Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2010 HSC ASSESSMENT TASK 4

Mathematics

Extension 1

General Instructions

- Working time – 50 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Ireland
- O Mr Lowe
- O Mr Rezcallah
- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number:________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	Total	Total %
Mark	—/6	—/10	—/6	—/9	—/14	—/5	—/50	—/100

Question 1 (6 marks) **Marks**

A particle moves linearly so that its displacement x metres from the origin O is given by $x = \sqrt{3}\cos 3t - \sin 3t$.

(i) Show that the equation can be expressed as $x = 2\cos(3t + \frac{\pi}{6})$. **2**

(ii) Show that the motion is simple harmonic. **2**

(iii) Find the time at which the particle first passes through the origin. **2**

Question 2 (10 marks)

When a particle moves along the x-axis with velocity v at time t, its acceleration is given by $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by the expression $v^2 = 28 + 24x - 4x^2$.

(i) Between which two points is the particle oscillating? **2**

(ii) What is the amplitude of the motion? **1**

(iii) Find the acceleration in terms of x. **2**

(iv) What is the centre of the motion? **1**

(v) Find the period of the oscillation. **2**

(vi) If the particle starts from the position furthest to the right, find the displacement function in terms of t. **2**

Question 3 (6 marks)

The velocity of a particle moving in a straight line at position x is given by $v = 2e^{-x}$. The particle is initially at the origin.

(i) Show that the acceleration at position x is given by $a = -4e^{-2x}$. **2**

(ii) What is the initial acceleration? **1**

(iii) Find the displacement function giving x at time t. **3**

Question 4 (9 marks)

(a) $(1 + \sqrt{3})^4 = a + b\sqrt{3}$, where a, b are integers. By expansion, find a and b. **3**

(b) Find the value of n such that when $(1 + x)^n$ is expanded in ascending powers of x, the coefficient of x^4 is twice the coefficient of x^3. **3**

(c) Find the exact value of the coefficient of x^{12} in the expansion of

$$\left(2x - \frac{1}{x^2}\right)^{30}$$ **3**

Question 5 (14 marks)

(a) Isaac calculates that he will need \$1 000 000 if he is to retire in 20 years' time and maintain his present lifestyle. He decides to invest a fixed amount \$$P$ at the beginning of each quarter in an investment account that pays interest at an annual rate of 8% compounding quarterly.

Let A_n be the amount in dollars in Isaac's investment account at the end of the n^{th} quarter.

(i) Show that $A_2 = \$P \times (1.02 + 1.02^2)$ **2**

(ii) Find a similar expression for the value of his investment after 20 years and hence calculate the value of P needed to realise the amount of \$1 000 000 needed for his retirement. *(Show all appropriate working).* **3**

(b) Mr Wu decides to borrow \$250 000 to buy a unit. Interest is calculated monthly on the balance still owing, at a rate of 6% per annum. The loan is to be repaid at the end of 15 years with equal monthly repayments of \$$M$. Let A_n be the amount owing after the n^{th} repayment.

(i) Show that $A_{60} = 250\,000(1.005)^{60} - M(1 + 1.005 + \cdots + 1.005^{59})$ **2**

(ii) Find the value of M, showing all appropriate working. **2**

(iii) Hence calculate the amount still owing after 5 years of payment at this rate. **2**

(iv) At the end of five years the interest rate is increased to 7·2% per annum and Mr Wu changes his payments to \$2400 per month.
How many months are required to pay off the remainder of the loan? **3**

Question 6 (5 marks)

(a) By considering the expansion of the identity $(1 + x)^6\,(1 + x)^6 \equiv (1 + x)^{12}$, show that

$$\sum_{k=0}^{6} \binom{6}{k}^2 = \binom{12}{6}$$ **3**

(b) Solve the inequality $x^4 + 4x^3 + 6x^2 + 4x + 1 \leq x^3 + 3x^2 + 3x + 1$ **2**

END OF EXAMINATION

$\boxed{\text{Q1}}$

(i) $x = \sqrt{3}\cos 3t - \sin 3t$

$= 2\left(\frac{\sqrt{3}}{2}\cos 3t - \frac{1}{2}\sin 3t\right)$

$= 2\cos(3t + \alpha)$

where $\cos\alpha = \frac{\sqrt{3}}{2}$, $\sin\alpha = \frac{1}{2}$ $\therefore \alpha = \frac{\pi}{6}$

$\therefore x = 2\cos\left(3t + \frac{\pi}{6}\right)$

✓ Derives amplitude (Also: $a = \sqrt{3+1} = 2$.

✓ Derives auxiliary angle correctly (showing in working that it is in 1^{st} quadrant)

(ii) $\dot{x} = -6\sin\left(3t + \frac{\pi}{6}\right)$

$\ddot{x} = -18\cos\left(3t + \frac{\pi}{6}\right)$

But $x = 2\cos\left(3t + \frac{\pi}{6}\right)$

$\therefore \ddot{x} = -9x$. ✓

This is of form $\ddot{x} = -n^2 x$ ✓

$\therefore$ motion is S.H.M.

(iii) At $t=0$, $x = 2\cos\frac{\pi}{6} = \sqrt{3}$

and $v = -6\sin\frac{\pi}{6} = -3$ m/s

So at $t=0$ the particle is moving towards the origin.

At $x=0$ we have $\cos\left(3t + \frac{\pi}{6}\right) = 0$

$\therefore 3t + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ ✓

$\therefore t = \frac{\pi}{9}, \frac{4\pi}{9}, \ldots$

So first through origin at $t = \frac{\pi}{9}$ secs. ✓

6

Q2

(i) $v^2 = 28 + 24x - 4x^2$

$= 4(7 + 6x - x^2)$

$= 4(7-x)(1+x)$

Since $v^2 \geq 0$, $\therefore 4(7-x)(1+x) \geq 0$ ✓

$\therefore -1 \leq x \leq 7$ ✓

(ii) amplitude $= \dfrac{7-(-1)}{2} = 4$ ✓

(iii) $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$= \dfrac{d}{dx}\left(14 + 12x - 2x^2\right)$ ✓

$= 12 - 4x$ ✓

$\ddot{x} = -4(x-3)$

(iv) Centre of motion is $x = 3$ ✓

(v) $T = \dfrac{2\pi}{n}$.

Now $\ddot{x} = -n^2(x-3)$ $\therefore n = 2$ ✓ for n

$\therefore T = \dfrac{2\pi}{2} = \pi$ seconds. ✓ for T

(vi) $x = 3 + 4\cos(2t + \alpha)$ ✓ appropriate start

At $t=0$, $x = 7$

$\therefore 7 = 3 + 4\cos\alpha$

$\cos\alpha = 1$

$\alpha = 0$

$\therefore x = 3 + 4\cos 2t$. ✓ or equivalent equation

10

$\boxed{3}$ $V = 2e^{-x}$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2} \cdot 4 \cdot e^{-2x}\right)$ ✓ Correct start

$= -2 \cdot 2e^{-2x}$

$\therefore a = -4e^{-2x}$ ✓ shows correct working

(ii) When $t = 0$, $x = 0$

$\therefore a = -4e^0$

$= -4$

$\therefore$ initial acceleration is -4 units. ✓ for -4

(iii) $\frac{dx}{dt} = 2e^{-x}$

$\therefore \frac{dt}{dx} = \frac{e^x}{2}$ ✓

$t = \frac{1}{2}e^x + C$

When $t = 0$, $x = 0$

$\therefore 0 = \frac{1}{2}e^0 + C$

$\therefore C = -\frac{1}{2}$ ✓

$\therefore t = \frac{1}{2}e^x - \frac{1}{2}$

$2t = e^x - 1$

$e^x = 2t + 1$

$x = \ln(2t+1)$ ✓

6

$\boxed{4}$

(a) $(1+\sqrt{3})^4 = 1^4 + 4\cdot 1^3\cdot\sqrt{3} + 6\cdot 1^2\cdot(\sqrt{3})^2 + 4\cdot 1\cdot(\sqrt{3})^3 + (\sqrt{3})^4$

$= 1 + 4\sqrt{3} + 18 + 12\sqrt{3} + 9$

$= 28 + 16\sqrt{3}$

$\therefore\ a = 28$
$b = 16$

(students who only write $28+16\sqrt{3}$ without $a=28$, $b=16$ get 2 marks)

✓ correctly uses Pascal or binomial [...]

✓ for a=28
✓ for b=16

(b) Coefficient of x^4 is ${}^nC_4$, coeff. of x^3 is ${}^nC_3$

$\therefore\ {}^nC_4 = 2\cdot{}^nC_3$ ✓

students who write $2\,{}^nC_4 = {}^nC_3$ and $n=5$ get only 1 mark. (Answer not checked).

$$\frac{n!}{(n-4)!\,4!} = 2\cdot\frac{n!}{(n-3)!\,3!}$$ ✓

$n-3 = 2\cdot 4$

$n = 11$ ✓

Check: ${}^{11}C_4 = 2\times{}^{11}C_3$. ✓ works.

(c) The general term in $(a+b)^n$ is $T_{r+1} = {}^nC_r\,a^{n-r}\,b^r$

Thus general term in $\left(2x - \frac{1}{x^2}\right)^{30}$

equals ${}^{30}C_r\,(2x)^{30-r}\left(-\frac{1}{x^2}\right)^r$

a correct general term (expand[...]) ✓

$= {}^{30}C_r\cdot 2^{30-r}\cdot(-1)^r\cdot x^{30-r}\cdot x^{-2r}$

$= {}^{30}C_r\cdot 2^{30-r}\cdot(-1)^r\cdot x^{30-3r}$

We want x^{12}, $\therefore\ 30-3r = 12$

$\therefore r = 6$

✓ or $r=$[...] starting fro[...]

$\therefore$ coefficient desired is ${}^{30}C_6\cdot 2^{24} \doteq 9.96\times 10^{24}$ ✓

Also ${}^{30}C_{24}\times 2^{24}$

${}^{30}C_{12}\,(2x)^{30-12}\,(-1)^{12}(x^{-2})^{12}$ gets no marks !!

2nd method: 7th term from expansion and ${}^{30}C_6\,2^{24}$

9

5 (a) 0.08 p.a. $\equiv 0.02$ per quarter

(i) After 1 quarter:

$$A_1 = P + P(0.02)$$
$$= P(1.02)$$

After 2 quarters:

$$A_2 = A_1(1.02) + P(1.02)$$
$$= P(1.02)(1.02) + P(1.02)$$
$$= P(1.02^2 + 1.02)$$, as required.

* To get the marks the derivation (ie the process) must be shown

(ii) Similarly, after 20 years (= 80 quarters),

$$A_{80} = P(1.02^{80} + 1.02^{79} + \dots + 1.02^2 + 1.02)$$
$$= P(\underbrace{1.02^1 + 1.02^2 + \dots + 1.02^{80}}_{\text{a G.P. with } a = 1.02,\ r = 1.02,\ n = 80})$$

$$\therefore S_{80} = \frac{1.02(1.02^{80} - 1)}{1.02 - 1}$$
$$= 51(1.02^{80} - 1)$$

So if we want $A_{80} = 1\,000\,000$,

then $P = \dfrac{1\,000\,000}{51(1.02^{80} - 1)}$

$\therefore P = \$5\,059.52$ (nearest cent).

$\boxed{5}$ - (b)

(i) $A_1 = 250\,000(1.005) - M$

$A_2 = A_1(1.005) - M$

$= [250\,000(1.005) - M](1.005) - M$

$= 250\,000(1.005)^2 - M(1 + 1.005)$

$\vdots$

$\therefore A_{60} = 250\,000(1.005)^{60} - M(1 + 1.005 + \cdots + 1.005^{59})$ as required ✓✓ must show process

(ii) 15 years = 180 months, $\therefore A_{180} = 0$

$\therefore 250\,000(1.005)^{180} - M(\underbrace{1 + 1.005 + \cdots + 1.005^{179}}_{\text{a G.P. with } a=1,\ r=1.005,\ n=180}) = 0$ ✓

$\therefore S_{180} = \dfrac{1(1.005^{180} - 1)}{1.005 - 1}$

$\therefore M = 250\,000(1.005)^{180} \times \dfrac{0.005}{1.005^{180} - 1}$

$= \$2\,109.64$ (nearest cent) ✓

(iii) Amount still owing after 5 years

$= A_{60} = 250\,000(1.005)^{60} - 2109.64\left[\dfrac{1.005^{60} - 1}{1.005 - 1}\right]$ ✓

$= \$190\,022.89$ ✓

$\boxed{5}$-(b) continued

(iv) 7.2% p.a. $= 0.6\%$ per month.

We must pay off \$190 022.89 given

$r = 0.006$, $M = \$2400$:-

Suppose it is paid off after n months, then

$$190\,022.89\,(1.006)^n - 2400\,\frac{(1.006^n - 1)}{0.006} = 0$$

correct set-up start

$$\therefore\ 190\,022.89\,(1.006)^n - 400\,000\,(1.006^n - 1) = 0$$

$$\therefore\ 1.006^n = \frac{400\,000}{400\,000 - 190\,022.89}$$

$$= \frac{400\,000}{209\,977.11}$$

correct process

$$n \log_{10} 1.006 = \log_{10} \frac{400\,000}{209\,977.11}$$

$$n = \frac{\log_{10}\left(\frac{400\,000}{209\,977.11}\right)}{\log_{10} 1.006}$$

$$\doteqdot 107.7$$

correct answer

ie. 108 months are required.

(accept

$\boxed{6}$

(a) The coefficient of x^6 in $(1+x)^{12}$ is ${}^{12}C_6 = \binom{12}{6}$ ✓

The coefficient of x^6 in $(1+x)^6 \cdot (1+x)^6$ is

$${}^6C_0 \cdot {}^6C_6 + {}^6C_1 \cdot {}^6C_5 + {}^6C_2 \cdot {}^6C_4 + {}^6C_3 \cdot {}^6C_3 + {}^6C_4 \cdot {}^6C_2 + {}^6C_5 \cdot {}^6C_1 + {}^6C_6 \cdot {}^6C_0$$ ✓

$$= ({}^6C_0)^2 + ({}^6C_1)^2 + ({}^6C_2)^2 + ({}^6C_3)^2 + ({}^6C_4)^2 + ({}^6C_5)^2 + ({}^6C_6)^2$$ ✓ uses pascal relation

$$\left[\text{since } {}^6C_k = {}^6C_{6-k}\right]$$

$$= \sum_{k=0}^{6} ({}^6C_k)^2.$$

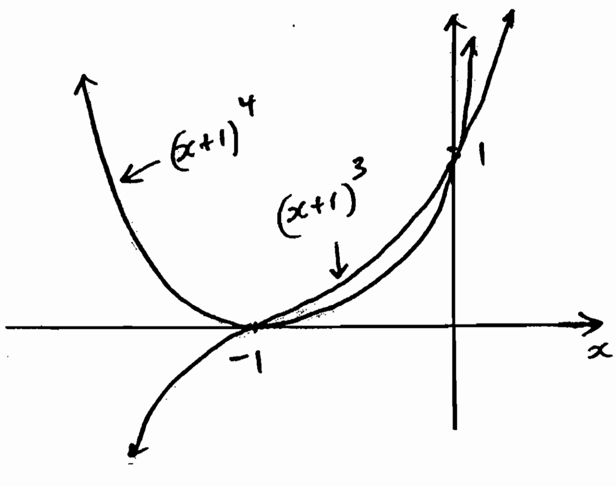
Hence $\displaystyle\sum_{k=0}^{6} \binom{6}{k}^2 = \binom{12}{6}$

(b) $x^4 + 4x^3 + 6x^2 + 4x + 1 \leq x^3 + 3x^2 + 3x + 1$

$\therefore \quad (1+x)^4 \leq (1+x)^3$

ie $\quad (x+1)^4 \leq (x+1)^3$ ✓ correct start

$\therefore -1 \leq x \leq 0$ ✓ correct answer



5