\vartheta,\ 5\tan\vartheta)$ lies on a hyperbola. Find the equation of the hyperbola. **4**

ii) Find an expression for the gradient of the tangent at the point P as a single trigonometric ratio.

Question 2

a) Evaluate $\int_0^2 \sqrt{4-x^2}\, dx$ **1**

b) i) Find real numbers a and b such that **4**

$$\frac{2x^2+3x+5}{(x-1)(x^2+4)} = \frac{a}{x-1} + \frac{b}{x^2+4}$$

ii) Hence find $\int \frac{2x^2+3x+5}{(x-1)(x^2+4)}\, dx$

c) The point $Q\ (ct, \frac{c}{t})$ in the first quadrant lies on the hyperbola $xy = c^2$. **6**

i) What are the possible values of the parameter t ?

ii) The line $y = \frac{c}{t}$ cuts the major axis of the hyperbola at T. Find the co-ordinates of M, the mid point of QT.

iii) Find the locus of M giving any restrictions.

d) i) Find the general solution of $\cos 5\vartheta = 1$. **6**

ii) Using the result $\cos 5\vartheta = 16\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta$,
solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$.

ii) Hence show without using direct calculation that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}.$$

Question 3

a) $P(x)$ is an even polynomial with real co-efficients having the following properties: **2**

α) it has a complex zero

β) $P(1) = 2$ *and* $P(2) = -1$.

State the least degree of *P(x)* giving reasons.

b) By using integration by parts or otherwise find $\int \cos^{-1} x \ dx$ **3**

c) Find $\int \frac{du}{\sqrt{u}+1}$ **3**

d) i) Show that the equation of the normal to the hyperbola $xy = 4$ **5**

at $(2t, \frac{2}{t})$ *is* $t^3 x - ty + 2(1 - t^4) = 0$.

ii) Find all the possible values of t if the normal passes through the point (16, ¼).

e) The polynomial $P(x)$ is given by $P(x) = x^5 - 5cx + 1$ where c is a real number. **4**

By considering the turning points of P(x), or otherwise, show that P(x) has three distinct real roots if $c > (\frac{1}{4})^{4/5}$

End of Exam

1 a) $\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{x^2-2x+1+4}}$

$= \int \frac{dx}{\sqrt{(x-1)^2+4}}$

$= \ln\left\{(x-1)+\sqrt{x^2-2x+5}\right\} + c$

b) $P(x) = x^5 + 2x^3 + ax + b$

$P(-1) = -1 \; 2 - a + b = 0$ ①

$P'(x) = 5x^4 + 6x^2 + a$

$P'(-1) = 0$ root (multiplicity 2)

$\therefore \; 5 + 6 + a = 0$ ②

From ② $a = -11$

From ① $b = -8$

c) i) $\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\beta^2\alpha\gamma} + \frac{1}{\gamma^2\alpha\beta} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha^2\beta^2\gamma^2}$

$= \frac{4}{m^2}$

ii) $\frac{1}{\alpha^2\beta\gamma} = \frac{1}{\alpha(\alpha\beta\gamma)}$

$= \frac{1}{\alpha m}$ since $\alpha\beta\gamma = m$

iii) Let $x = \frac{1}{\alpha m}$

$\therefore \alpha = \frac{1}{mx}$

since α is a root of $P(x) = x^3 + 4x - m$

then $P(x) = \left(\frac{1}{mx}\right)^3 + \frac{4}{mx} - m$

$= 1 + 4m^2x^2 - m^4x^3$

$\therefore P(x) = 1 + 4m^2x^2 - m^4x^3$

d) $\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sin^2 x}\,dx$

Let $u = \sin x$ $\quad x = \pi/2 \quad u = 1$

$du = \cos x\,dx$ $\quad x = \pi/6 \quad u = 1/2$

$= \int_{1/2}^{1} \frac{1-u^2}{u^2}\,du$

$= \int_{1/2}^{1} \frac{1}{u^2} - 1\,du$

$= \left[-u^{-1} + u\right]_{1/2}^{1}$

$= -2 - [-2 + 1/2]$

$= 0.5$

e) i) $x = 4\sec\theta$, $y = 5\tan\theta$

$\therefore \frac{x^2}{16} - \frac{y^2}{25} = 1$

ii) $x = 4\sec\theta$ $\qquad y = 5\tan\theta$

$\frac{dx}{d\theta} = 4\sec\theta\tan\theta$ $\qquad \frac{dy}{d\theta} = 5\sec^2\theta$

$\therefore \frac{dy}{dx} = \frac{5\sec^2\theta}{4\sec\theta\tan\theta}$

$= \frac{5\sec\theta}{4\tan\theta}$

$= \frac{5\operatorname{cosec}\theta}{4}$

3 i) complex roots exist in conjugate pairs

ii) There is at least one zero between 1 and 2

$\therefore$ zero between -1 and -2 because $P(x)$ is even

b) $\int \cos^{-1} dx = \int \cos^{-1}x \, \frac{d(x)}{dx} \, dx$

$= \cos^{-1}x \cdot x - \int x \cdot \frac{-1}{\sqrt{1-x^2}} \, dx$

$= x\cos^{-1}x - \sqrt{1-x^2} + c$

c) $\int \frac{du}{\sqrt{u}+1}$

let $u = x^2$

$\therefore du = 2x\,dx$

$\int \frac{2x\,dx}{x+1} = \int \frac{2(x+1)-2}{x+1} \, dx$

$= \int 2 - \frac{2}{x+1} \, dx$

$= 2x - 2\ln(x+1) + c$

$= 2\sqrt{u} - 2\ln(\sqrt{u}+1) + C$

d) $xy = 4 \Rightarrow y = \frac{4}{x}$

$\frac{dy}{dx} = \frac{-4}{x^2}$

at $(2t, \frac{2}{t})$

$= \frac{-4}{4t^2}$

$= \frac{-1}{t^2}$

$m_\perp = t^2$

Eqn normal. $y - \frac{2}{t} = t^2(x - 2t)$

$ty - 2 = t^3(x - 2t)$

$t^3x - ty + 2(1 - t^4) = 0$

$16t^3 -$ [illegible] $t + 2(1 - t^4) = 0$

$64t^3 - t + 8(1 - t^4) = 0$

$64t^3 - 8t^4 + 8 - t = 0$

$8t^3(8 - t) + 1(8 - t) = 0$

$(8 - t)(8t^3 + 1) = 0$

$\therefore t = 8$ or $-1/2$

e) $P(x) = x^5 - 5cx + 1$

$P'(x) = 5x^4 - 5c$

For stationary pt $P'(x) = 0$

$\therefore 5x^4 - 5c = 0$

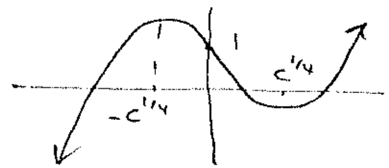
$5(x^4 - c) = 0$

$5(x^2 - c^{1/2})(x^2 + c^{1/2}) = 0$

$5(x - c^{1/4})(x + c^{1/4})(x^2 + c^{1/2}) = 0$

$\therefore x = c^{1/4}$ or $-c^{1/4}$

$\therefore$ Stationary pt at $(c^{1/4}, 1 - 4c^{5/4})$ $(-c^{1/4}, 1 + 4c^{5/4})$



For 3 root $1 - 4c^{5/4} < 0$

$-4c^{5/4} < -1$

$c^{5/4} > 1/4$

$\therefore c^{5/4} < -1/4$ or $c^{5/4} > 1/4$

For $c^{5/4} < -1/4$ maximum is below x-axis

$\therefore$ does not apply

For $c^{5/4} > 1/4$

$c > (1/4)^{4/5}$.

2. a) $\int_0 \sqrt{4-x^2}\,dx$

$\therefore \int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4} \times \pi \times 2^2$

$= \pi$

b) i) $\frac{2x^2+3x+5}{(x-1)(x^2+4)} = \frac{a(x^2+4)+b(x-1)}{(x-1)(x^2+4)}$

$= \frac{ax^2+bx+4a-b}{(x-1)(x^2+4)}$

equating co-efficients

$a = 2$

$b = 3$

ii) $\int \frac{2x^2+3x+5}{(x-1)(x^2+4)} = \int \frac{2}{x-1}\,dx + \int \frac{3}{x^2+4}\,dx$

$= 2\ln(x-1) + \frac{3}{2}\tan^{-1}\frac{x}{2} + c$

c) i) $t > 0$

ii) when $y = x$ $\therefore T\left(\frac{c}{t}, \frac{c}{t}\right)$

$\therefore M = \left(\frac{ct + \frac{c}{t}}{2}, \frac{\frac{c}{t}+\frac{c}{t}}{2}\right)$

$= \left(\frac{ct^2+c}{2t}, \frac{c}{t}\right)$

iii) Now $y = \frac{c}{t} \rightarrow t = \frac{c}{y}$

$x = \frac{c\left(\frac{c^2}{y^2}\right)+c}{\frac{2c}{y}}$

$= \frac{c^3+cy^2}{y^2} \times \frac{y}{2c}$

$= \frac{c^2+y^2}{2y}$

$\therefore 2xy - y^2 = c^2$ for $x, y > 0$

$\therefore 16x^5 - 20x^3 + 5x = 1$

Solving $\cos 5\theta = 1$

$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi \ldots$

$\theta = 0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5$

$\therefore 16x^5 - 20x^3 + 5x - 1 = 0$ has solution

$\cos 0, \cos 2\pi/5, \cos 4\pi/5, \cos 6\pi/5, \cos 8\pi/5$

ii) Now

$\cos 0 + \cos 2\pi/5 + \cos 4\pi/5 + \cos 6\pi/5 + \cos 8\pi/5 = 0$

but $\cos 0 = 1$

and $\cos 8\pi/5 = \cos 2\pi/5$

$\cos 6\pi/5 = \cos 4\pi/5$

$\therefore 1 + 2\cos 2\pi/5 + 2\cos 4\pi/5 = 0$

$2(\cos 2\pi/5 + \cos 4\pi/5) = -1$

$\therefore \cos 2\pi/5 + \cos 4\pi/5 = -1/2$