SYDNEY TECHNICAL HIGH SCHOOL

Mathematics Extension 2

HSC ASSESSMENT TASK
JUNE 2009

General Instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions

NAME :________________________

TEACHER :____________________

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL

Question 1 (17 marks) Marks

a) Find $\int \frac{15}{x^2+3x-4}\,dx$ 3

b) Evaluate $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ 3

c) Evaluate $\int_0^{\ln 2} \frac{e^{2x}}{e^x+1}\,dx$ 4

d) Two of the roots of the equation $x^3+ax^2+15x-7=0$ 3

are equal and rational. Find the value of a.

e) The equation $x^3-4x+5=0$ has roots α , β and δ.

Find the value of i) $\alpha^3+\beta^3+\delta^3$ 2

ii) $(\alpha+\beta)^2(\alpha+\delta)^2(\beta+\delta)^2$ 2

Question 2 (17 marks) Marks

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ using the substitution $t = \tan\frac{\theta}{2}$ 4

b) Find $\int \frac{2x}{x^2+4x+5}\,dx$ 4

c) If $I_n = \int_0^2 (x^3-8)^n dx$. where n is a positive integer, 4

show that $I_n = \frac{-24n}{3n+1} I_{n-1}$.

d) Let α be the complex root of $z^7 = 1$ with smallest positive argument.

i) Show that $1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6 = 0$ 2

ii) If $x^3+ax^2+bx+c=0$ is a cubic equation with roots 3

$\alpha+\alpha^6$, $\alpha^2+\alpha^5$ and $\alpha^3+\alpha^4$,

find the values of a , b and c .

Question 3 (17 marks) Marks

a) Evaluate $\displaystyle\int_2^4 \frac{dx}{x\sqrt{x-1}}$ 3

b) Find $\int \sin^4 x \cos^3 x \, dx$ 3

c) $P(x)$ is a cubic polynomial with real coefficients. 4

One zero of $P(x)$ is $1+2i$, the constant term is -15, and $P(2)=5$.

Find the equation of the polynomial $P(x)$.

d) The polynomial $P(x)=x^4+x^3-3x^2-5x-2$ 3

has a root of multiplicity 3.

Solve $P(x)=0$.

e) Let α , β , δ be the roots of the cubic equation $x^3+px^2+q=0$, 4

where p , q are real.

The equation $x^3+ax^2+bx+c=0$ has roots $\dfrac{1}{\alpha+1}, \dfrac{1}{\beta+1}, \dfrac{1}{\delta+1}$.

Find expressions for a , b and c in terms of p and q.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SOLUTIONS

1. a. $\displaystyle\int \frac{15}{(x+4)(x-1)}$

$$\frac{a}{x+4} + \frac{b}{x-1} = \frac{15}{(x+4)(x-1)}$$

$\therefore\ a(x-1) + b(x+4) = 15$

let $x=1$ $\quad 5b = 15$

$b = 3$

$x=-4$ $\quad -5a = 15$

$a = -3$

$$= \int \frac{3}{x-1} - \frac{3}{x+4}\, dx$$

$$= 3\ln(x-1) - 3\ln(x+4)$$

$$= 3\ln\left(\frac{x-1}{x+4}\right) + c$$

b. $\displaystyle\int_0^{\frac{\pi}{4}} x \sin 2x\, dx$

$u = x \quad u' = 1$

$v = -\frac{1}{2}\cos 2x \quad v' = \sin 2x$

$$= \left[-\tfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tfrac{1}{2}\cos 2x\, dx$$

$$= 0 + \left[\tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}$$

c. $\displaystyle\int_0^{\ln 2} \frac{e^x . e^x}{1+e^x}$

let $u = 1 + e^x$

$du = e^x\, dx$

$$= \int_2^3 \frac{u-1}{u}\, du$$

$$= \int_2^3 1 - \frac{1}{u}\, du$$

$$= u - \ln u\Big]_2^3$$

$$= (3 - \ln 3) - (2 - \ln 2)$$

$$= 1 + \ln\frac{2}{3}$$

$\left(\text{or} \quad 1 - \ln\frac{3}{2}\right)$

d. let roots be α, α, β

$$\therefore \quad 2\alpha + \beta = -a \qquad (1)$$
$$\alpha^2 + 2\alpha\beta = 15 \qquad (2)$$
$$\alpha^2\beta = 7 \qquad (3)$$

sub (3) into (2)

$$\alpha^2 + 2\alpha\left(\frac{7}{\alpha^2}\right) = 15$$

$$\alpha^3 - 15\alpha + 14 = 0$$
$$\alpha = 1 \quad \text{(by inspection)}$$
$$\therefore \beta = 7$$

$$\therefore \quad a = -9$$

e. i) α, β, δ are solutions

$$\therefore \quad \alpha^3 - 4\alpha + 5 = 0$$
$$\beta^3 - 4\beta + 5 = 0$$
$$\delta^3 - 4\delta + 5 = 0 \qquad \text{adding}$$

$$\alpha^3 + \beta^3 + \delta^3 = 4(\alpha + \beta + \delta) - 15$$
$$= 4 \times 0 - 15$$
$$= -15$$

ii) as $\alpha + \beta + \delta = 0$

$$(\alpha+\beta)^2(\alpha+\delta)^2(\beta+\delta)^2$$
$$= (-\delta)^2(-\beta)^2(-\alpha)^2$$
$$= (\alpha\beta\delta)^2$$
$$= (-5)^2$$
$$= 25$$

2. a. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ $\qquad t = \tan\frac{\theta}{2}$

$\cos\theta = \dfrac{1-t^2}{1+t^2}$

$d\theta = \dfrac{2\,dt}{1+t^2}$

$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{3+t^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}}\Big]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

b. $\displaystyle\int \frac{2x}{x^2+4x+5}\,dx$

$$= \int \frac{2x+4}{x^2+4x+5} - \int \frac{4}{(x+2)^2+1}$$

$$= \ln(x^2+4x+5) - 4\tan^{-1}(x+2) + c$$

c. $\displaystyle I_n = \int_0^2 (x^3-8)^n\,dx$

$u = (x^3-8)^n \qquad u' = 3nx^2(x^2-8)^{n-1}$

$v = x \qquad v' = 1$

$$= \left[x(x^3-8)^n\right]_0^2 - 3n\int_0^2 x^3(x^2-8)^{n-1}\,dx$$

$$= 0 - 3n\int_0^2 (x^3-8)(x^2-8)^{n-1} + 8(x^3-8)^{n-1}\,dx$$

$$= -3n\,I_n - 24n\,I_{n-1}$$

$$(3n+1)I_n = -24n\,I_{n-1}$$

$$I_n = \frac{-24n}{3n+1}\,I_{n-1}$$

d. i. $z^7 - 1 = 0$

$$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$$

α u complex $\therefore$ $\alpha - 1 \neq 0$

$$\therefore \quad \alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

ii. Sum of roots 1 at a time:

$$= \alpha + \alpha^6 + \alpha^2 + \alpha^5 + \alpha^3 + \alpha^4$$
$$= -1$$

Sum of roots 2 at a time

$$= (\alpha + \alpha^6)(\alpha^2 + \alpha^5) + (\alpha + \alpha^6)(\alpha^3 + \alpha^4) + (\alpha^2 + \alpha^5)(\alpha^3 + \alpha^4)$$

$$= \alpha^3 + \alpha^6 + \alpha^8 + \alpha^{11} + \alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10} + \alpha^5 + \alpha^6 + \alpha^8 + \alpha^9$$
$$= \alpha^3 + \alpha^6 + \alpha + \alpha^4 + \alpha^4 + \alpha^5 + \alpha^2 + \alpha^3 + \alpha^5 + \alpha^6 + \alpha + \alpha^2$$

$$= 2(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)$$
$$= 2(-1)$$
$$= -2$$

sum of roots 3 at a time

$$= (\alpha + \alpha^6)(\alpha^2 + \alpha^5)(\alpha^3 + \alpha^4)$$

$$= \alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^{11} + \alpha^{12} + \alpha^{14} + \alpha^{15}$$
$$= \alpha^6 + 1 + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + 1 + \alpha$$
$$= 2 - 1$$
$$= 1$$

$\therefore$ equation is

$$x^3 + x^2 - 2x - 1 = 0$$

$$\therefore \quad a = 1, \quad b = -2, \quad c = -1$$

3. a. $\displaystyle\int_2^4 \frac{dx}{x\sqrt{x-1}}$

let $u = \sqrt{x-1}$

$du = \dfrac{dx}{2\sqrt{x-1}}$

$u^2 = x-1$

$= \displaystyle\int_1^{\sqrt{3}} \frac{2\,du}{u^2+1}$

$= 2\tan^{-1} u \Big]_1^{\sqrt{3}}$

$= 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1)$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{6}$

b. $\displaystyle\int \sin^4 x \cos^3 x \, dx$

$= \displaystyle\int \sin^4 x\,(1-\sin^2 x)\cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \displaystyle\int u^4 - u^6 \, du$

$= \frac{1}{5}u^5 - \frac{1}{7}u^7$

$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

c. 2 roots are $1+2i$, $1-2i$ $\therefore$ $1+2i+1-2i = 2$

$(1+2i)(1-2i) = 5$

$\therefore$ $P(x) = (x^2 - 2x + 5)(ax+b)$

constant term $\Rightarrow$ $5b = -15$

$b = -3$

$\therefore$ $P(x) = (x^2 - 2x + 5)(ax - 3)$

but $P(2) = 5$

$\therefore$ $5(2a - 3) = 5$

$a = 2$

$\therefore$ $P(x) = (x^2 - 2x + 5)(2x - 3)$

d. $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$P'(x) = 4x^3 + 3x^2 - 6x - 5$

$P''(x) = 12x^2 + 6x - 6$ triple root is root of $P''(x)$

$6(2x^2 + x - 1) = 0$

$6(2x - 1)(x + 1) = 0$

$x = \frac{1}{2}, -1$

$P'(-1) = 0$

$\therefore$ -1 is triple root

$\therefore$ $-1 + -1 + -1 + \alpha = -1$ sum of roots

$\alpha = 2$

$\therefore$ Solutions are $-1, -1, -1, 2$

e. let $y = \frac{1}{x+1} \Rightarrow x = \frac{1}{y} - 1$

$\therefore$ required polynomial is

$$P(y) = \left(\frac{1}{y} - 1\right)^3 + p\left(\frac{1}{y} - 1\right)^2 + q$$

$$= \frac{1}{y^3} - \frac{3}{y^2} + \frac{3}{y} - 1 + \frac{p}{y^2} - \frac{2p}{y} + p + q$$

$$= 1 - 3y + 3y^2 - y^3 + py - 2py^2 + py^3 + qy^3$$

$$= (p + q - 1)y^3 + (3 - 2p)y^2 + (p - 3)y + 1$$

$$= y^3 + \frac{3 - 2p}{p + q - 1}y^2 + \frac{p - 3}{p + q - 1}y + \frac{1}{p + q - 1}$$

$\therefore$ $a = \frac{3 - 2p}{p + q - 1}$ $b = \frac{p - 3}{p + q - 1}$

$c = \frac{1}{p + q - 1}$