Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

Mathematics Extension 2

HSC TASK 2

JUNE 2007

TIME ALLOWED: 70 minutes

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.

(FOR MARKERS USE ONLY)

Q1	Q2	Q3	TOTAL
/20	/20	/20	/60

QUESTION 1: (20 MARKS)

Marks

3 (a) Find $\int \frac{2x}{\sqrt{x+1}}dx$

2 (b) Find $\int \frac{dx}{\sqrt{4x-x^2}}$

3 (c) Find $\int x^2 e^x dx$

4 (d) Evaluate $\int_0^{\pi} \sin^3 x dx$

3 (e) Find $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}d\theta$

3 (f) (i) Find values of A, B and C for which

$$A(x+1)^2 + B(x-1) + C(x+1)(x-1) = 8x-4$$

and hence, or otherwise, express $\frac{8x-4}{(x-1)(x+1)^2}$ in the form

$$\frac{A}{x-1} + \frac{B}{(x+1)^2} + \frac{C}{x+1}$$

2 (ii) Using your solution to (i) above, find

$$\int \frac{8x-4}{(x-1)(x+1)^2}dx$$

QUESTION 2: (20 MARKS)

Marks

(a) For the hyperbola $\frac{x^2}{2} - \frac{y^2}{2} = 1$ find

2 (i) The foci

2 (ii) The equations of the directrices

1 (iii) The equations of the asymptotes

1 (iv) The length of the major axis

(b) The point $P(3t, \frac{3}{t})$ lies on the hyperbola $xy = 9$

3 (i) Prove that P is the midpoint of the line MN where M and N are the points where the tangent to the hyperbola at P cuts the x and y axes respectively.

2 (ii) Show that the midpoint of PM lies on another hyperbola and give its equation

4 (c) (i) **Derive** the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P *(asecθ, btanθ)*

Give your answer in the form Ax+By=1

3 (ii) The tangents at the points P *(asecθ, btanθ)* and Q *(asecα, btanα)* meet at right angles.

Prove that $\sin\theta\sin\alpha = -\frac{b^2}{a^2}$

2 (d) P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose foci are S and S' .

Use the focus-directrix definition of the hyperbola to prove that $|PS - PS'|$ is a constant

QUESTION 3: (20 MARKS)

Marks

3 (a) Find the values of a and b so that 2 is a double root of the polynomial

$$x^4 + ax^3 - 3x^2 - bx + 4 = 0$$

2 (b) (i) Find (in expanded form) the equation whose roots exceed by 1 the roots of

$$x^3 + 6x^2 - 3x + 1 = 0$$

4 (ii) If α, β and γ are the roots of the polynomial $x^3 - 2x^2 + 3x - 4$ prove that $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = -7$.

Show that the polynomial whose roots are $\beta^2\gamma^2, \gamma^2\alpha^2$ and $\alpha^2\beta^2$ is $x^3 + 7x^2 - 32x - 256$

4 (c) Solve the quartic equation $x^4 + 2x^3 + x^2 - 1 = 0$ given that one root is $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$

3 (d) (i) Using deMoivre's Theorem, or otherwise, obtain an expression for cos4θ in terms of cosθ
(**NOTE:** $(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$)

4 (ii) By considering the roots of the equation $16x^4 - 16x^2 + 1 = 0$ and using the substitution x = cosθ and your answer to part (i) above,

show that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$

1 PAPER

$$P(x) = \left(x + \tfrac{1}{2} - \tfrac{i\sqrt{3}}{2}\right)\left(x + \tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}\right) Q(x)$$

$$= \left[\left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right] Q(x)$$

$$= (x^2 + x + 1)\, Q(x) \quad \leftarrow 1$$

$$\begin{array}{r|l} & x^2 + x - 1 \quad \leftarrow 1 \\ x^2 + x + 1 & x^4 + 2x^3 + x^2 \qquad - 1 \\ & \underline{x^4 + x^3 + x^2} \\ & x^3 \qquad\qquad - 1 \\ & \underline{x^3 + x^2 + x} \\ & -x^2 - x - 1 \\ & \underline{-x^2 - x - 1} \\ & \ldots \end{array}$$

Now $x^2 + x - 1 \Rightarrow x = -\frac{1 \pm \sqrt{5}}{2}$

$$= -\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2} \quad \leftarrow 1$$

$\therefore$ Roots are $-\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2}$

OR

Sum of Roots is $-1 + \alpha + \beta = -2$

$\therefore \quad \alpha + \beta = -1 \quad (1) \quad$ 1

Product is $\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\alpha\beta = -1$

$\therefore \left(\frac{1}{4} + \frac{3}{4}\right)\alpha\beta = -1$

$\alpha\beta = -1 \quad (2) \quad$ 1

Into (1) $\quad \alpha - \frac{1}{\alpha} + 1 = 0$

$$\alpha^2 - 1 + \alpha = 0$$

$$\alpha = -\frac{1 \pm \sqrt{5}}{2}$$

$$= -\tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2} \quad \leftarrow 1$$

$\therefore$ Roots are $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

d) $\displaystyle\int_0^{\pi} \sin^3 x\, dx = \int_0^{\pi} \sin^2 x \cdot \sin x\, dx$

$$= \int_0^{\pi} 1 - \cos^2 x \cdot \sin x\, dx$$

Let $u = \cos x \qquad x = \pi \quad u = -1$

$du = -\sin x \qquad x = 0 \quad u = 1$

$$= \int_1^{-1} u^2 - 1\, du$$

$$= 2\int_0^1 1 - u^2\, du$$

$$= 2\left[u - \frac{u^3}{3}\right]_0^1$$

$$= 2\left[\tfrac{2}{3}\right]$$

$$= \tfrac{4}{3}$$

e) $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sin\theta}\, d\theta$. Let $t = \tan\theta/2$

$\tan^{-1} t = \theta/2$

$\theta = 2\tan^{-1} t$

$d\theta = \frac{2}{1+t^2}\, dt$

$\sin\theta = \frac{2t}{1+t^2}$

$\theta = \pi/2 \quad t = 1$

$\theta = 0 \quad t = 0$

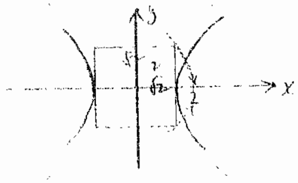
$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{1 + \frac{2t}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{t^2 + 2t + 1}$$

$$= 2\int_0^1 \frac{dt}{(t+1)^2}$$

QUESTION 2

a)



i) foci $(\pm 2, 0)$

ii) $ae = 2$

$\therefore e = \sqrt{2}$

directrices at $x = \pm \frac{a}{e}$ ie. $\frac{\sqrt{2}}{\sqrt{2}}$

$\therefore x = \pm 1$

iii) $y = \pm x$

iv) $2\sqrt{2}$

b) i) Eqn of tangent at P :

$x + t^2 y = 6t$

At M $(y = 0)$ $x = 6t$

At N $(x = 0)$ $y = \frac{6}{t}$

$\therefore$ Mid point is $\left(\frac{6t+0}{2}, \frac{\frac{6}{t}+0}{2}\right)$

ie. $(3t, \frac{3}{t})$ which is point P.

ii) $P(3t, \frac{3}{t})$, $M(6t, 0)$

Mid point of PM is $(\frac{9t}{2}, \frac{3}{2t})$

ie. If $X = \frac{9t}{2}$ & $Y = \frac{3}{2t}$ then $XY = \frac{27}{4}$

which is the equation of a hyperbola.

c) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating :

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

ie. $\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ and at P

$m = \frac{b\sec\theta}{a\tan\theta}$

So the equation of the tangent at P is:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

ie. $\frac{y\tan\theta}{b} - \tan^2\theta = \frac{x\sec\theta}{a} - \sec^2\theta$

ie. $\sec^2\theta - \tan^2\theta = \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b}$

ie. $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

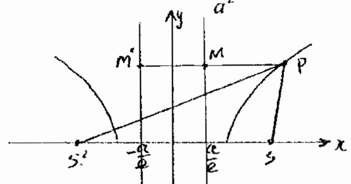
ii) Since tangents are perpendicular:

$\frac{b\sec\theta}{a\tan\theta} \times \frac{b\sec\alpha}{a\tan\alpha} = -1$

ie. $\frac{b^2}{a^2}\frac{1}{\cos\theta}\frac{1}{\cos\alpha} = -\frac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}$

ie. $\sin\theta\sin\alpha = -\frac{b^2}{a^2}$

d)



$|PS - PS'| = |ePM - ePM'|$

$= e|PM - PM'|$

$= e \times \frac{2a}{e}$

$= 2a$ which is constant

3 (a) $P(x) = x^4 + ax^3 - 3x^2 - bx + 4 = 0$

$P'(x) = 4x^3 + 3ax^2 - 6x - b$

$P(2) = 16 + 8a - 12 - 2b + 4 = 0$ (1)

$\Rightarrow 8a - 2b + 8 = 0$

$4a - b + 4 = 0$ (1)

$P'(2) = 32 + 12a - 12 - b = 0$ (1)

$\therefore 12a - b + 20 = 0$ (2)

(2)−(1) $8a + 16 = 0$

$a = -2$

$b = -4$ (1)

(b) (i) $P(x-1) = (x-1)^3 + 6(x-1)^2 - 3(x-1) + 1$ (1)

$\therefore Q(x) = x^3 - 3x^2 + 3x - 1 + 6x^2 - 12x + 6 - 3x + 3 + 1$

$Q(x) = x^3 + 3x^2 - 12x + 9$ (1)

(ii) $(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma + 2$

$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= (3)^2 - 2(4)(2)$

$= -7$

Sum of Roots is $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 = -7$ (1)

Sum of Roots × 2 is $\beta^2\gamma^4\alpha^2 + \alpha^2\beta^4\gamma^2 + \alpha^4\beta^2\gamma^2$

$= \beta^2\gamma^2\alpha^2(\gamma^2 + \beta^2 + \alpha^2)$

$= (4)^2\left[(\gamma + \beta + \alpha)^2 - 2(\alpha\beta + \alpha\right.$

$= 16\left[(2)^2 - 2(3)\right]$

$= 16[-2]$ (1)

$= -32$

Product of Roots is $\alpha^4\beta^4\gamma^4$

$= (\alpha\beta\gamma)^4$

$= (4)^4$

$= 256$ (1)

$\therefore$ Poly is $x^3 + 7x^2 - 32x - 256$

(c) One root is $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (1)

another root is $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Question 1

a) $\int \frac{2x}{\sqrt{x+1}}\,dx$

let $u^2 = x+1$

$\therefore 2u\,du = dx$

$= \int \frac{2(u^2-1)}{u} 2u\,du$

$= 4\int u^2 - 1\,du$

$= 4\left[\frac{u^3}{3} - u\right]$

$= 4\left[\frac{(x+1)^{3/2}}{3} - (x+1)^{1/2}\right] + c$

b) $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x^2-4x+4)}}$

$= \int \frac{dx}{\sqrt{4-(x-2)^2}}$

$= \sin^{-1}\left(\frac{x-2}{2}\right) + c$

c) $\int x^2 e^x\,dx = \int x^2 \frac{d(e^x)}{dx}\,dx$

$= x^2e^x - \int 2xe^x$

$= x^2e^x - 2\int x\frac{d(e^x)}{dx}\,dx$

$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2e^x + c$

$$= -2\left[(t+1)^{-1}\right]_0^1$$
$$= -2\left[2^{-1} - 1^{-1}\right]$$
$$= -2\left[1/2 - 1\right]$$
$$= 1$$

f) i) $A(x+1)^2 + B(x-1) + C(x+1)(x-1) = 8x-4$

$x=1$ $\quad 4A = 4$

$\quad A = 1$

$x=-1$ $\quad -2B = -12$

$\quad B = 6$

Comparing co-efficient of x^2

$\quad 1 + C = 0$

$\quad C = -1$

$$\therefore \frac{8x-4}{(x-1)(x+1)^2} = \frac{1}{x-1} + \frac{6}{(x+1)^2} - \frac{1}{x+1}$$

ii) $$\int \frac{8x-4}{(x-1)(x+1)^2}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{6\,dx}{(x+1)^2} - \int \frac{1}{x+1}\,dx$$
$$= \ln(x-1) - \ln(x+1) - 6(x+1)^{-1} + c$$
$$= \ln\left[\frac{x-1}{x+1}\right] - \frac{6}{x+1} + c$$

(d) (i) $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\ldots$
$+ i^4\sin^4\theta$ (1)

(1)

$\therefore \cos 4\theta + i\sin 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$+ 4i(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

Comparing real parts,

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 + \cos^4\theta - 2\ldots$
$= 8\cos^4\theta - 8\cos^2\theta + 1$ (1)

(ii) Let $x = \cos\theta$

$\therefore 16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$\therefore 2(8\cos^4\theta - 8\cos^2\theta + 1) - 1 = 0$ (1)

$\therefore 2\cos 4\theta - 1 = 0$

$\cos 4\theta = 1/2$ (1)

$\therefore 4\theta = \pi/3,\ 5\pi/3,\ 7\pi/3$ or $11\pi/3$ ONLY 4 ...

$\theta = \pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12$ (1)

$\therefore x = \cos\theta = \cos\pi/12, \cos 5\pi/12, \cos 7\pi/12, \cos 11\pi/12$

PRODUCT OF ROOTS $= 1/16$. (1)

$\therefore \cos\pi/12 \cos 5\pi/12 \cos 7\pi/12 \cos 11\pi/12 = 1/16$