SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION 2

HSC ASSESSMENT TASK

JUNE 2010

General Instructions:

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions.

NAME:________________________________

TEACHER: ______________________________

Question 1	Question 2	Question 3	Total

Question 1 **(15 marks)** **Marks**

a) Find $\int \frac{1}{e^x+e^{-x}}\,dx$ 2

b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+cosx+sinx}$ 3

c) Find $\int \frac{sin^{-1}x}{\sqrt{1+x}}\ dx$ 3

d) i) Find real numbers a and b such that

$\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$ 3

ii) Hence find $\int \frac{5-3x}{(x+1)(x^2+1)}dx$ 2

e) Find $\int cosec\ x\, dx$ 2

Question 2 **(14 marks)**

a) The points $P(2t,\frac{2}{t})$ and $Q(2s,\frac{2}{s})$ lie on the hyperbola $xy = 4$.

$(t \neq 0, s \neq 0, t^2 \neq s^2)$.

i) Prove that the equation of the tangent to the hyperbola at the point P is

$x + t^2y = 4t$ 2

ii) Prove that the tangents at P and Q intersect at

$M(\frac{4st}{s+t},\frac{4}{s+t})$ 2

iii) Suppose that $s = \frac{-1}{t}$. Prove that the locus of M is a straight line and state any restrictions that may apply. 2

b) Sketch without using calculus showing all important features: 2

$y = sin^{-1}(sinx)$ $D: -\pi \leq x \leq \pi$

Marks

c) The equation $x^3 + x^2 - 2x + 1 = 0$ has roots α, β, γ.

i) Show that α, β, γ are not integers. 2

ii) Find the monic equation with roots $\alpha + 1, \beta + 1, \gamma + 1$ 2

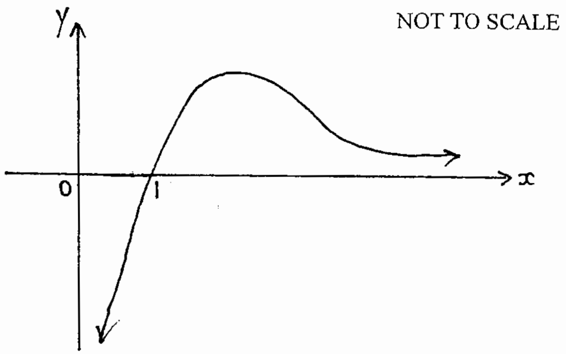
iii) Hence using both polynomials above or otherwise find the value of $(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)$ 2

Question 3 **(15 marks)**

a) Consider the polynomial $P(x) = x^4 + 2x^3 + x^2 - 1$

It is given that one zero is $\frac{-1+i\sqrt{3}}{2}$. Find the other 3 zeros. 3

b) The curve $y = f(x) = \frac{\log_e x}{x}$ is shown below.



Given the maximum turning point is $(e, \frac{1}{e})$, sketch the following curves showing essential features, using at least $\frac{1}{3}$ page for each.

i) $y = f(x + 1)$ 1

ii) $y = f(|x|)$ 1

iii) $y = \frac{1}{f(x)}$ 3

c) i) Given $cos3\theta = 4cos^3\theta - 3cos\theta$, deduce $8x^3 - 6x - 1 = 0$ has solutions $x = cos\theta$ where $cos3\theta = \frac{1}{2}$ 2

ii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $cos\theta$. 3

iii) Hence evaluate $cos\frac{\pi}{9}cos\frac{2\pi}{9}cos\frac{4\pi}{9}$ 2

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x, \quad x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax}, \quad a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax, \quad a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax, \quad a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax, \quad a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax, \quad a \neq 0$
$\int \frac{1}{a^2 + x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$	$= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Solutions 2010 Ext. 2 Task 2

Question 1

a) $\int \frac{1}{e^x + e^{-x}}\,dx$

$\int \frac{e^x}{(e^x)^2+1}\,dx$

Let $u = e^x$

$du = e^x dx$

$\int \frac{du}{u^2+1}$ ①

$= \tan^{-1}(e^x) + c$ ①

b) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x + \sin x}$

$\int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$ ①

$\int \frac{2\,dt}{1+\cancel{t^2}+1-\cancel{t^2}+2t}$

$\int \frac{dt}{1+t}$

$= \left[\log_e\left(1+\tan\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$ ①

$= \log_e(1+1) - \log_e 1$

$= \log_e 2$ ①

c) $\int \frac{\sin^{-1}x}{\sqrt{1+x}}\,dx$

$\int \sin^{-1}x \times \frac{d}{dx}\left(2\sqrt{1+x}\right)dx$ ①

$= \sin^{-1}x \times 2\sqrt{1+x} - \int \frac{1}{\sqrt{1-x^2}} \times 2\sqrt{1+x}\,dx$ ①

$= 2\sqrt{1+x}\,\sin^{-1}x - \int \frac{2}{\sqrt{1-x}}\,dx$

$= 2\sqrt{1+x}\,\sin^{-1}x + 4\sqrt{1-x} + c$ ①

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d)(i) $\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$

$5-3x = a(x^2+1) + (bx+c)(x+1)$

$= ax^2 + a + bx^2 + bx + cx + c$

$-3x+5 = (a+b)x^2 + (b+c)x + (a+c)$

$a+b=0 \qquad b+c=-3 \qquad a+c=5$

$a=-b \quad \therefore \quad b+c=-3$

$-b+c=5$

$2c=2$

$c=1$ ①

$\therefore b=-4$ ① $\qquad \therefore a=4$ ①

(ii) $\int \frac{4}{x+1} + \frac{-4x+1}{x^2+1}\,dx$ ①

$= 4\log_e|x+1| - 2\log_e(x^2+1) + \tan^{-1}x + c$ ①

e) $\int \operatorname{cosec} x\,dx$

$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}\,d\theta$ ① $= \log_e|\operatorname{cosec} x - \cot x|$ ①

Question 2

a)(i) $xy = 4$

$y = 4x^{-1}$

$y' = \frac{-4}{x^2}$

$\therefore$ At $x = 2t$

$y' = \frac{-4}{4t^2}$

$m_{tangent} = \frac{-1}{t^2}$

$y - \frac{2}{t} = \frac{-1}{t^2}(x-2t)$

$t^2 y - 2t = -x + 2t$

$x + t^2 y = 4t$

(ii) Tangents at P and Q are

$x + t^2 y = 4t$
$x + s^2 y = 4s$ } Solve simultaneously

$y(t^2 - s^2) = 4(t-s)$

$y = \frac{4}{s+t}$

$\therefore x = 4t - t^2 \times \frac{4}{s+t}$

$= \frac{4ts + \cancel{4t^2} - \cancel{4t^2}}{s+t}$

$= \frac{4st}{s+t}$

So M is

$$\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$$

(iii) If $st = -1$

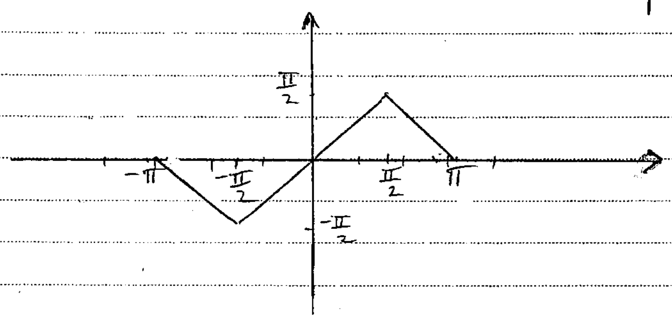
$$x = \frac{-4}{s+t}, \quad y = \frac{4}{s+t}$$

$\therefore x = -y$ is locus of M but $s \neq 0$ and $t \neq 0$ $\therefore (0,0)$ is not part of locus.

b) $y = \sin^{-1}(\sin x)$ $\quad D: -\pi \le x \le \pi$

has range the same as $y = \sin^{-1} x$

ie: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. It is also odd since $\sin x$ is odd. Also period is 2π



c) (i) $x^3 + x^2 - 2x + 1 = 0$ has roots α, β, γ.

By Factor Theorem roots must be factors of 1 ie: 1 or -1 if they are integer.

$P(1) = 1^3 + 1^2 - 2 + 1 \neq 0$

$P(-1) = (-1)^3 + (-1)^2 - 2(-1) + 1 \neq 0$

$\therefore$ Roots are not integers.

(ii) Replace x with $x-1$ in original:

$(x-1)^3 + (x-1)^2 - 2(x-1) + 1 = 0$

$x^3 - 3x^2 + 3x - 1 + x^2 - 2x + 1 - 2x + 2 + 1 = 0$

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$x^3 - 2x^2 - x + 3 = 0$ is monic equation.

(iii) From original polynomial,

$\alpha + \beta + \gamma = -1$

$\therefore \alpha + \beta = -1 - \gamma$

$\alpha + \gamma = -1 - \beta$

$\beta + \gamma = -1 - \alpha$

$\therefore (\alpha+\beta)(\alpha+\gamma)(\beta+\gamma) = -(\alpha+1) \times -(\beta+1) \times -(\gamma+1)$

$= -(\alpha+1)(\beta+1)(\gamma+1)$

which is the negative of the product of the roots in $x^3 - 2x^2 - x + 3 = 0$

ie: $-1 \times \frac{-3}{1} = 3$.

Question 3

a) $P(x) = x^4 + 2x^3 + x^2 - 1$

One zero is $\frac{-1 + i\sqrt{3}}{2}$ $\therefore$ another root is $\frac{-1 - i\sqrt{3}}{2}$ as polynomial has real coefficients.

This quadratic factor must be:

$$x^2 - \left(\frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2}\right)x + \frac{1+3}{4} = 0$$

$x^2 + x + 1 = 0$

$\therefore$ By inspection:

$x^4 + 2x^3 + x^2 - 1 = (x^2 + x + 1)(x^2 + x - 1)$

Other zeros are $\frac{-1 \pm \sqrt{1 - 4 \times 1 \times -1}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

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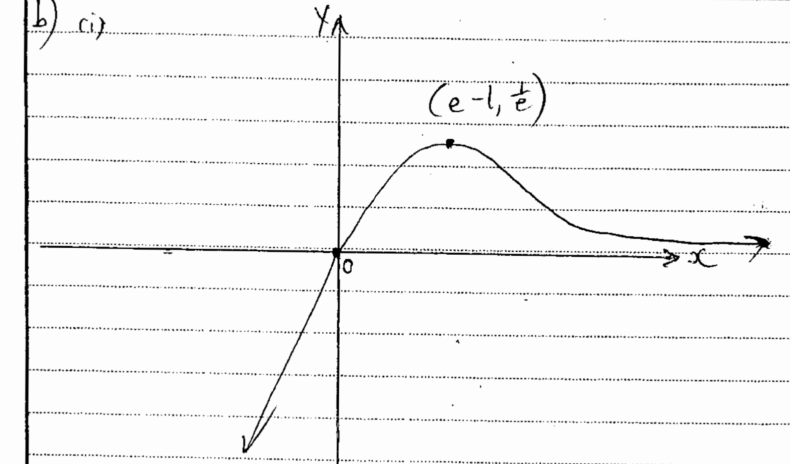
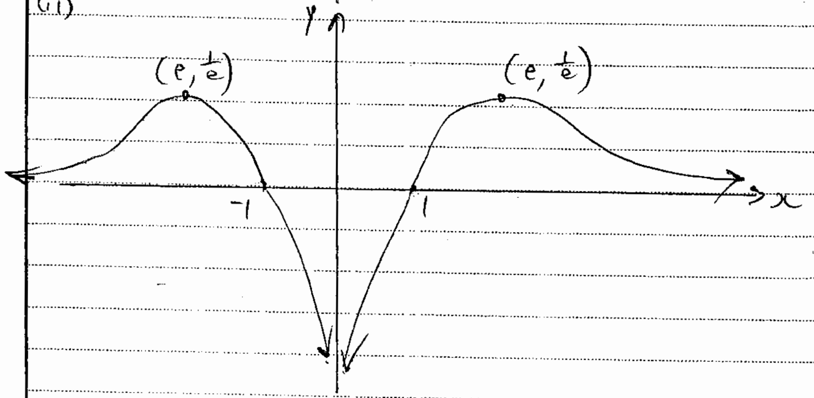
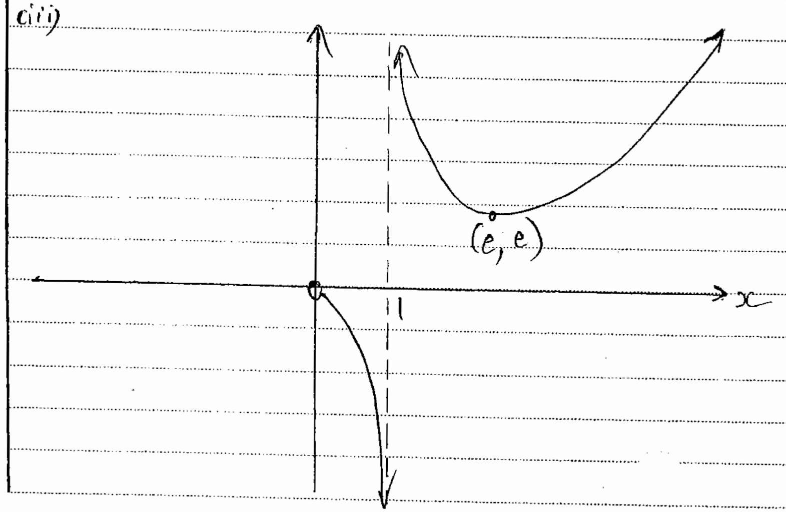
b) (i)



(ii)



(iii)



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c) (i) If $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$2\cos 3\theta = 8\cos^3\theta - 6\cos\theta$

$2\cos 3\theta - 1 = 8\cos^3\theta - 6\cos\theta - 1$

So in $8x^3 - 6x - 1 = 0$ if $x = \cos\theta$, solution for θ are the same as for

$2\cos 3\theta - 1 = 0$

$2\cos 3\theta = 1$

$\underline{\cos 3\theta = \frac{1}{2}}$

c(ii) $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$

$\therefore \theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

$\therefore$ Solutions are

$\cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}, \cos\frac{11\pi}{9}, \cos\frac{13\pi}{9}, \cos\frac{17\pi}{9}$

SAME

$\therefore$ Roots are $\underline{\cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}}$

c(iii) $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$, $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ is the product of the roots in this equation and is calculated by $\frac{-d}{a}$

$\therefore \underline{\frac{1}{8}}$