SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION 2

Assessment No. 2

June 2013

TIME ALLOWED: **70 minutes**

Instructions:

- Write your name and class at the top of this page, and on all your answer booklet
- Hand in your answer booklet and this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- Approved calculators may be used.
- A set of Standard Integrals is provided at the rear of thgis question sheet. It may be detached at any time, but must be handed in.
- PART A is a multiple choice section worth 5 marks.
  - It should take you about 7 minutes.
  - The answer sheet for Part A is the first page in our answer booklet. Do not detach it.

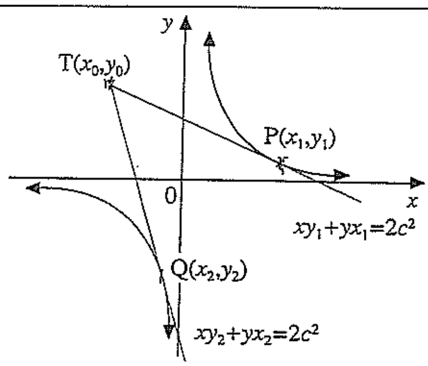
In Part B, each question is to be started on a new page

PART A

MULTIPLE CHOICE

Choose the correct answer from among the choices and fill in the appropriate circle on the Multiple Choice Answer Sheet included in your Answer Booklet.

Each question is worth 1 mark

QUESTION	
1	Where there are vertical tangents on a curve $y = f(x)$, then, in the expression for $\frac{dy}{dx}$ A. The numerator equals zero B. The denominator equals zero C. Both the numerator and denominator equal zero D. None of the above
2	 For the diagram above, it is true that: A. $x_0y_1 + y_0x_1 = 2c^2$ B. $x_0y_2 + y_0x_2 = 2c^2$ C $xy_0 + yx_0 = 2c^2$ D. All of the above
3	If $P(x)$ is a polynomial with real coefficients, A. Irrational roots occur in conjugate pairs B. All roots will be real C. Complex roots will occur in conjugate pairs D. Both (A) and (C) will be true.

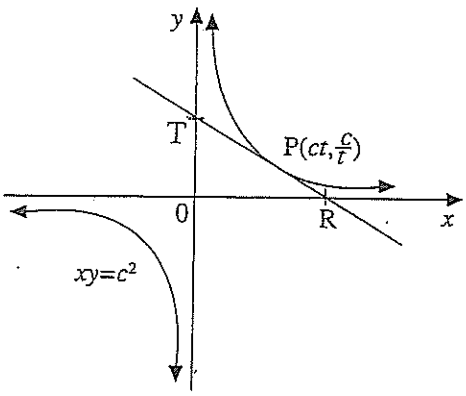
4	If $y = f(x)$ is an even function, the graph of $y = \{f(x)\}^3$ will be: A. Odd B. Even C. Neither D. It cannot be determined.
5	The definite integral $\int_0^3 \sqrt{9 - x^2}\, dx$ A. Could be evaluated by the substitution $x = 3tan\theta$ B. Could be evaluated by the substitution $x = 3sec\theta$ C. Could be evaluated by the substitution $x = 3cosec\theta$ D. Equals $\frac{9\pi}{4}$

PART B

START EACH QUESTION ON A NEW PAGE

QUESTION 6: (15 Marks) **Marks**

(a)



(i) For the diagram above, show that the equation of the tangent at the point $P(ct,\frac{c}{t})$ is $x+t^2y=2ct$ **2**

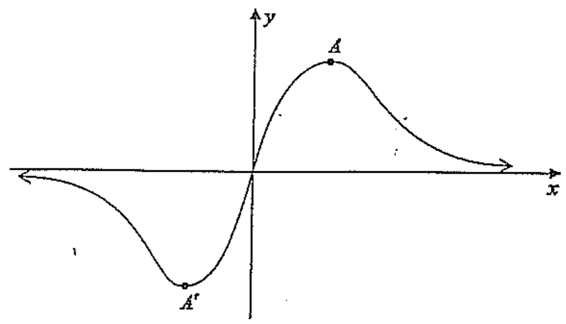
(ii) Find the co-ordinates of R and T **2**

(iii) Find the area of ΔOTR, and state the significance of this area with regards to the position of the point P **2**

QUESTION 6 continues overleaf......)

QUESTION 6 continued......)

(b) Drawn below is the graph of $y = \frac{2x}{1+x^2}$



(i) Find the co-ordinates of the turning points A and A' **2**

(DO NOT TEST THEIR NATURE)

(ii) On separate diagrams, sketch graphs of the following functions, showing all important features:

I. $y = \frac{|2x|}{1+x^2}$ **1**

II. $y = \frac{1+x^2}{2x}$ **2**

III. $y^2 = \frac{2x}{1+x^2}$ **2**

IV. $y = ln\ \left(\frac{2x}{1+x^2}\right)$ **2**

QUESTION 7: (15 Marks) **(Start a new page)**

Marks

(a) Find $\int \frac{x}{\sqrt{2-x}}\,dx$ by using the substitution $u = \sqrt{2-x}$ **3**

(b) Evaluate $\int_0^1 tan^{-1}x\ dx$ **3**

(c) (i) Find numbers A, B, and C, such that **2**

$$\frac{x^2}{4x^2-9} \equiv A + \frac{B}{2x-3} + \frac{C}{2x+3}$$

(ii) Hence, or otherwise, find $\int \frac{x^2}{4x^2-9}\,dx$ **2**

(d) (i) By using the substitution $x = -u$, show that $\int_{-a}^{0} f(x)dx = \int_0^a f(-x)dx$ **1**

(ii) Deduce that **1**

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]\,dx$$

(iii) Hence, or otherwise, evaluate **3**

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}dx$$

QUESTION 8: (15 marks) (Start a new page)

Marks

(a) (i) Show that 1 and -1 are both roots of multiplicity 2 of the polynomial **2**

$$P(x) = x^6 - 3x^2 + 2$$

(ii) Express $P(x)$ as the product of irreducible factors over the Complex Field **1**

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β, and γ.

In each of the following cases, find an equation <u>with integer coefficients</u> having roots of:

(i) $-\alpha$, $-\beta$ and $-\gamma$ **1**

(ii) α, $-\alpha$, β, $-\beta$, γ, and $-\gamma$ **1**

(iii) α^2, β^2, and γ^2 **2**

(c) (i) Given the result $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and using De Moivre's Theorem for *cis* θ , or otherwise, show that **3**

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

(ii) Using the above result, solve the equation **3**

$$16x^4 - 16x^2 + 1 = 0$$

(iii) Use the above results to show that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$ **2**

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Solutions to June 2013 Ext. 2 Assessment Task

1\. B 2. D 3. C 4. B 5. D

6\. a(i) $x = ct$ $y = \frac{c}{t}$

$\frac{dx}{dt} = c$ $\frac{dy}{dt} = \frac{-c}{t^2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{-c/t^2}{c}$

$= \frac{-1}{t^2}$

Tangent at P is

$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$ (1)

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$ as required. (1)

(ii) R is when $y = 0$

$x = 2ct$

$\therefore$ R is $(2ct, 0)$ (1)

T is when $x = 0$

$y = \frac{2ct}{t^2} = \frac{2c}{t}$

$\therefore$ T is $(0, \frac{2c}{t})$ (1)

(iii) Area is

$\frac{1}{2}bh$

$= \frac{1}{2} \times 2ct \times \frac{2c}{t}$

$= 2c^2$ (1)

(iv) The area of a triangle formed when the tangent intersects the axes is constant. (1)

b) (i) $y = \frac{2x}{x^2+1}$

$\frac{dy}{dx} = \frac{(x^2+1)2 - 2x \times 2x}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2} = 0$

at turning pts

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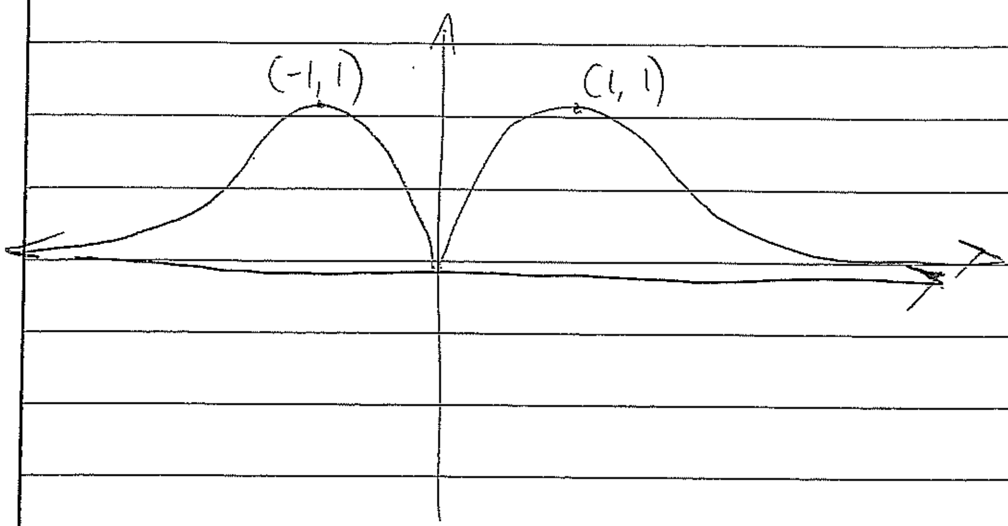
$2x^2 = 2$ A is $(1, 1)$ (1)

$x^2 = 1$ B is $(-1, -1)$ (1)

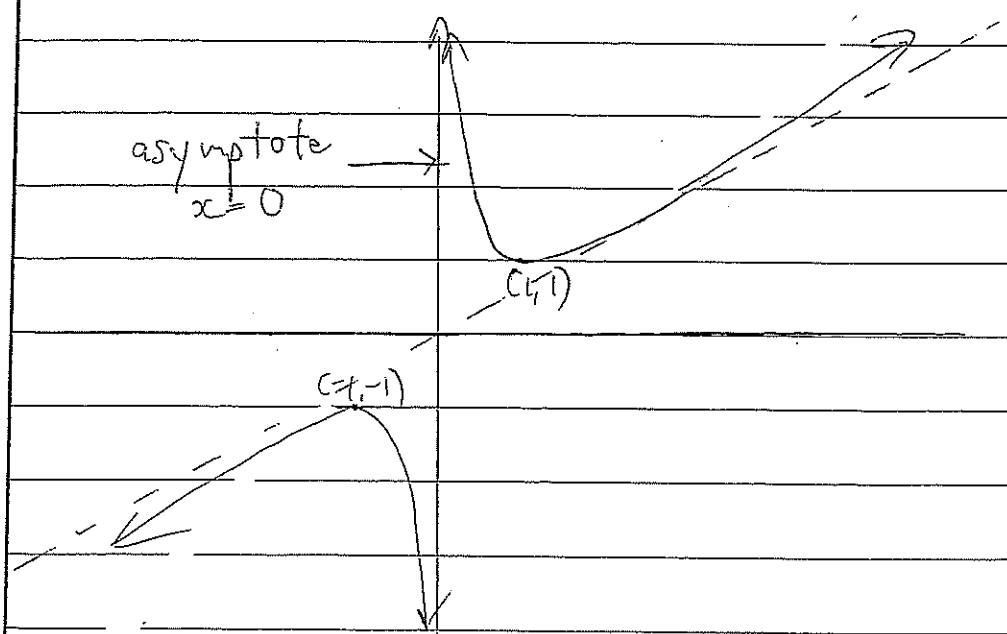
$x = \pm 1$

(ii) 1. $y = \dfrac{|2x|}{1 + x^2}$



(1)

2. $y = \dfrac{1 + x^2}{2x}$



(2)

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$\therefore$ 4 solutions are

$\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$, $\cos\frac{7\pi}{12}$, $\cos\frac{11\pi}{12}$,

c(iii) Product of roots:

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$$

$$\cos\frac{\pi}{12}\times\cos\frac{5\pi}{12}\times-\cos\frac{5\pi}{12}\times-\cos\frac{\pi}{12} = \frac{1}{16}$$

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4} > 0.$$

7. a) $\int \frac{x}{\sqrt{2-x}}\,dx$ $u = \sqrt{2-x}$

$u^2 = 2-x$

$2u\,du = -dx$

$$\int \frac{2-u^2}{u} \times -2u\,du$$

$$-\int 4-2u^2\,du$$

$$\int 2u^2-4\,du$$

$$\frac{2}{3}u^3 - 4u + c$$

$$\frac{2}{3}(2-x)^{\frac{3}{2}} - 4\sqrt{2-x} + c$$

b) $\int_0^1 \tan^{-1}x\,dx$

$$\int_0^1 \frac{d}{dx}(x)\tan^{-1}x\,dx$$

$$\left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$\frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$$

$$\frac{\pi}{4} - \frac{1}{2}\left[\ln 2 - \ln 1\right]$$

$$\frac{\pi}{4} - \frac{1}{2}\ln 2$$

c) $\frac{x^2}{4x^2-9} = A + \frac{B}{2x-3} + \frac{C}{2x+3}$

$x^2 = A(4x^2-9) + B(2x+3) + C(2x-3)$

Let $x = 1\frac{1}{2}$

$2\frac{1}{4} = B\times 6$

$B = \frac{9}{4}\times\frac{1}{6} = \frac{3}{8}$

Let $x = -1\frac{1}{2}$

$2\frac{1}{4} = -6C$

$C = -\frac{3}{8}$

Let $x=0$ $0 = -9A + \frac{9}{8} + \frac{9}{8}$ $9A = \frac{9}{4}$ $\therefore A = \frac{1}{4}$

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(ii) $\int \frac{x^2}{4x^2-9}\,dx$

$= \int \frac{1}{4} + \frac{\frac{3}{8}}{2x-3} - \frac{\frac{3}{8}}{2x+3}\,dx$

$= \frac{x}{4} + \frac{3}{16}\ln(2x-3) - \frac{3}{16}\ln(2x+3)$

$= \frac{x}{4} + \frac{3}{16}\ln\left|\frac{2x-3}{2x+3}\right| + C$

d) (i) $\int_{-a}^{0} f(x)\,dx$

Let $x = -u$

$dx = -du$

$\therefore \int_{a}^{0} f(-u) \times -du$

$\int_{0}^{a} f(-u)\,du$

$= \int_{0}^{a} f(-x)\,dx$ as req'd.

(ii) $\int_{-a}^{a} f(x)\,dx$

$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$

$= \int_{0}^{a} f(x) + f(-x)\,dx$

(iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx$

$= \int_{0}^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)}\,dx$

$= \int_{0}^{\frac{\pi}{4}} \frac{1 - \sin x + 1 + \sin x}{1-\sin^2 x}\,dx$

$= 2\int_{0}^{\frac{\pi}{4}} \sec^2 x\,dx = 2\left[\tan x\right]_{0}^{\frac{\pi}{4}} = 2$

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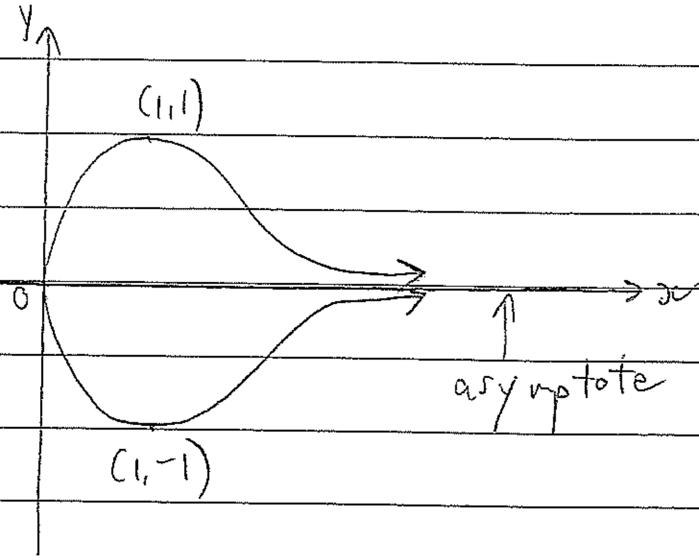
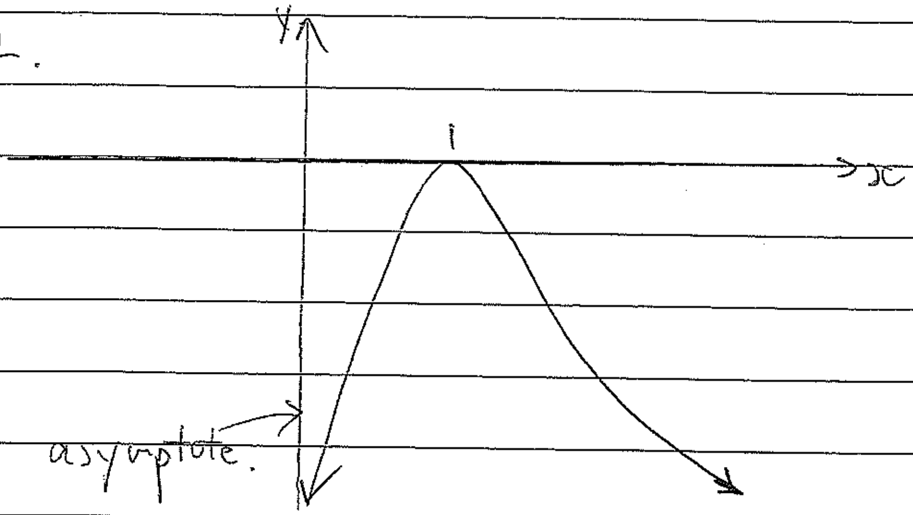
3 $y^2 = \dfrac{2x}{1+x^2}$



(2)

4.



(2)

$y = \ln\left(\dfrac{2x}{1+x^2}\right)$

8. (i) $P(x) = x^6 - 3x^2 + 2$

$P'(x) = 6x^5 - 6x$

$P'(1) = 6 - 6 = 0$
$P'(-1) = -6 + 6 = 0$ $\Big\}$ $\therefore$ Roots of multiplicity 2.

(ii) $P(x) = (x-1)^2(x+1)^2$

$(x^4 - 2x^2 + 1)(x^2 + 2)$ by inspection

$= (x-1)^2(x+1)^2(x-\sqrt{2}i)(x+\sqrt{2}i)$

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b) (i) $x^3 + 2x - 1 = 0$

Let $x = -x$

$(-x)^3 + 2(-x) - 1 = 0$

$-x^3 - 2x - 1 = 0$

$x^3 + 2x + 1 = 0$

(ii) $(x^3 + 2x + 1)(x^3 + 2x - 1)$

$(x^3 + 2x)^2 - 1$

$x^6 + 4x^4 + 4x^2 - 1 = 0$

(iii) Let $x = \sqrt{x}$

$(\sqrt{x})^3 + 2\sqrt{x} - 1 = 0$

$x^{3/2} + 2x^{\frac{1}{2}} - 1 = 0$

$x^{\frac{1}{2}}(x + 2) = 1$

$x(x+2)^2 = 1$

$x^3 + 4x^2 + 4x - 1 = 0$

c) (i) By De Moivre's Theorem

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta \times -\sin^2\theta$

$+ 4\cos\theta\,(i)^3\sin^3\theta + \sin^4\theta$

Equating real parts:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ as req'd

(ii) Let $x = \cos\theta$

$16\cos^4\theta - 16\cos^2\theta = -1$

$8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$