Name:____________________ Maths Class Teacher:____________________

SYDNEY TECHNICAL HIGH SCHOOL

Extension 2 Mathematics

HSC Assessment Task 2

June 2012

General Instructions

- Working time – 70 minutes
- Write using **black or blue pen**
- Board-approved calculators may be used
- **All necessary working** should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Start each question on a **new page**.
- Place your papers **in order** with the question paper on top and staple or pin them.

Total Marks - 52

- Attempt Questions 1 – 4
- Mark values are shown with the questions.

Question 1 **13 Marks**

(a) Find $\int x\cos(x^2)\,dx$ **2**

(b) Find $\int \frac{dx}{\sqrt{5-4x-x^2}}$ **2**

(c) Use partial fractions to find $\int \frac{-2dx}{x^2+3x-4}$ **3**

(d) Using integration by parts, evaluate $\int_1^e \ln x\,dx$ **2**

(e) Using the substitution $x = 2\sin\theta$ or otherwise, evaluate $\int_0^1 \sqrt{4-x^2}\,dx$ **4**

leaving your answer in exact form.

Question 2 **13 Marks**

(a) Find $\int \frac{x^2+2x-3}{x+1}\,dx$ **3**

(b) Find $\int \frac{x-1}{\sqrt{x+1}}\,dx$ **2**

(c) Consider the rectangular hyperbola, R: $xy = c^2$

(i) Find the foci. **2**

(ii) Write the equations of the directrices. **1**

(iii) Find the equation of the tangent to R at P $(ct, \frac{c}{t})$. **2**

(iv) This tangent cuts the coordinate axes at A and B. Prove that $PA = PB$. **3**

(Questions 3 and 4 on reverse of page)

Question 3

13 Marks

(a) If $f(x) = x - \frac{1}{x}$, provide separate half page sketches of the graphs of the following:

(i) $y = f(x)$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y = f'(x)$ **2**

(v) $y = f(|x|)$ **2**

(b) Solve the equation $4x^3 - 8x^2 + 5x - 1 = 0$ given that it has a double root. **3**

Question 4

13 Marks

(a) If $2x^3 - 4x^2 + 6x - 1 = 0$ has roots α, β and γ, find:

(i) $\alpha^3 + \beta^3 + \gamma^3$ **3**

(ii) $\alpha^4 + \beta^4 + \gamma^4$ **2**

(iii) $\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$ **2**

(b) $1 - i$ and $2 + i$ are zeroes of a monic polynomial, $P(x)$, with real coefficients and degree 4.

(i) Express $P(x)$ as a product of two real quadratic factors. **2**

(ii) Explain briefly why the polynomial $P(x)$ cannot take negative values. **1**

(c) The equation $x^3 - 6x^2 + 7x - 3 = 0$ has roots α, β and γ.

(i) Write an equation which has the roots α^2, β^2 and γ^2. **2**

(ii) It is known that the solution to a given problem is the average of the roots of the equation $x^3 - 6x^2 + 7x - 3 = 0$. **1**

Without finding the roots, determine the solution to the problem.

End of Exam

HSC 2012 Ass 2 Extension 2 – June SOLUTIONS

Q1 a) $\int x\cos(x^2)\,dx = \frac{1}{2}\int 2x\cos(x^2)\,dx$
$= \frac{1}{2}\sin(x^2) + c$

b) $\int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

$= \sin^{-1}\left(\frac{x+2}{3}\right) + c$

c) $\int \frac{-2\,dx}{x^2+3x-4} = \int \frac{-2\,dx}{(x+4)(x-1)}$

$= \int \frac{A}{x+4} + \frac{B}{x-1}\,dx$

$A(x-1) + B(x+4) = -2$
Let $x=1$, $5B = -2$
$\therefore B = \frac{-2}{5}$
Let $x=-4$, $-5A = -2$
$\therefore A = \frac{2}{5}$

$= \frac{2}{5}\int \frac{dx}{x+4} - \frac{2}{5}\int \frac{dx}{x-1}$

$= \frac{2}{5}\ln(x+4) - \frac{2}{5}\ln(x-1) + c$

$= \frac{2}{5}\ln\frac{(x+4)}{(x-1)} + c$

d) $\int_1^e \ln x\,dx = \left[x\ln x\right]_1^e - \int_1^e x\cdot\frac{1}{x}\,dx$

let $u = \ln x$
$du = \frac{1}{x}\,dx$
let $v = x$
$dv = dx$

$= e - \left[x\right]_1^e$
$= e - (e-1)$
$= 1$

e) $\int_0^1 \sqrt{4-x^2}\,dx =$

let $x = 2\sin\theta$
$\therefore dx = 2\cos\theta\,d\theta$
When $x=1$, $\sin\theta = \frac{1}{2}$
$\therefore \theta = \frac{\pi}{6}$
When $x=0$, $\theta = 0$

$= \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta}\;2\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\,d\theta$

$= \int_0^{\frac{\pi}{6}} 2(\cos 2\theta + 1)\,d\theta$

$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{6}}$

$= 2\left[\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right) - (0+0)\right]$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

Q3 a) $\int \frac{x^2+2x-3}{x+1}\,dx$

$$\begin{array}{r} x+1 \\ x+1\overline{)\,x^2+2x-3} \\ \underline{x^2+x} \\ x-3 \\ \underline{x+1} \\ -4 \end{array}$$

$= \int\left((x+1) - \frac{4}{x+1}\right)dx$

$= \frac{x^2}{2}+x - 4\ln(x+1)+c$

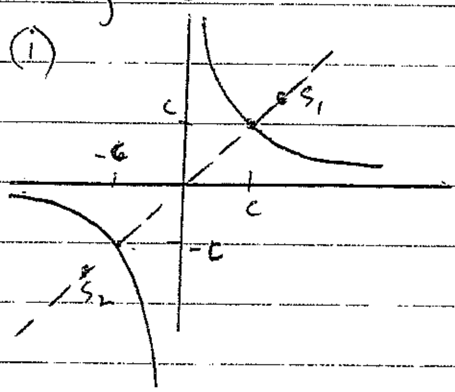
b) $\int \frac{x-1}{\sqrt{x+1}}\,dx = \int \frac{x+1}{\sqrt{x+1}}\,dx - \int\frac{2}{\sqrt{x+1}}\,dx$

$= \int \sqrt{x+1} - \frac{2}{\sqrt{x+1}}\,dx$

$= \frac{2}{3}\left(\sqrt{x+1}\right)^{\frac{3}{2}} - 4\sqrt{x+1}+c$

c) $xy=c^2$

(i)



$S_1 = \sqrt{2}c, \sqrt{2}c$

$S_2 = -\sqrt{2}c, -\sqrt{2}c$

(ii) directrices $\Rightarrow x+y=\pm\sqrt{2}$

(iii) tangent to R at $\left(ct, \frac{c}{t}\right)$

$y=\frac{c^2}{x}$

$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$

$= -\frac{c^2}{c^2t^2}$ when $x=ct$

$= \frac{-1}{t^2}$

$\therefore$ Equation of tangent is...

$y-\frac{c}{t} = -\frac{1}{t^2}(x-ct)$

$t^2y - tc = -x+ct$

$\therefore x+t^2y = 2ct$

(iv) When $x=0$, $t^2y=2ct$

$\therefore y = \frac{2c}{t}$ $A\left(0,\frac{2c}{t}\right)$

When $y=0$, $x=2tc$

$\therefore B(2ct, 0)$

Midpoint $AB = \left(\frac{0+2ct}{2}, \frac{\frac{2c}{t}+0}{2}\right)$

$= \left(ct, \frac{c}{t}\right)$

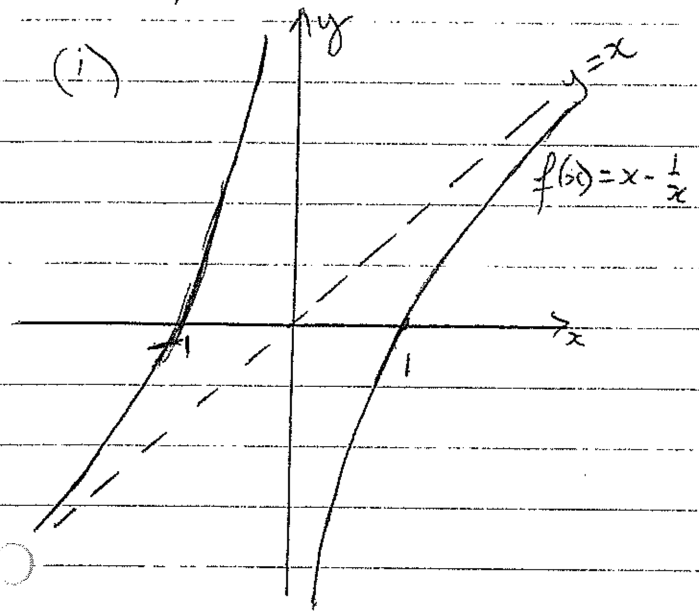
$= P$

$\therefore PA = PB$

Q3

a) $f(x) = x - \frac{1}{x}$

(i)



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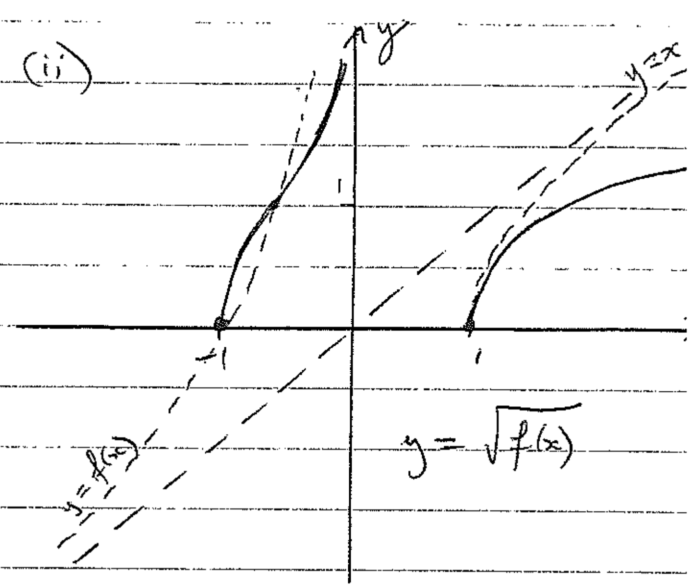

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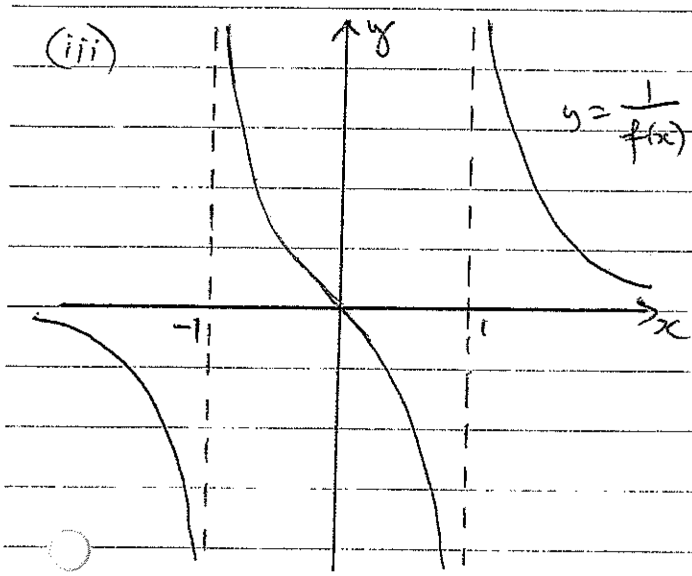
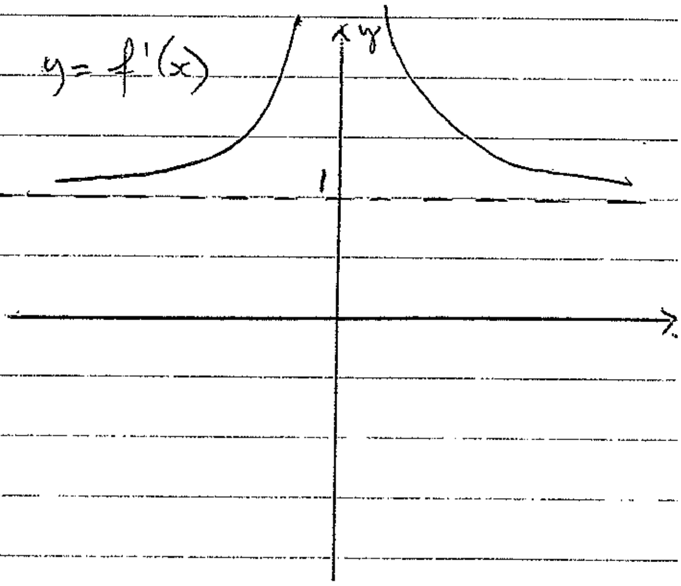
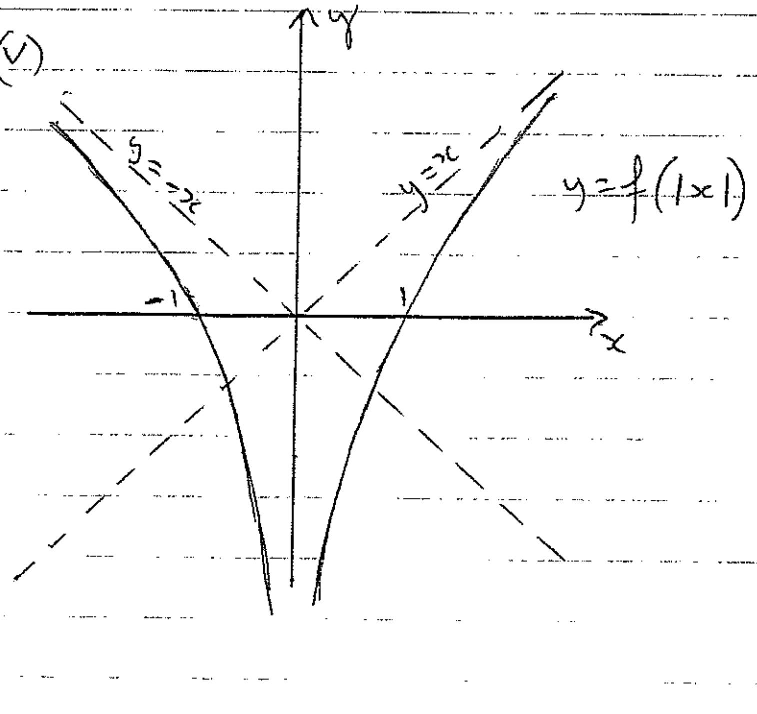


(iv) $y = f'(x)$



(v)



b) Let $P(x) = 4x^3 - 8x^2 + 5x - 1$

$\therefore P'(x) = 12x^2 - 16x + 5$

$= (2x-1)(6x-5)$

$= 0$ when $x = \frac{1}{2}, \frac{5}{6}$

Check $x = \frac{1}{2}$ for double root of $P(x)$

$P(\frac{1}{2}) = \frac{1}{2} - 2 + \frac{5}{2} - 1$

$= 0$

Product of roots of $P(x) = \frac{-d}{a}$

$= \frac{1}{4}$

$\therefore$ Other root is 1 as $\frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}$.

$\therefore$ Roots of $P(x)$ are $x = \frac{1}{2}, \frac{1}{2}, 1$.

Q4

a) Let $2x^3-4x^2+6x-1=0$ have roots α, β and γ.

$\sum\alpha = \frac{-b}{a} = 2$

$\sum\alpha\beta = \frac{c}{a} = 3$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{1}{2}$

(i) Now $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$= 4-6$

$= -2$

Also as α, β and γ are roots...

$2\alpha^3-4\alpha^2+6\alpha-1=0$

$2\beta^3-4\beta^2+6\beta-1=0$

$2\gamma^3-4\gamma^2+6\gamma-1=0$

By addition, $2\sum\alpha^3-4\sum\alpha^2+6\sum\alpha-3=0$

$\therefore 2\sum\alpha^3+8+12-3=0$

$\therefore \cancel{\sum\alpha^3} = -17$

$\therefore \sum\alpha^3 = \frac{-17}{2}$

(ii) Multiplying by x and adding equa

$2\sum\alpha^4-4\sum\alpha^3+6\sum\alpha^2-\sum\alpha=0$

$\therefore 2\sum\alpha^4+34-12-2=0$

$\therefore 2\sum\alpha^4 = -20$

$\therefore \sum\alpha^4 = -10$

(iii) Now, $(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (\alpha^2\beta+\alpha^2\gamma+\alpha\beta\gamma+\beta^2\alpha+\alpha\beta\gamma+\beta^2\gamma$
$+\alpha\beta\gamma+\gamma^2\alpha+\gamma^2\beta)$

$= \sum\alpha^2\beta + 3\alpha\beta\gamma$

$\therefore 2\times3 = \sum\alpha^2\beta + \frac{3}{2}$

$\therefore \sum\alpha^2\beta = 4\frac{1}{2}$

b) (i) $x-(1-i)$ and $x-(1+i)$ are factors, and so are $x-(2+i)$ and $x-(2-i)$.

$\therefore P(x) = (x-(1-i))(x-(1+i))(x-(2+i))(x-(2-i))$

$= (x^2-2x+2)(x^2-4x+5)$

(ii) $\Delta_1 = 4-4.1.2$
<0

$\Delta_2 = 16-4\times5$
<0

As both quadratics are positive definite, all values of $P(x)$ are greater than or equal to 0.

c) (i) Let $x=\sqrt{y}$

$\therefore (\sqrt{y})^3-6(\sqrt{y})^2+7\sqrt{y}-3=0$

$\therefore y\sqrt{y}-6y+7\sqrt{y}-3=0$

$\therefore y\sqrt{y}+7\sqrt{y}=6y+3$

$\therefore y^3+14y^2+49y=36y^2+36y+9$

$\therefore y^3-\cancel{22}y^2+13y-9=0$

$\therefore$ Equation is $x^3-22x^2+13x-9=0$

(ii) solution $= \frac{\sum\alpha}{3}$

$= \frac{6}{3}$

$= 2$