SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 3

JUNE 2016

Mathematics Extension 2

General Instuctions

- Working time - 90 minutes
- Write using black pen only
- Board-approved calculators may be used
- All necessary working should be shown in questions 6 to 9
- Start each question on a new page
- A Board of Studies reference sheet is provided

Total marks - 60

Section 1 - 5 marks

Attempt Questions 1 – 5.
Allow about 8 minutes for this section.

Section 2 - 55 marks

Attempt Questions 6 – 9.
Allow about 82 minutes for this section.

Name : ______________________________

Teacher : ______________________________

Section 1 (5 marks)

Attempt Questions 1 – 5

Use the multiple-choice answer sheet in your answer booklet for Questions 1 – 5.
Do not remove the multiple-choice answer sheet from your answer booklet.

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1. Which of the following is equivalent $\int xsec^2(x^2)\,dx$?

(A) $2\tan(x^2) + c$

(B) $\frac{1}{2}\tan(x^2) + c$

(C) $\frac{1}{6}\,sec^3(x^2) + c$

(D) $\frac{1}{3}\,sec^3(x^2) + c$

2. What is the multiplicity of the root $x = -1$ of the equation

$3x^5 - 5x^4 - 35x - 27 = 0$?

(A) 1

(B) 2

(C) 3

(D) 4

3. If $y = \cos^{-1}(e^x)$ then $\frac{dy}{dx}$ equals

(A) $-cosec\,y$

(B) $-\tan y$

(C) $-\cot y$

(D) $-\sec y$

4. The equation $2x^3 - 7x + 1 = 0$ has roots α , β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) 0

(B) $\frac{43}{4}$

(C) $-\frac{1}{2}$

(D) $-\frac{3}{2}$

5. Which integral has the smallest value ?

(A) $\int_0^{\frac{\pi}{4}} sin^2x\, dx$

(B) $\int_0^{\frac{\pi}{4}} cos^2x\, dx$

(C) $\int_0^{\frac{\pi}{4}} sinx\, cosx\, dx$

(D) $\int_0^{\frac{\pi}{4}} sinx\, tanx\, dx$

Section 2 (55 marks)

Attempt Questions 6 – 9
Start each question on a new page

Question 6 (14 marks)

a) Find $\int \frac{dx}{\sqrt{6x-x^2}}$ 2

b) Evaluate $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\, dx$ using the substitution $u = 1 + x^2$ 3

c) Use integration by parts to evaluate $\int_1^e \frac{lnx}{\sqrt{x}}\, dx$ 3

d) If α , β and γ are the roots of the equation $x^3 - 5x - 3 = 0$ 3

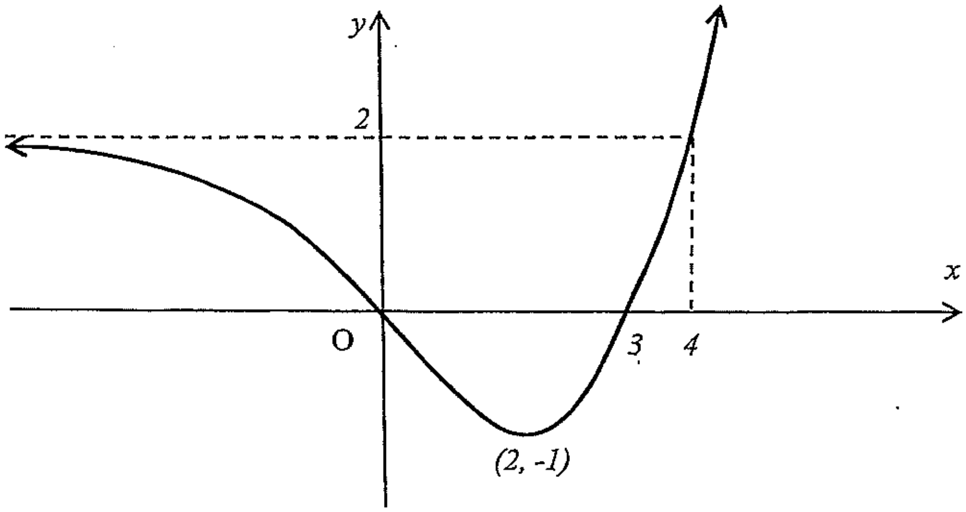
find the monic polynomial equation whose roots are $\frac{\alpha}{\beta\gamma}$, $\frac{\beta}{\alpha\gamma}$ and $\frac{\gamma}{\alpha\beta}$

e) Find the real values of a and b given that $3 + i$ is a root 3

of the equation $z^3 + az^2 + bz + 10 = 0$

Question 7 (13 marks) (Start a new page)

a) The diagram below shows the graph of $y = f(x)$.



Draw neat sketches, on separate diagrams, of the following;

i) $y = f(-x)$ 2

ii) $y = \frac{1}{f(x)}$ 2

iii) $y = [f(x)]^3$ 2

b) Use the substitution $t = tan\frac{\theta}{2}$ to find $\int \frac{\tan\theta}{1+\cos\theta} d\theta$ 3

c) For what values of k does the equation $x^3 - 3x^2 - 24x + k = 0$ have exactly one real root ? 4

Question 8 (14 marks) (start a new page)

a) Without the use of calculus, sketch the curve $y^2 = x^2(4 - x^2)$ **2**

b) Find $\int \frac{x+6}{x^2+4x+29}\, dx$ **4**

c) If the polynomial $ax^{n+1} + bx^n + 1$ is divisible by $(x-1)^2$, **3**

find expressions for a and b in terms of n.

d) i) If $I_n = \int_0^1 x^n e^{x^2} dx$ show that $I_n = \frac{e}{2} - \left(\frac{n-1}{2}\right) I_{n-2}$ **3**

ii) Evaluate $\int_0^1 x^5 e^{x^2} \mathrm{dx}$ **2**

Question 9 (14 marks) (start a new page)

a) Use partial fractions to find $\int \frac{x-3}{x^2+6x+5} dx$ **3**

b) Find $\int cos^5 x\, dx$ **3**

c) i) Solve $z^5 - 1 = 0$ over the complex field. **2**

ii) Considering the sum of the roots found in part i), or otherwise,

determine the exact value of $cos\frac{2\pi}{5}$. **2**

d) The equation $x^3 + ax^2 + bx + c = 0$ **4**

has one root equal to the sum of the other two roots.

Show that $a^3 - 4ab + 8c = 0$.

End of Paper

SOLUTIONS

. B 2. B 3. C 4. D 5. A

a) $\int \frac{dx}{\sqrt{9-(x-3)^2}}$

$= \sin^{-1}\frac{x-3}{3} + c$

b) $\int_0^{\sqrt{3}} \frac{x^3}{(1+x^2)^{\frac{3}{2}}}\,dx$ $\qquad u = 1 + x^2$, $du = 2x\,dx$

$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2\, 2x\,dx}{(1+x^2)^{\frac{3}{2}}}$

$= \frac{1}{2}\int_1^4 \frac{u-1}{u^{\frac{3}{2}}}\,du$

$= \frac{1}{2}\int_1^4 u^{-\frac{1}{2}} - u^{-\frac{3}{2}}\,du$

$= \frac{1}{2}\left[2u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right]_1^4$

$= \frac{1}{2}\left[(4+1) - (4)\right]$

$= \frac{1}{2}$

c) $\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$ $\qquad u = \ln x \quad v = 2\sqrt{x}$, $u' = \frac{1}{x} \quad v' = \frac{1}{\sqrt{x}}$

$= \left[2\sqrt{x}\ln x\right]_1^e - \int_1^e 2x^{-\frac{1}{2}}\,dx$

$= 2\sqrt{e} - \left[4x^{\frac{1}{2}}\right]_1^e$

$= 2\sqrt{e} - 4\sqrt{e} + 4$

$= 4 - 2\sqrt{e}$

$\frac{\alpha}{\beta\gamma} = \frac{\alpha^2}{\alpha\beta\gamma}$

$= \frac{\alpha^2}{3}$

$\therefore$ roots $\frac{\alpha^2}{3}, \frac{\beta^2}{3}, \frac{\gamma^2}{3}$

replace x with $\sqrt{3x}$

$\therefore (\sqrt{3x})^3 - 5\sqrt{3x} - 3 = 0$

$3x\sqrt{3x} - 5\sqrt{3x} = 3$

$(3x\sqrt{3x} - 5\sqrt{3x})^2 = 9$

$27x^3 - 90x^2 + 75x - 9 = 0$

$x^3 - \frac{10}{3}x^2 + \frac{25}{9}x - \frac{1}{3} = 0$

e) roots $3+i, 3-i, \alpha$

$\therefore (3+i)(3-i)\alpha = -10$

$10\alpha = -10$

$\alpha = -1$

$\therefore 3 + i + 3 - i - 1 = -a$

$a = -5$

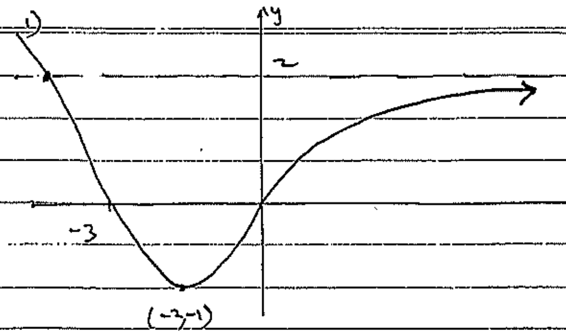
sub $x = -1$

$-1 - 5 - b + 10 = 0$

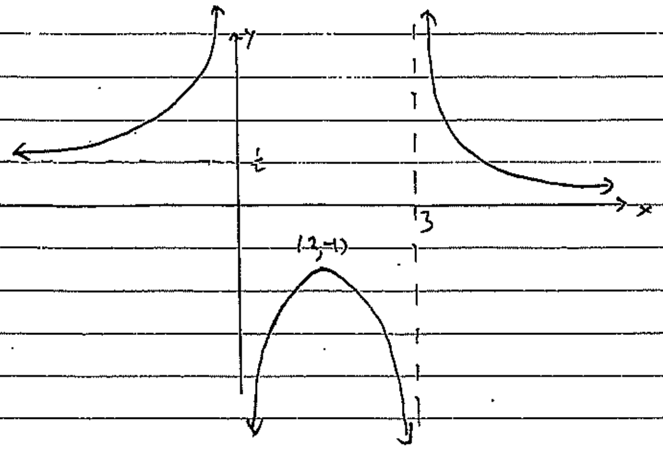
$b = 4$

$\therefore a = -5, b = 4$

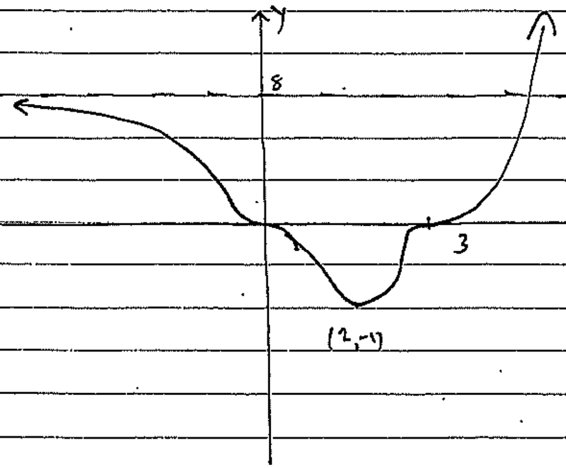
7. a) i)



ii)



iii)



$$\int \frac{\frac{2t}{1-t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$

$$= \int \frac{\frac{4t}{1-t^2}\,dt}{1+t^2+1-t^2}$$

$$= \int \frac{4t}{2(1-t^2)}\,dt$$

$$= -\int \frac{-2t}{1-t^2}\,dt$$

$$= -\ln(1-t^2)$$

$$= -\ln\left(1-\tan^2\frac{\theta}{2}\right) + c$$

c) St pt $3x^2-6x-24=0$

$3(x^2-2x-8)=0$

$3(x-4)(x+2)=0$

$x=4,-2$

$\therefore$ $x=4$ $\Rightarrow$ $y=64-48-96+k$

$= k-80$

$x=-2$ $y=-8-12+48+k$

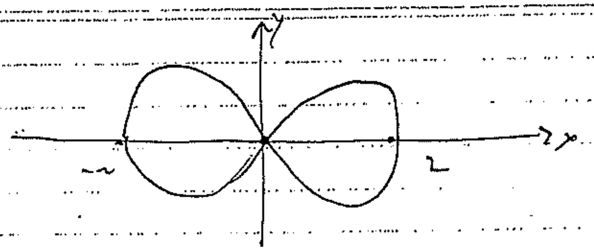
$= 28+k.$

$\therefore$ $(k-80)(28+k)>0$

$k>80,\ k<-28$

8.

a)



b) $\int \frac{x+6}{x^2+4x+29}\,dx$

$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+29}\,dx + \int \frac{4}{x^2+4x+29}\,dx$

$= \frac{1}{2}\ln|x^2+4x+29| + \int \frac{4}{(x+2)^2+25}\,dx$

$= \frac{1}{2}\ln|x^2+4x+29| + \frac{4}{5}\tan^{-1}\left(\frac{x+2}{5}\right) + C$

c) 1 is a double root

$(n+1)ax^n + nbx^{n-1} = 0$ at $x=1$

$a(n+1) + nb = 0$

and $a+b+1=0$

$b = -a-1$

$a(n+1) + n(-a-1) = 0$

$an + a - an - n = 0$

$\underline{a = n}$

$\therefore \underline{b = -n-1}$

d) i) $I_n = \int_0^1 x^n e^{x^2}\,dx$

$= \frac{1}{2}\int_0^1 x^{n-1}\, 2x e^{x^2}\,dx$

$u = x^{n-1}$ $\quad v = e^{x^2}$

$u' = (n-1)x^{n-2}$ $\quad v' = 2xe^{x^2}$

$= \frac{1}{2}\left[x^{n-1}e^{x^2}\right]_0^1 - \int_0^1 (n-1)x^{n-2}e^{x^2}\,dx$

$\therefore I_n = \frac{1}{2}e - \left(\frac{n-1}{2}\right) I_{n-2}$

ii) $I_5 = \frac{1}{2}e - 2I_3$

$= \frac{1}{2}e - 2\left[\frac{1}{2}e - I_1\right]$

$= \frac{1}{2}e - e + 2\int_0^1 xe^{x^2}\,dx$

$= \frac{1}{2}e - e + \left[e^{x^2}\right]_0^1$

$= \frac{1}{2}e - e + e - 1$

$= \underline{\frac{1}{2}e - 1}$

9. $\frac{x-3}{(x+5)(x+1)} = \frac{A}{x+5} + \frac{B}{x+1}$

a) $\therefore A(x+1) + B(x+5) = x-3$

let $x=-1$ $\quad B=-1$

let $x=-5$ $\quad A=2$

$\therefore \int \frac{2}{x+5} - \frac{1}{x+1}\, dx$

$= 2\ln|x+5| - \ln|x+1| + c$

b) $\int \cos^5 x\, dx$

$= \int \cos x \cos^4 x\, dx$

$= \int \cos x (1-\sin^2 x)^2\, dx$ $\quad$ let $u = \sin x$

$du = \cos x\, dx$

$= \int (1-u^2)^2\, du$

$= \int 1 - 2u^2 + u^4\, du$

$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$

c) i) Solutions are $z_1 = 1$

$z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_4 = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$z_5 = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

these can be written in many different ways.

ii) $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$\therefore 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$

$\cos\frac{2\pi}{5} + \left(2\cos^2\frac{2\pi}{5} - 1\right) = -\frac{1}{2}$ $\quad$ double angle result

$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$

$\cos\frac{2\pi}{5} = \frac{-2 \pm \sqrt{2^2 - 4\times 4\times -1}}{8}$

$= \frac{-2 \pm \sqrt{20}}{8}$

$= \frac{-1 \pm \sqrt{5}}{4}$

but $\cos\frac{2\pi}{5} > 0$

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

d) let roots be $\alpha, \beta, \alpha+\beta$

sum of roots 1 at a time $\quad 2\alpha + 2\beta = -a$ ①

2 at a time $\quad 3\alpha\beta + \alpha^2 + \beta^2 = b$ ②

3 at a time $\quad \alpha\beta(\alpha+\beta) = -c$ ③

from ③ and ①

$\alpha\beta\left(\frac{-a}{2}\right) = -c$

$\alpha\beta = \frac{2c}{a}$

from ②

$3\alpha\beta + \alpha^2 + \beta^2 = b$

$3\alpha\beta + (\alpha+\beta)^2 - 2\alpha\beta = b$

$\alpha\beta + (\alpha+\beta)^2 = b$

$$\therefore \quad \frac{2c}{a} + \left(\frac{-a}{2}\right)^2 = b$$

$$\frac{2c}{a} + \frac{a^2}{4} = b$$

$$8c + a^3 = 4ab$$

$$a^3 - 4ab + 8c = 0$$