Name: ______________________ Class: ______________

SYDNEY TECHNICAL HIGH SCHOOL

Year 12 Mathematics
Extension 2
HSC Assessment Task 2
June 2005

Instructions:

- Begin each question on a new page.
- Write only on the front of each page. Single column only.
- Staple these questions to the front of your answers.
- Full marks may not be awarded for careless or incomplete work.
- Marks indicated in each question are a guide only and may change slightly during the marking process.

Time allowed: ***70 mins***

Q1	Q2	Q3	TOTAL

Question 1.

a) Factorise x^3+x^2-x+15 over the complex field given that $1-2i$ is a zero. 3

b) Find i) $\int \frac{e^{\tan x}}{\cos^2 x}dx$ 1

ii) $\int \sec^6 x\ dx$ 3

iii) $\int \frac{x-2}{\sqrt{x^2-9}}dx$. 3

c) $P(2p,\frac{2}{p})$ and $Q(2q,\frac{2}{q})$ where $p,\ q>0$ are two distinct points on the hyperbola $xy=4$. 7

i) Show that the equation of the tangent to $xy=4$ at $P(2p,\frac{2}{p})$ is

$$x+p^2y-4p=0.$$

ii) If the tangents at $P(2p,\frac{2}{p})$ and $Q(2q,\frac{2}{q})$ intersect at T, find the coordinates of T.

iii) If the tangent at Q passes through the point $(2p,\ 0)$ show that $p=2q$.

iv) Find the equation of the locus of T.

Question 2.

a) Find $\int \sin^2 2x \cos x \, dx$ 4

b) Find $\int \frac{\ln x}{x^2} dx$ using integration by parts. 3

c) Find $\int_0^{1/2} \sqrt{1-x^2} \, dx$ using a suitable substitution 3

d) The cubic equation $x^3 + kx + 1 = 0$, where k is a constant, 7

has roots p, q *and* r.

For each non negative integer n, $S_n = p^n + q^n + r^n$.

i) State the value of S_1.

ii) Express S_2 in terms of k.

iii) Show that for all n, $S_{n+3} + k S_{n+1} + S_n = 0$.

iv) Hence, or otherwise, express $p^5 + q^5 + r^5$ in terms of k.

Question 3.

a) Evaluate $\int_{4}^{9} \frac{\sqrt{x}}{x-1} dx$ using the substitution $u^2 = x$ 5

b) If α, β and δ are the roots of $x^3 + 2x - 1 = 0$, form the equation whose roots are 5

i) α, $-\alpha$, β, $-\beta$, δ and $-\delta$

ii) $\beta + \delta - 2\alpha$, $\delta + \alpha - 2\beta$ and $\alpha + \beta - 2\delta$.

c) A polynomial $P(x)$ is given by $P(x) = x^5 - 1$. 6

Let α ($\alpha \neq 1$) be that complex root of $P(x)$ which has the smallest positive argument.

i) Show that $P(x) = (x-1)(1 + x + x^2 + x^3 + x^4)$

ii) Show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

iii) If $p = \alpha + \alpha^4$ and $q = \alpha^2 + \alpha^3$ find the values of $p + q$ and pq.

iv) Show that $p = \frac{-1+\sqrt{5}}{2}$ and $q = \frac{-1-\sqrt{5}}{2}$. Justify your answer.

End of paper.

Question 1

a) (3)

$$x^3+x^2-x+15$$
$$=(x-(1-2i))(x-(1+2i))(x-?)$$
$$=(x^2-2x+5)(x+3)$$
$$\therefore =(x-(1-2i))(x-(1+2i))(x+3)$$

b) i) (1)

$$\int \frac{e^{\tan x}}{\cos^2 x}\,dx$$
$$=\int e^{\tan x}\cdot \sec^2 x\,dx$$
$$=e^{\tan x}+c$$

ii) (3)

$$\int \sec^6 x\,dx$$
$$=\int \sec^4 x \sec^2 x\,dx$$
$$=\int (1+\tan^2 x)^2\cdot \sec^2 x\,dx \quad (\text{put } u=\tan x)$$
$$=\int (1+2u^2+u^4)\,du$$
$$=u+\frac{2u^3}{3}+\frac{u^5}{5}+c$$
$$=\tan x+\frac{2\tan^3 x}{3}+\frac{\tan^5 x}{5}+c$$

iii) (3)

$$\int \frac{x-2}{\sqrt{x^2-9}}\,dx=\int \frac{x\,dx}{\sqrt{x^2-9}}-2\int\frac{dx}{\sqrt{x^2-9}}$$
$$=\sqrt{x^2-9}-2\ln\left(x+\sqrt{x^2-9}\right)+c$$

c) i) (7)

$$y=\frac{4}{x}\rightarrow y'=\frac{-4}{x^2}$$

when $x=2p$, $m=-\frac{1}{p^2}$

Eqn of Tangent at $\left(2p,\frac{2}{p}\right)$ is

$$y-\frac{2}{p}=-\frac{1}{p^2}(x-2p)$$

ie $yp^2-2p=-x+2p$

ie $x+yp^2-4p=0$

ii) $x+yp^2-4p=0$ tangent at P — (1)

$x+yq^2-4q=0$ — tangent at Q — (2)

$\therefore$ $y(p^2-q^2)=4(p-q)$

ie $y=\frac{4}{p+q}$

$\therefore$ $x=4p-p^2\left(\frac{4}{p+q}\right)$

$$=\frac{4pq}{p+q}$$

$\therefore$ T is $\left(\frac{4pq}{p+q},\frac{4}{p+q}\right)$ — (3)

iii) $(2p,0)$ lies on (2)

$\therefore$ $2p+0=4q$

$\therefore$ $p=2q$ — (4)

iv) From (3), $x=y\cdot pq$

but using (4), $pq=2q^2$ and

putting this in (3), $y=\frac{4}{3q}\Rightarrow q=\frac{4}{3y}$

$$\therefore x=y\cdot 2\left(\frac{4}{3y}\right)^2$$
$$=\frac{32}{9y}$$

ie $xy=\frac{32}{9}$

Question 2

a) $\int \sin^2 2x \cdot \cos x \, dx$ (4)

$= \int (2\sin x \cos x)^2 \cdot \cos x \, dx$

$= 4\int \sin^2 x \cos^2 x \cdot \cos x \, dx$

$= 4\int \sin^2 x (1-\sin^2 x) \cdot \cos x \, dx$

put $u = \sin x$

$= 4\int u^2(1-u^2)\, du$

$= 4\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c$

$= \frac{4}{3}\sin^3 x - \frac{4}{5}\sin^5 x + c$

b) $\int \frac{\ln x}{x^2}\, dx \Rightarrow$ (3)

| $u = \ln x$
| $v = -\frac{1}{x}$
| $du = \frac{dx}{x}$
| $dv = \frac{1}{x^2}dx$

$= \ln x \cdot -\frac{1}{x} - \int -\frac{1}{x^2} \cdot dx$

$= -\frac{\ln x}{x} - \frac{1}{x} + c$

c) $\int_0^{1/2} \sqrt{1-x^2}\, dx \qquad x = \sin\theta$ (3)

$= \int_0^{\pi/6} \cos\theta \cdot \cos\theta \, d\theta$

$= \int_0^{\pi/6} \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/6}$

$= \frac{\pi}{2} + \frac{\sqrt{3}}{8}$

d) i) $S_1 = p+q+r = 0$ (7)

ii) $S_2 = p^2+q^2+r^2$

$= (p+q+r)^2 - 2(pq+pr+qr)$

$= 0 - 2(k)$

$= -2k$

iii) From $x^3 + kx + 1 = 0$ we obtain $x^{n+3} + kx^{n+1} + x^n = 0$.

Now $p^{n+3} + kp^{n+1} + p^n = 0$,

$q^{n+3} + kq^{n+1} + q^n = 0$

and $r^{n+3} + kr^{n+1} + r^n = 0$.

Summing: $S_{n+3} + kS_{n+1} + S_n = 0$ as req^d.

(iv) When $n = 0$, $S_0 = 1+1+1 = 3$

and $S_3 + kS_1 + S_0 = 0$

ie $S_3 = -S_0 - kS_1$

$= -$

When $n = 2$

$S_5 + kS_3 + S_2 = 0$

ie $S_5 = -kS_3 - S_2$

$= -k \cdot -3 + 2k$

$= 5k$

$\therefore p^5 + q^5 + r^5 = 5k$

Question 3

a) $\displaystyle\int_4^9 \frac{\sqrt{x}}{x-1}\,dx$ (5) $\qquad u^2 = x,\ u = \sqrt{x}$, $2u\,du = dx$

$$= \int_2^3 \frac{2u^2\,du}{u^2-1}$$

$$= 2\int_2^3 \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right) du$$

$$= 2\int_2^3 \left(1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\right) du$$

$$= \Big[2u + \ln(u-1) - \ln(u+1)\Big]_2^3$$

$$= \left(6 + \ln\tfrac{1}{2}\right) - \left(4 + \ln\tfrac{1}{3}\right)$$

$$= 2 + \ln\tfrac{3}{2}$$

(2)

b) i) $x^3 + 2x - 1 = 0$ has roots α, β, δ

$(-x)^3 + 2(-x) - 1 = 0$ has roots $-\alpha, -\beta, -\delta$

ie $-x^3 - 2x - 1 = 0$

$(x^3 + 2x - 1)(-x^3 - 2x - 1) = 0$ has roots $\alpha, -\alpha, \beta, -\beta, \delta, -\delta$.

ii) Since $\alpha + \beta + \delta = 0$ (3)

$\alpha + \beta = -\delta$

So $\alpha + \beta - 2\delta = -3\delta$.

Similarly $\beta + \delta - 2\alpha = -3\alpha$

and $\alpha + \delta - 2\beta = -3\beta$ and

the equation with these roots is

$$\left(-\frac{x}{3}\right)^3 + 2\left(-\frac{x}{3}\right) - 1 = 0$$

c) i) $(x-1)(1 + x + x^2 + x^3 + x^4)$ (1)

$= x + x^2 + x^3 + x^4 + x^5$
$\quad -1 - x - x^2 - x^3 - x^4$

$= x^5 - 1 = P(x)$.

(ii) $P(\alpha) = 0$ and $\alpha \neq 1$ (1)

$\therefore (\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 0$

Since $\alpha \neq 1$, $\alpha - 1 \neq 0$

$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$ —— (1)

(iii) $p + q = \alpha + \alpha^4 + \alpha^2 + \alpha^3$ (1)

$= -1$ from (1)

and $pq = (\alpha + \alpha^4)(\alpha^2 + \alpha^3)$

$= \alpha^3 + \alpha^4 + \alpha^6 + \alpha^7$

But $\alpha^6 = \alpha$ and $\alpha^7 = \alpha^2$

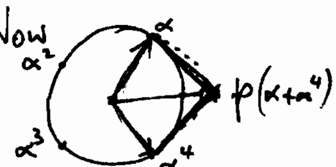
$\therefore pq = -1$ (1)

(iv) p and q are the roots of a quadratic equation in which sum of roots $= -1$ and product of roots $= -1$

ie $x^2 + x - 1 = 0$

$\therefore x = \dfrac{-1 \pm \sqrt{1+4}}{2}$ (1)

$= \dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}$

Now 

p is the greater of the two

$\therefore p = \dfrac{-1+\sqrt{5}}{2}$ (1)

and $q = \dfrac{-1-\sqrt{5}}{2}$.